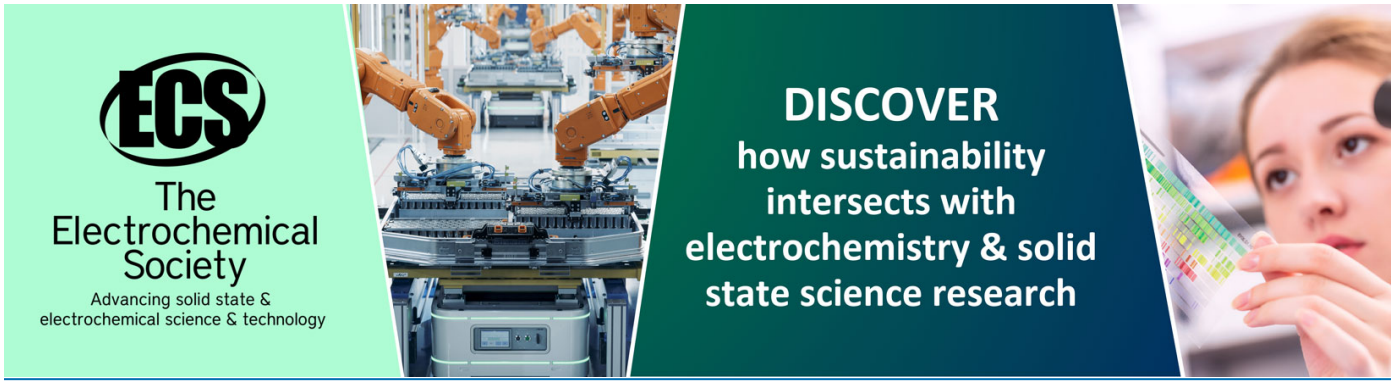
On position-dependent mass harmonic oscillators



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On position-dependent mass harmonic oscillators

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Abstract. The classical and quantum position-dependent mass harmonic oscillators are constructed by means of the supersymmetric approach. The correspondence between the classical and the quantum Hamiltonians is used to fix the ordering of the kinetic term in the quantum framework. Some examples are given for different types of mass function.

1. Introduction

In the study of the electronic properties of semiconductors, it is well known that the envelope wave function of a single electron fulfills the Schrödinger equation (the effective mass wave equation) with a Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + E + V(x)\,, \tag{1}$$

where E is the edge of the conduction band and $V(x)$ a potential term which includes the interaction with an impurity (if any) inside the semiconductor, an external potential, etc [1, 2]. In this Hamiltonian, m stands for the effective mass of the electron in the lattice. In the case of nonuniform semiconductors, both, the effective mass m, as well as the edge of the conduction band E, depend on the position. Therefore, it is necessary to establish the effective mass wave equation in an appropriate form [3–8], as it is clear that the Hamiltonian (1) is not a Hermitian operator when $m = m(x)$. The position-dependence of the edge of the conduction band can be absorbed into the potential term, but since the effective mass $m(x)$ and the momentum operator p do not commute any longer, a problem arises in stating the correct form of the kinetic term. A general expression for a Hermitian kinetic energy operator is

$$T = \frac{1}{2}m^a p\, m^{2b} p\, m^a, \tag{2}$$

where $a + b = -1/2$. For solving a particular problem we require, therefore, to know the value of a or b. Nevertheless, it is not clear how to fix both parameters.

In general, the Hamiltonians with position-dependent mass (PDM) have been widely studied due to its multiple applications, not only in the theory of semiconductors. Indeed, they also appear in the study of quantum dots [9] and quantum liquids [10], as well as in the description

of non linear oscillators [11, 12], to mention just a few fields. A very important dilemma in this context is how to deal with the ordering ambiguity of the kinetic term (2) in the quantum case. This question has been addressed using different approaches (see for instance [13, 14]). The aim of this work is to establish a kind of criterion to chose a proper ordering in the particular case of the PDM harmonic oscillator. Our treatment will be based on a comparison between the classical and the quantum cases. We will also show that this choice enables to extend, to this system, many of the well known properties of the constant mass (CM) case in a very simple way, reproducing some results already obtained from other points of view.

The paper is organized as follows. In Section 2, we present a short review of the classical CM harmonic oscillator. Section 3 is devoted to the study of the classical PDM harmonic oscillator by means of the canonical transformation connecting this problem with the CM case. In Section 4 we illustrate the formalism by considering some particular mass functions. In Section 5 it is shown that there is a particular ordering in the kinetic term of the quantum PDM Hamiltonian giving an expression for the potential which coincides with the classical one. This allows us to solve the problem in a very simple form by means of the correspondence between CM and PDM systems. In Section 6 these results are illustrated using the same mass functions introduced before in the classical analysis. Finally, we present some concluding remarks.

2. The classical constant mass harmonic oscillator

Since the comparison between the CM and the PDM harmonic oscillators will be the keystone of our approach, let us start by presenting some basic features of the CM harmonic oscillator. Indeed, let us consider the Hamiltonian (choosing, without loss of generality, $m = \omega = 1$)

$$\mathcal{H} = \frac{P^2}{2} + \frac{X^2}{2}\,, \tag{3}$$

where X, P stand for the classical position and momentum variables. The phase space trajectories can be constructed in a purely algebraic way: observe that $\mathcal{H}$ can be factorized in the form

$$\mathcal{H} = a^+ a^- = a^- a^+\,, \qquad a^\pm = \frac{1}{\sqrt{2}}\left(\mp iP + X\right), \tag{4}$$

with $a^\pm$ fulfilling $(a^-)^* = a^+$ (* denoting complex conjugation). It turns out that these functions close the Heisenberg algebra with Poisson brackets [15]

$$i\left\{a^-, a^+\right\} = 1\,, \qquad i\left\{\mathcal{H}, a^\pm\right\} = \pm a^\pm. \tag{5}$$

This algebraic structure allows us to construct non-autonomous integrals of motion of the form

$$Q^\pm = a^\pm e^{\mp it} = \frac{1}{\sqrt{2}}\left(\mp iP + X\right) e^{\mp it} \tag{6}$$

satisfying $(Q^-)^* = Q^+$, and $Q^+Q^- = \mathcal{H}$. On the other hand, it is well known that the Hamiltonian is a conserved quantity, indeed the total energy E of the system. Therefore, letting $Q^+ = \sqrt{E}e^{i\phi}$, with ϕ a constant phase fixed by the initial conditions, it is possible to obtain the well known phase trajectories for the CM harmonic oscillator, namely

$$X(t) = \sqrt{2E}\cos(t+\phi)\,, \qquad P(t) = -\sqrt{2E}\sin(t+\phi)\,, \tag{7}$$

which are concentric circumferences centered at the origin and having radius $\sqrt{2E}$. The position and momentum of a particle in this potential can take arbitrarily large values just by choosing the energy E in the proper way.

3. The classical position dependent mass harmonic oscillator

A generic PDM classical Hamiltonian has the standard form

$$\mathcal{H} = \frac{p^2}{2m(x)} + \mathcal{V}(x), \tag{8}$$

where $m(x)$ is an arbitrary position-dependent function and the potential $\mathcal{V}(x)$, which depends on the form of $m(x)$, has to be determined. With this aim, let us suppose that the Hamiltonian $\mathcal{H}$ can be factorized in terms of two functions $\mathcal{A}^{\pm}(x,p)$ of the form

$$\mathcal{A}^{\pm}(x,p) = \mp i\frac{p}{\sqrt{2m(x)}} + \mathcal{W}(x). \tag{9}$$

In this way

$$\mathcal{H} = \mathcal{A}^{+}\mathcal{A}^{-} = \mathcal{A}^{-}\mathcal{A}^{+} = \frac{p^2}{2m(x)} + \mathcal{W}^2(x), \tag{10}$$

and then, the potential $\mathcal{V}(x)$ and the position-dependent function $\mathcal{W}(x)$ are related by

$$\mathcal{V}(x) = \mathcal{W}^2(x). \tag{11}$$

Next, we demand that the functions $\mathcal{H}$, $\mathcal{A}^{\pm}$ close the Heisenberg algebra with Poisson brackets as in the CM case (5). It is not difficult to show that

$$i\left\{\mathcal{A}^{-},\mathcal{A}^{+}\right\} = \frac{2\mathcal{W}(x)'}{\sqrt{2m(x)}}\,, \qquad i\left\{\mathcal{H},\mathcal{A}^{\pm}\right\} = \pm\frac{2\mathcal{W}(x)'}{\sqrt{2m(x)}}\,\mathcal{A}^{\pm}\,, \tag{12}$$

so the former condition fixes $\mathcal{W}(x)$ as

$$\mathcal{W}(x) = \frac{1}{\sqrt{2}}\left(\int^{x}\sqrt{m(t)}dt + X_0\right), \tag{13}$$

where X_0 is an integration constant. The potential $\mathcal{V}(x)$ has then the form

$$\mathcal{V}(x) = \frac{1}{2}\left(\int^{x}\sqrt{m(t)}dt + X_0\right)^2, \tag{14}$$

and hence, the classical PDM harmonic oscillator Hamiltonian (8) is

$$\mathcal{H} = \frac{p^2}{2m(x)} + \frac{1}{2}\left(\int^{x}\sqrt{m(t)}dt + X_0\right)^2. \tag{15}$$

Notice that it has a very similar form to the CM Hamiltonian (3). Indeed, if we perform the canonical transformation

$$X(x) = \int^{x}\sqrt{m(t)}dt + X_0\,, \qquad P(x,p) = \dot{X}(x) = \frac{p}{\sqrt{m(x)}}\,, \tag{16}$$

the PDM Hamiltonian (15) transforms into the CM one (3). Observe that the constant X_0 in (13) determines the position of the origin of the potential in the variable X. We fix this constant in such a way that

$$\mathcal{V}(x=0) = 0. \tag{17}$$

The classical PDM problem, for the harmonic oscillator, can then be reduced to the CM problem, where the new "position" and "momentum" variables are related to the mass function by means of (16). For instance, the phase space trajectories $x(t)$, $p(t)$ are determined as

$$x(t) = X^{-1}\left(\sqrt{2E}\cos(t+\phi)\right), \qquad p(t) = -\sqrt{2E}\sqrt{m(x(t))}\sin(t+\phi), \tag{18}$$

and can be either harmonic or not, depending on the form of $m(x)$. It is worthwhile to mention that they could also be obtained by means of the integrals of motion

$$\mathcal{Q}^{\pm} = \frac{1}{\sqrt{2}}\left(\mp iP(x,p) + X(x)\right)e^{\mp it} = \frac{1}{\sqrt{2}}\left(\mp i\frac{p}{\sqrt{m(x)}} + \int^{x}\sqrt{m(t)}dt + X_0\right)e^{\mp it} = \mathcal{A}^{\pm}e^{\mp it}. \tag{19}$$

In order to construct the phase space trajectories for some particular cases, it is necessary to specify the form of the mass function. We can choose it basically in an arbitrary way; however, since our treatment is based on the correlation between the CM and the PDM systems, it is more natural to choose $m(x)$ in such a way that we can make a complete correspondence between both harmonic oscillators. We refer particularly to the canonical transformation (16) connecting them. Observe that for some choices of $m(x)$, the first equation in (16) may not map the mass natural domain $\mathcal{D}(m)$ onto the whole real line, as it is required if $X(x)$ represent the position of the CM harmonic oscillator. With this in mind, we should take into account the following facts [16]:

(A) Suppose the mass function has $\mathbb{R}$ as its domain. Then, $m(x)$ should be such that $X(x)$ becomes divergent as $|x| \to \infty$; in this way, we assure that the range $\mathcal{R}(X) = \mathbb{R}$. For instance, if $m(x) \sim 1/x^q$ (with $q > 0$) as $|x|$ grows, then

$$X(x) \sim \frac{1}{x^{(q-2)/2}} \tag{20}$$

is divergent for $q < 2$. If $q = 2$, then $X(x) \sim \ln|x|$.

(B) Suppose, on the contrary, that $m(x)$ is only defined on a finite interval (a, b). Then, $\mathcal{R}(X) = \mathbb{R}$ if $X(x)$ is divergent in both limits of this interval. For instance, if $m(x)$ presents a singularity, *i.e.*, $m(x) \sim 1/\left(x - x_0\right)^q$ (with $q > 0$) as $x \to x_0$, then

$$X(x) \sim \frac{1}{\left(x - x_0\right)^{(q-2)/2}}, \tag{21}$$

is now divergent for $q > 2$. Again, if $q = 2$, then $X(x) \sim \ln|x|$.

If $m(x)$ does not satisfy these conditions, this approach is still valid but the solutions to the problem will present some differences with respect to the CM harmonic oscillator solutions. We will see that this fact also has consequences on the wave functions of the corresponding quantum PDM Hamiltonian.

4. Examples of classical position dependent mass harmonic oscillator

As illustrative examples, we present now the cases of some masses considered before, for some authors, in the study of PDM systems in different contexts. These include some forms of $m(x)$ depending on two real parameters, m_0 and λ, which do not satisfy the conditions mentioned at the end of the previous section.

In the first instance, let us consider the mass

$$m_1(x) = m_0\left(\frac{1+\lambda+x^2}{1+x^2}\right)^2, \tag{22}$$

which is a function free of singularities in the real line, *i.e.*, $\mathcal{D}(m_1) = \mathbb{R}$. It takes its maximum value $m_0(1+\lambda)^2$ at $x = 0$ and tends to a constant value m_0 as $|x| \to \infty$. The function (22) is represented in Figure 1(a). The "position" is obtained from (16)–(17). It is not difficult to show that $X_0 = 0$ and that

$$X_1(x) = \sqrt{m_0}\,[x + \lambda \arctan x] \tag{23}$$

takes arbitrary values in $\mathbb{R}$. The potential has the form

$$\mathcal{V}_1(x) = \frac{m_0}{2}\,[x + \lambda \arctan x]^2\,, \tag{24}$$

and it can be seen from Figure 1(b), that it is just a slight deformation of the CM harmonic oscillator potential. Also, some plots of the phase trajectories are represented in Figure 1(c). They are given by the expression

$$\frac{m_0}{2}\,[x + \lambda \arctan x]^2 + \frac{1}{2m_0}\left(\frac{1+x^2}{1+\lambda-x^2}\right)^2 p^2 = E, \tag{25}$$

and they seem quite similar to the usual harmonic oscillator. Notice that, in the limit $\lambda \to 0$, the CM harmonic oscillator is recovered. The mass (22) has been used in many works due to its physical behavior [17–20], since it represent a system which has almost a constant mass all over the space except for a small region near the origin, where its mass softly increases.

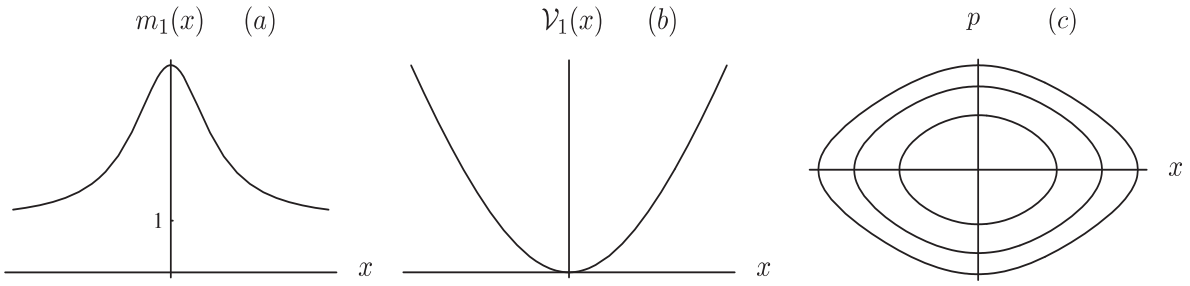

Figure 1. The mass $m_1(x)$ (a), the potential $\mathcal{V}_1(x)$ (b) and some phase space trajectories (c) for the following values of parameters: $m_0 = \lambda = 1$, $\phi = 0$ and $E = 0.3, 0.7, 1.1$.

In the second place, we consider the mass

$$m_2(x) = m_0 \tanh^2(\lambda x)\,. \tag{26}$$

This function, represented in Figure 2(a), is also free of singularities in the real line ($\mathcal{D}(m_2) = \mathbb{R}$). It is constant in almost all the straight real line, but contrary to the previous case, it rapidly vanishes in a small region near the origin.

In this case

$$X_2(x) = \frac{\sqrt{m_0}}{\lambda}\,\mathrm{sign}(x)\;\ln\left(\cosh \lambda x\right)\,, \tag{27}$$

takes also arbitrary values in $\mathbb{R}$. The potential

$$\mathcal{V}_2(x) = \frac{m_0}{2\lambda^2}\;\ln^2\left(\cosh \lambda x\right), \tag{28}$$

is shown in Figure 2(b) and the phase trajectories

$$\frac{m_0}{2\lambda^2}\;\ln^2\left(\cosh \lambda x\right) + \frac{1}{2m_0}\frac{p^2}{\tan^2 \lambda x} = E \tag{29}$$

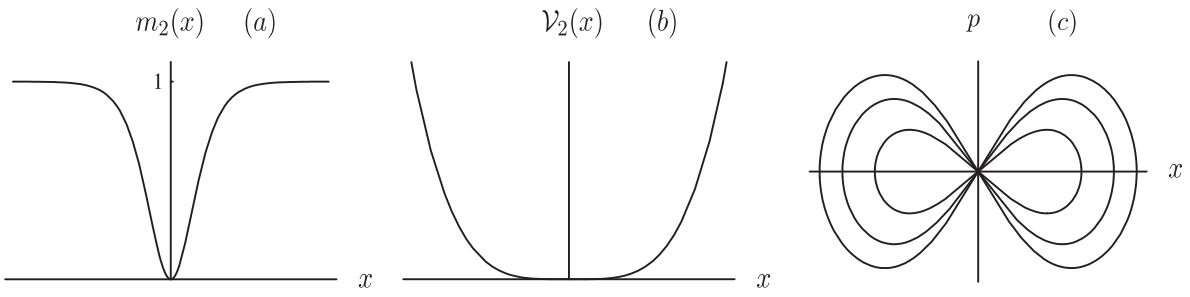

Figure 2. The mass $m_2(x)$ (a), the potential $\mathcal{V}_2(x)$ (b) and some phase space trajectories (c) for the following values of parameters: $m_0 = \lambda = 1$, $\phi = 0$ and $E = 0.3, 0.7, 1.1$.

are radically deformed near the origin, showing the fact that the mass vanishes at $x = 0$ (see Figure 2(c)). The CM case is now recovered in the limit $\lambda \to \infty$. The mass (26) has been used to illustrate the mechanism of evaluating the Green functions for PDM systems [21].

Finally, we consider the case

$$m_3(x) = \frac{m_0}{1 - (\lambda x)^2}\,. \tag{30}$$

This mass function has two singularities, its domain is now $\mathcal{D}(m_3) = (-1/\lambda, 1/\lambda)$, and it rapidly grows as x tend to the limits of the domain, while it reaches its minimum value at $x = 0$ (see Figure 3(a)).

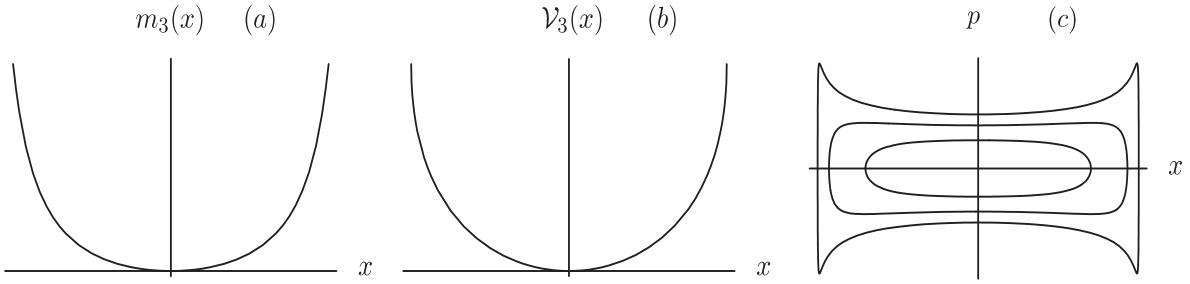

Figure 3. The mass $m_3(x)$ (a), the potential $\mathcal{V}_3(x)$ (b) and some phase space trajectories (c) for the following values of parameters: $m_0 = \lambda = 1$, $\phi = 0$ and $E = 0.3, 0.7, 1.1$.

Observe that, even when the mass presents singularities at $|x| = 1/\lambda$, the variable

$$X_3(x) = \frac{\sqrt{m_0}}{\lambda} \arcsin \lambda x \tag{31}$$

is not divergent at this points and takes values only in the interval $\left[-\frac{\sqrt{m_0}\pi}{2\lambda}, \frac{\sqrt{m_0}\pi}{2\lambda}\right]$. The finite range potential

$$\mathcal{V}_3(x) = \frac{m_0}{2\lambda^2} \arcsin^2 \lambda x, \tag{32}$$

takes its maximum value $\mathcal{V}_{max} = \dfrac{m_0\pi^2}{8\lambda^2}$ at $x = \pm 1$, as it is shown in Figure 3(b). Nevertheless, due to the fact that the mass becomes infinite at this points, the particle is still confined in the region determined by $\mathcal{D}(m_3)$, even when its energy $E > \mathcal{V}_{max}$. The plots of the phase space trajectories

$$\frac{m_0}{2\lambda^2} \arcsin^2 \lambda x + \frac{1}{2m_0} \left(1 - (\lambda x)^2\right) p^2 = E \tag{33}$$

illustrate well this situation in Figure 3(c): the motion is confined to a zone from which the system can not escape since at the boundaries its mass tends to infinity. This fact becomes more evident as the energy of the system takes larger values, in such a way that, for $E > \mathcal{V}_{max}$, the momentum becomes infinity. The CM harmonic oscillator is recovered, in this case, in the limit $\lambda \to 0$. Masses of this kind appears in the study of singular oscillators (see, *e.g.* [11, 12]).

Observe that, in the previous examples, the "position" $X(x)$ and the potential $\mathcal{V}(x)$ are, respectively, odd and even functions due to the fact that $m(x)$ is an even function of x in all the cases.

5. The quantum position-dependent mass harmonic oscillator

It is worthwhile, as in the classical framework, to begin this section with some words about the CM quantum harmonic oscillator. The Hamiltonian has the form ($\hbar = m = \omega = 1$)

$$h = -\frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{2}y^2 = b^+b^- + \frac{1}{2}, \tag{34}$$

where y and $-id/dy$ are the position and momentum operators in the coordinate representation. The creation and annihilation operators

$$b^\pm = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dy} + y\right), \tag{35}$$

together with the Hamiltonian h, close the Heisenberg algebra

$$\left[b^-, b^+\right] = 1\,, \qquad \left[h, b^\pm\right] = \pm b^\pm. \tag{36}$$

These commutation relations allow us to determine the spectral values and wave functions:

$$E_n = n + \frac{1}{2}\,, \qquad \phi_n(y) = \frac{1}{\sqrt{n!}}\left(b^+\right)^n \phi_0(y) = \frac{1}{\sqrt{\pi^{1/2}2^n n!}}\, e^{-y^2/2} H_n(y), \tag{37}$$

with $\phi_0(y)$ the ground state (the eigenstate state annihilated by b^-), and $H_n(y)$ the Hermite polynomials.

In the PDM case, as stated before, there exist an ordering ambiguity of p and $m(x)$ in the kinetic term of the Hamiltonian due to the fact that, these variables do not commute in this framework. A general, Hermitian, kinetic term has the form (2)

$$T_a = -\frac{1}{2}m^a\frac{d}{dx}m^{2b}\frac{d}{dx}m^a, \tag{38}$$

with $a + b = -1/2$. Then, the generic PDM Hamiltonian is

$$H_a = -\frac{1}{2}m^a\frac{d}{dx}m^{2b}\frac{d}{dx}m^a + V_a(x), \tag{39}$$

where the potential $V_a(x)$, as in the classical case, is a position-dependent function which has to be determined.

It turns out that the ambiguity parameters can be transferred from the kinetic to the potential term just by choosing a particular ordering and redefining the potential in the proper form. In many works, this particular ordering is fixed *a priori* without taking into account the particular problem which is to be solved. Here, we will first determine the potential, and its form will suggest the proper ordering for solving this problem.

Using the supersymmetric approach to construct the potential $V_a(x)$ [16–18], we suppose that H_a can be factorized in terms of two linear operators $A_a^{\pm}$:

$$H_a = A_a^+ A_a^- + \frac{1}{2}. \tag{40}$$

These operators have the form

$$A_a^- = \frac{1}{\sqrt{2}} m^b \frac{d}{dx} m^a + W_a(x), \tag{41}$$

$$A_a^+ = -\frac{1}{\sqrt{2}} m^a \frac{d}{dx} m^b + W_a(x), \tag{42}$$

where $W_a(x)$ is a position-dependent function to be determined. Observe $A_a^{\pm}$ can also be written as

$$A_a^- = \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{m}} \frac{d}{dx} + \frac{a}{\sqrt{m}} (\ln m)' \right] + W_a \,, \tag{43}$$

$$A_a^+ = -\frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{m}} \frac{d}{dx} + \frac{b}{\sqrt{m}} (\ln m)' \right] + W_a \,, \tag{44}$$

so defining a new differential operator

$$\mathbf{D} = \frac{1}{\sqrt{m}} \frac{d}{dx}, \tag{45}$$

they take very simple forms:

$$A_a^- = \frac{1}{\sqrt{2}} \left[\mathbf{D} + a \left(\mathbf{D} \ln m \right) \right] + W_a \,, \tag{46}$$

$$A_a^+ = -\frac{1}{\sqrt{2}} \left[\mathbf{D} + b \left(\mathbf{D} \ln m \right) \right] + W_a \,. \tag{47}$$

Equation (40) implies that $V_a(x)$ and $W_a(x)$ are related by the expression

$$V_a(x) = \frac{1}{\sqrt{2}} \left[\frac{4a+1}{2} W_a \left(\mathbf{D} \ln m \right) - \mathbf{D} W_a \right] + W_a^2 + \frac{1}{2}. \tag{48}$$

It is not difficult to show that

$$\left[A_a^-, A_a^+ \right] = \sqrt{2} \mathbf{D} W_a + \frac{4a+1}{4} \left(\mathbf{D}^2 \ln m \right); \tag{49}$$

then, demanding that $A_a^{\pm}$ fulfill the Heisenberg algebra, we find the following expression for $W_a(x)$

$$W_a(x) = \frac{1}{\sqrt{2}} \left[\int^x \sqrt{m} dt + y_0 - \frac{4a+1}{4} \left(\mathbf{D} \ln m \right) \right], \tag{50}$$

where y_0 is an integration constant which has to be chosen (as in the classical case) in such a way that the origin of the potential occurs at $x = 0$. The potential has then the form:

$$V_a(x) = \frac{1}{2} \left[\left(\int^x \sqrt{m} dt + y_0 \right)^2 + \frac{4a+1}{4} \left(\mathbf{D}^2 \ln m \right) - \left(\frac{4a+1}{4} \mathbf{D} \ln m \right)^2 \right]. \tag{51}$$

At this point it is important to mention that $V_a(x)$ depends explicitly on the parameter a, and the subindex labels different potentials for different choices of the ordering in (38). However, the Hamiltonian H_a does not explicitly depends on this parameter, and the subindex only labels a particular ordering of T_a. This means that the eigenvalues and eigenvectors of H_a will not depend on a, so there is some freedom in choosing this parameter. Remember, on the other hand that this ambiguity is not present in the classical case, where the dynamical variables are commuting functions of x, p. This fact suggest that we can fix the ordering in the quantum case by making a correspondence of the potential $V_a(x)$ with its classical counterpart $\mathcal{V}(x)$. Observe that, for $a = -1/4$, both, classical and quantum potentials has the same form, as it happens in the CM harmonic oscillator. In this sense we state that the PDM harmonic oscillator Hamiltonian is given by

$$H = -\frac{1}{2}\sqrt[4]{m}\,\mathbf{D}^2\frac{1}{\sqrt[4]{m}} + \frac{1}{2}\left(\int^x \sqrt{m}dt + y_0\right)^2 = A^+A^- + \frac{1}{2}, \tag{52}$$

where

$$A^\pm = \frac{1}{\sqrt{2}}\left(\mp\sqrt[4]{m}\,\mathbf{D}\frac{1}{\sqrt[4]{m}} + \int^x \sqrt{m}dt + y_0\right). \tag{53}$$

The algebraic properties of $A^\pm$ have the following consequences: (a) the spectrum E_n of H is the same as for the CM harmonic oscillator, (b) the wave functions $\psi_n(x)$ are given by:

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(A^+\right)^n\psi_0(x), \tag{54}$$

where $\psi_0(x)$ is the ground state defined as the wave function annihilated by A^-:

$$A^-\psi_0(x) = \frac{1}{\sqrt{2}}\left(\sqrt[4]{m}\,\mathbf{D}\frac{1}{\sqrt[4]{m}} + \int^x \sqrt{m}dt + y_0\right)\psi_0(x) = 0. \tag{55}$$

In order to integrate this equation we can make the substitution

$$\psi_0(x) = \sqrt[4]{m}\,\phi_0\left(\int^x \sqrt{m}dt + y_0\right),$$

then (55) transforms into

$$\frac{1}{\sqrt{2}}\left(\mathbf{D} + \int^x \sqrt{m}dt + y_0\right)\phi_0\left(\int^x \sqrt{m}dt + y_0\right) = 0\,. \tag{56}$$

We can easily observe that this equation is the same defining the ground state ϕ_0 for the CM harmonic oscillator, where y is now replaced by the integral of the square root of the mass function (actually, if one makes the change of variable $y = \int\sqrt{m}dx + y_0$, then $\mathbf{D} \equiv d/dy$). The rest of the wave functions can be constructed by applying consecutively the operator A^+ to the ground state:

$$\psi_n(x) = \sqrt[4]{m}\frac{1}{\sqrt{n!}}\left(\mathbf{D} + \int^x \sqrt{m}dt + y_0\right)^n\phi_0\left(\int^x \sqrt{m}dt + y_0\right). \tag{57}$$

By comparing (57) with (37) we can conclude that the wave functions $\psi_n(x)$ of the PDM harmonic oscillator are given in terms of the CM harmonic oscillator wave functions $\phi_n(x)$ in the form

$$\psi_n(x) = \sqrt[4]{m}\,\phi_n\left(\int^x \sqrt{m}dt + y_0\right). \tag{58}$$

This last expression is a well known result called the *point canonical transformation* [22, 23], we should emphasize that in this formalism it comes directly from the algebraic approach used to construct the wave functions.

It is not difficult to show that the eigenstates (58) are square integrable, though, not necessarily normalized functions. Let

$$y(x) = \int^x \sqrt{m}\, dt + y_0\,, \tag{59}$$

where $\mathcal{D}(y) = \mathcal{D}(m)$ and the range of $y(x)$ is $\mathcal{R}(y)$. Then

$$\int_{\mathcal{D}(m)} |\psi_n(x)|^2\, dx = \int_{\mathcal{R}(y)} |\phi_n(y)|^2\, dy \leq 1 \tag{60}$$

since the CM harmonic oscillator wave functions $\phi_n(y)$ are normalized. Therefore, in the case $y(x)$ (which corresponds to $X(x)$ in the classical framework) ranges all over $\mathbb{R}$, the functions (58) are square integrable as well as normalized eigenstates of the PDM harmonic oscillator; in any other case, even when they are square integrable functions, they will not be normalized.

6. Examples of quantum position dependent mass harmonic oscillator

In this section we will present the cases of the masses chosen before, in the classical approach, in order to illustrate our results in the quantum case. Figures 4, 5 and 6 show the potential $V(x)$ and the first three wave functions $\psi_n(x)$ and probability densities $\rho_n(x) = |\psi_n(x)|^2$ for each case, with $\lambda = 1$.

In the first place we consider the mass $m_1(x)$ given in (22). The wave functions

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} \sqrt{\frac{1+\lambda+x^2}{1+x^2}}\; e^{-\frac{1}{2}[x+\lambda \arctan x]^2} H_n(x + \lambda \arctan x)\,, \tag{61}$$

look quite similar to the usual harmonic oscillator eigenfunctions $\phi_n(y)$ given in (37) (see Figure 4). In this case $\mathcal{R}(y) = \mathbb{R}$, so the eigenstates are bounded, square integrable and normalized functions.

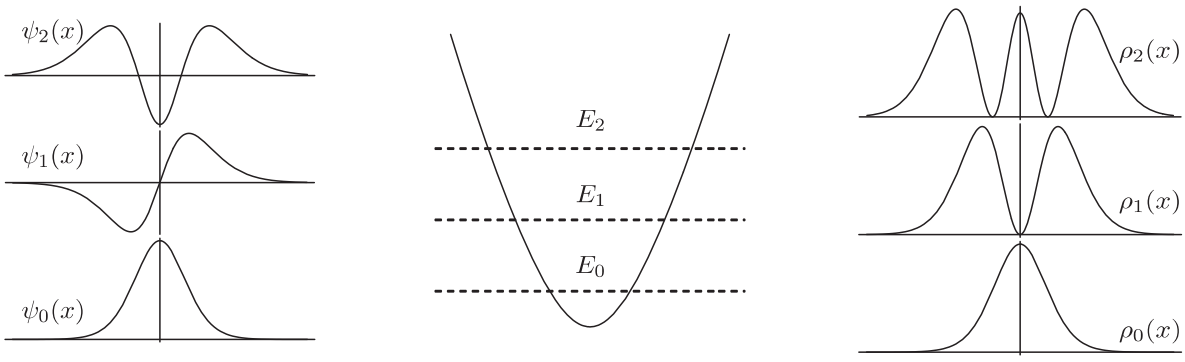

Figure 4. The potential $V(x)$ and its first three wave functions $\psi_n(x)$ and probability densities $\rho_n(x) = |\psi_n(x)|^2$ for the mass $m_1(x)$ with parameters $m_0 = \lambda = 1$.

In the second place, for $m_2(x)$ in (26), the wave functions

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} \sqrt{|\tanh \lambda x|}\; e^{-\frac{1}{2}\left[\frac{1}{\lambda} \ln \cosh \lambda x\right]^2} H_n\left(\frac{1}{\lambda} \operatorname{sign}(x)\, \ln(\cosh \lambda x)\right), \tag{62}$$

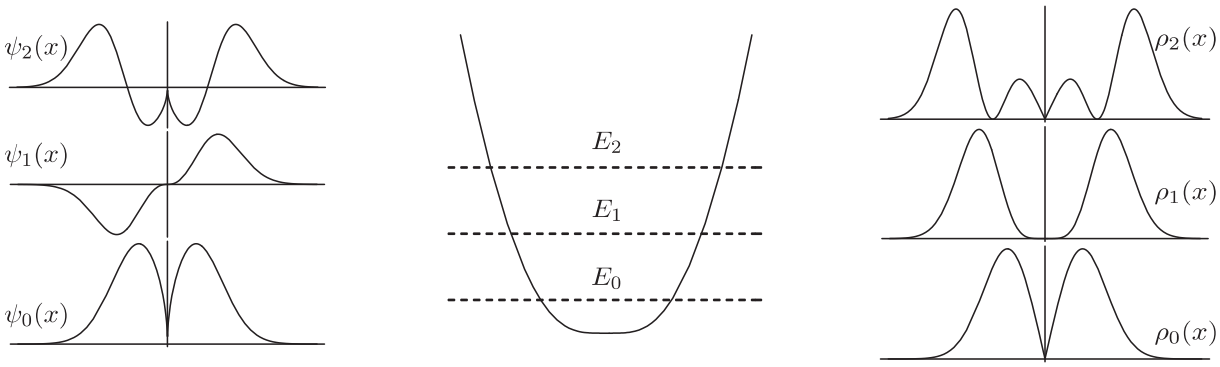

Figure 5. The potential $V(x)$ and its first three wave functions $\psi_n(x)$ and probability densities $\rho_n(x) = |\psi_n(x)|^2$ for the mass $m_2(x)$ with parameters $m_0 = \lambda = 1$.

are bounded, square integrable and also normalized functions since the range $\mathcal{R}(y) = \mathbb{R}$ (see Figure 5). Observe, additionally, that the probability of finding the system near the origin tends to zero due to the factor $\sqrt[4]{m}$ in the wave functions.

Next, for $m_3(x)$ in (30), it is evident from Figure 6 that the corresponding wave functions given by

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} \frac{1}{\sqrt[4]{1-(\lambda x)^2}}\, e^{-\frac{1}{2}\left[\frac{1}{\lambda}\arcsin \lambda x\right]^2} H_n\left(\frac{1}{\lambda}\arcsin \lambda x\right), \tag{63}$$

are not bounded functions in $\mathcal{D}(m_3)$. This is because $\mathcal{R}(y) \neq \mathbb{R}$, and the exponential factor can not compensate the divergencies of the factor $\sqrt[4]{m}$ near the boundaries. Even so, it can be easily checked that these wave functions are square integrable, though, not normalized.

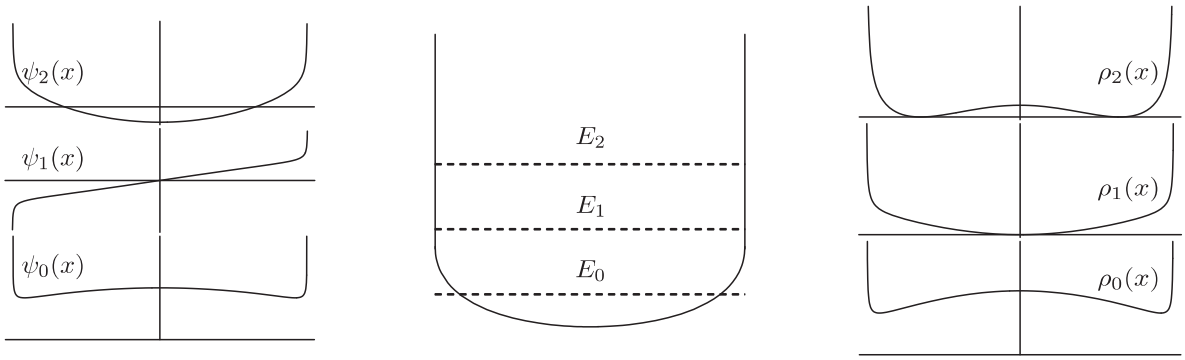

Figure 6. The potential $V(x)$ and its first three wave functions $\psi_n(x)$ and probability densities $\rho_n(x) = |\psi_n(x)|^2$ for the mass $m_3(x)$ with parameters $m_0 = \lambda = 1$.

Finally, it is worthwhile to mention that, correspondingly to the classical case, the parity of the mass function $m(x)$ in these examples imply that the wave functions $\psi_n(x)$ and the probability densities $\rho_n(x)$ are, respectively, odd and even functions of x in all cases.

7. Concluding remarks

He have studied the PDM harmonic oscillator from the classical as well as the quantum points of view by means of the supersymmetric approach. The correspondence to the CM counterpart is given by a canonical transformation in the classical case. The quantum Hamiltonian can be decomposed in many different forms, corresponding to different orderings of the kinetic term.

However, only a very particular decomposition gives the same potential term as for the classical case. Since the quantum Hamiltonian is constructed by making this correlation to the classical case, this formalism is equivalent to the point canonical transformation approach. In both frameworks we have illustrated, with some examples, how this formalism works for different types of the mass function.

Acknowledgments

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