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On a Curvature-Statistics Theorem

M. Calixto[1,2] **and V. Aldaya**[2]

[1]Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Paseo Alfonso XIII 56, 30203 Cartagena, Spain

[2]Instituto de Astrofísica de Andalucía, Apartado Postal 3004, 18080 Granada, Spain

E-mail: Manuel.Calixto@upct.es

Abstract. The spin-statistics theorem in quantum field theory relates the spin of a particle to the statistics obeyed by that particle. Here we investigate an interesting correspondence or connection between curvature ($\kappa = \pm 1$) and quantum statistics (Fermi-Dirac and Bose-Einstein, respectively). The interrelation between both concepts is established through vacuum coherent configurations of *zero modes* in quantum field theory on the compact $O(3)$ and noncompact $O(2,1)$ (spatial) isometry subgroups of de Sitter and Anti de Sitter spaces, respectively. The high frequency limit, is retrieved as a (zero curvature) group contraction to the Newton-Hooke (harmonic oscillator) group. We also make some comments on the physical significance of the vacuum energy density and the cosmological constant problem.

1. Introduction

Quantum vacua are not really empty. We know that zero-point energy, like other non-zero vacuum expectation values, leads to observable consequences like, for instance, the Casimir effect [1], and influences the behaviour of the Universe at cosmological scales, where the vacuum (dark) energy is expected to contribute to the cosmological constant, which affects the expansion of the universe.

In Quantum Field Theory, one expects the vacuum state to be stable under some underlying symmetry group G (viz, the Poincaré group). Then the action of some spontaneously broken symmetry transformations can destabilize the vacuum and make it to *radiate*. Such is the case of the Planckian radiation of the Poincaré invariant vacuum under uniform accelerations, that is, the Unruh effect [2]. Here, the Poincaré invariant vacuum looks the same to any inertial observer but converts into a thermal bath of radiation with temperature

$$T = \hbar a / 2\pi c k_B$$

in passing to a uniformly accelerated frame (a denotes the acceleration, c the speed of light and k_B the Boltzmann constant). These radiation phenomena are usually linked to some kind of global mutilation of the spacetime (namely, existence of horizons). In the reference [3], it was shown that the reason for this radiation is more profound and related to the spontaneous breakdown of the conformal symmetry in *quantum* field theory. From this point of view, a Poincaré invariant vacuum can be regarded as a coherent state of conformal zero modes, which are undetectable ("dark") by inertial observers but unstable under relativistic uniform accelerations (special conformal transformations). There we used the conformal group in (1+1)-dimensions, $SO(2,2) \simeq SO(2,1) \times SO(2,1)$, which consists of two copies of the pseudo-orthogonal group $SO(2,1)$ (left- and right-moving modes, respectively). Here we study in general the breakdown of a symmetry G in quantum field theory and apply to the particular cases of $O(3), O(2,1)$ and Newton-Hooke groups (previously developed in [4]). We could think of $O(3)$ and $O(2,1)$ as isometry subgroups of the spatial part of de Sitter and Anti de Sitter spaces, with positive and negative curvature κ, respectively.

The organization of the paper is as follows. In Sec. 3, we construct $G = O(3), O(2,1)$ and Newton-Hooke invariant quantum field theories, in a unified manner. To be more precise, we shall work with their double covers $U(2)$ and $U(1,1)$ instead, for convenience. Then we shall break the original symmetry G by choosing a pseudo-vacuum $|\theta\rangle$, which turns out to be a coherent state of zero modes, invariant under a subgroup $G_0 \subset G$. We could think of zero modes as "virtual" particles without "bright" energy (see later on Sec. 3) and undetectable by G_0 observers. However, we shall show that a general symmetry transformation, which incorporates the broken symmetry generators in G/G_0, produces a "rearrangement" of the the G_0-invariant vacuum $|\theta\rangle$ and makes it to *radiate*. In other words, we shall associate a *thermal bath* with the excited (or, let us say, "polarized") G_0-vacuum and we shall show in Sections 4 and 5 that the *mean energy per mode* matches the Bose-Einstein (BE) and Fermi-Dirac (FD) expressions, for the non-compact $U(1,1)$ and compact $U(2)$ isometry groups ($\kappa = \mp 1$), respectively. The limit of high frequencies (or large interlevel energy spacing $\hbar\omega \gg k_B T$) is regained from a contraction to the Newton-Hooke group (zero curvature, $\kappa = 0$), as explained in Sect. 6. The last Section is devoted to some comments on the vacuum energy and its implications in cosmology.

Before, we describe in Sec. 2 the general construction of a G-invariant quantum field theory as a second quantization on a group G and the spontaneous breakdown of G to G_0, the stability subgroup of the degenerated vacuum. We shall provide the essential ingredients to calculate probability distributions and average energies of the ground state excitations.

2. The general context

Let $\mathcal{G} = \{X_\alpha, \alpha = 0, 1, 2, \ldots, l\}$ be the (Lie) algebra of observables of a given quantum system, among which we highlight X_0 as the Hamiltonian operator. Let $\mathcal{H}$ be the (Hilbert) carrier space of a unitary irreducible representation U of the Lie group G. Let us assume that the energy spectrum is discrete and bounded from bellow, that is, there is a vacuum vector $|0\rangle$ whose energy E_0 we set to zero, i.e. $X_0|0\rangle = 0$. Let $B(\mathcal{H}) = \{|n\rangle, n = 0, 1, 2, \ldots\}$ be a (finite or infinite) orthonormal basis of $\mathcal{H}$ made up of energy eigenvectors, $X_0|n\rangle = E_n|n\rangle$.

An important ingredient to construct a G-invariant quantum field theory, as a second-quantized (many-particle) theory, will be the irreducible matrix coefficients

$$U_{mn}(x) \equiv \langle m|U(x)|n\rangle, \tag{1}$$

of the representation $U(x) = e^{i\sum_\alpha x^\alpha X_\alpha}$ of G in the orthonormal basis $B(\mathcal{H})$, where $\{x^\alpha \in \mathbb{R}, \alpha = 0, \ldots, \dim(G) - 1\}$ stands for a coordinate system in G.

Given the Fourier expansion, in energy modes, of a field

$$|\psi\rangle = \sum_{n=0} a_n |n\rangle\,, \tag{2}$$

the Fourier coefficients a_n (resp. $a_n^\dagger$) are promoted to annihilation (resp. creation) operators of energy modes E_n in the second quantized theory, with commutation relations $[a_n, a_m^\dagger] = \delta_{n,m}$. The (finite) action of G on annihilation operators is:

$$a_m \to a'_m = \sum_{n=0} U_{mn}(x) a_n, \tag{3}$$

together with the conjugated expression for creation operators. The infinitesimal generators of this finite action are the second quantized version, $\hat{X}_\alpha$, of the basic observables X_α. They have the following explicit expression in terms of creation and annihilation operators:

$$\hat{X}_\alpha = -i \sum_{m,n=0} a_m^\dagger \left. \frac{\partial U_{mn}(x)}{\partial x^\alpha} \right|_{x=0} a_n. \tag{4}$$

For example, since $|n\rangle$ are energy E_n eigenstates and we have set $E_0 = 0$, the Hamiltonian operator is:

$$\hat{X}_0 = \sum_{n=1} E_n a_n^\dagger a_n. \tag{5}$$

The vacuum $|\hat{0}\rangle$ of the second quantized theory is characterized by being stable under the symmetry group G, i.e. it is annihilated by the symmetry generators

$$\hat{X}_\alpha |\hat{0}\rangle = 0,\ \alpha = 0, \dots, \dim(G) - 1 \tag{6}$$

and also by $a_n|0\rangle = 0, \forall n \geq 0$. Then an orthonormal basis for the Hilbert space of the second quantized theory is constructed by taking the orbit through the vacuum $|\hat{0}\rangle$ of the creation operators $a_n^\dagger$:

$$|q(n_1), \dots, q(n_p)\rangle \equiv \frac{(a_{n_1}^\dagger)^{q(n_1)} \dots (a_{n_p}^\dagger)^{q(n_p)}}{(q(n_1)! \dots q(n_p)!)^{1/2}} |\hat{0}\rangle, \tag{7}$$

where $q(n) \in \mathbb{N}$ denotes the occupation number of the energy level n.

Note that any multi-particle state (7) made up of an arbitrary content of *zero modes*, like

$$|\theta\rangle \equiv \sum_{q=0}^{\infty} \theta_q (a_0^\dagger)^q |\hat{0}\rangle,\ \ \theta_q \in \mathbb{C}, \tag{8}$$

has zero total energy, i.e. $\hat{X}_0|\theta\rangle = 0$, since $[\hat{X}_0, a_0] = 0$. It also verifies $a_n|\theta\rangle = 0, \forall n > 0$. Let us denote by $G_0 \subset G$ the maximal stability (isotropy) subgroup of this "degenerated vacuum" $|\theta\rangle$, of which the Hamiltonian $\hat{X}_0$ is one of its generators.

Actually, the operator a_0 conmutes with the unbroken symmetry generators $\hat{X}_\alpha^{(0)} \in \mathcal{G}_0$ and the creation operators:

$$[a_0, \hat{X}_\alpha^{(0)}] = 0 = [a_0, a_n^\dagger],\ \forall n > 0, \tag{9}$$

so that, by Shur lemma, a_0 behaves as (a multiple of) the identity operator in the broken theory. That is, it is natural to demand a_0 to leave the G_0-invariant vacuum (8) stable, which implies that:

$$a_0|\theta\rangle = \theta|\theta\rangle \Rightarrow |\theta\rangle = e^{\theta a_0^\dagger - \bar{\theta} a_0} |\hat{0}\rangle. \tag{10}$$

Thus, the vacuum of our (spontaneously) broken theory will be a *coherent state of zero modes* (see [5] and [6] for a thorough exposition on coherent states).

Now we show that a general unitary symmetry transformation (3), which incorporates broken symmetry generators in G/G_0, produces a "rearrangement" of this pseudo-vacuum $|\theta\rangle$ and causes it to radiate. In other words, we can associate a *quantum statistical ensemble* with the excited (or, let us say, "polarized") vacuum

$$|\theta'\rangle \equiv e^{\theta a_0'^\dagger - \bar{\theta} a_0'} |\hat{0}\rangle, \tag{11}$$

where a_0' is given by (3).

Indeed, one can compute the average number of particles with energy E_n inside $|\theta'\rangle$ as the expectation value:

$$N_n(x) = \langle\theta'|a_n^\dagger a_n|\theta'\rangle = |\theta|^2|U_{0n}(x)|^2, \tag{12}$$

hence, $|\theta|^2$ is the total average of particles of this quantum statistical ensemble. In the same way, the probability $P_n(p,x)$ of observing p particles with energy E_n in $|\theta'\rangle$ can be calculated as:

$$P_n(q,x) = |\langle q(n)|\theta'\rangle|^2 = \frac{e^{-|\theta|^2}}{q!}|\theta|^{2q}\,|U_{0n}(x)|^{2q} = \frac{e^{-|\theta|^2}}{q!}N_n^q(x). \tag{13}$$

Hence, the relative probability of observing a state with total energy E in the excited vacuum $|\theta'\rangle$ is:

$$P(E) = \sum_{\substack{q_0,\ldots,q_k:\\ \sum_{n=0}^k E_n q_n = E}} \prod_{n=0}^k P_n(q_n,x)\,. \tag{14}$$

For the cases studied in this paper, this distribution function can be factorized as $P(E) = \Omega(E)e^{-\tau E}$, where $\Omega(E)$ is a relative weight proportional to the number of states with energy E and the factor $e^{-\tau E}$ fits this weight properly to a temperature $T = k_B/\tau$.

We shall be primarily interested in the mean values of the basic observables (4). They can be calculated as:

$$\begin{aligned}\mathcal{X}_\alpha &= \langle\theta'|\hat{X}_\alpha|\theta'\rangle = -i|\theta|^2\sum_{m,n=0} U_{0m}(x)\frac{\partial U_{mn}(0)}{\partial x^\alpha}\bar{U}_{0n}(x)\\ &= -i|\theta|^2\left(U(x)\frac{\partial U(0)}{\partial x^\alpha}U^\dagger(x)\right)_{00}.\end{aligned} \tag{15}$$

In particular, the mean energy is simply:

$$\mathcal{X}_0 = |\theta|^2\sum_{n=1}|U_{0n}(x)|^2 E_n. \tag{16}$$

We shall see that $\mathcal{X}_0$ matches the usual FD and BE expressions for the compact $G = U(2)$ and non-compact $G = U(1,1)$ cases, respectively.

3. The case of $G_\kappa = O(3), O(2,1)$ and Newton-Hooke groups

The Lie algebra commutators of our basic symmetry group G_κ are:

$$[A_+, A_-] = 2\kappa H - \Xi,\;\; [H, A_\pm] = \pm A_\pm,\;\; [\Xi, \text{all}] = 0, \tag{17}$$

where H represents the *Hamiltonian*, Ξ plays the role of the *zero-point energy* (or the total number of particles operator in second quantization), $A_\pm$ are *ladder* creation and annihilation operators and $\kappa = \pm 1, 0$ is the *curvature* parameter for $U(2)$, $U(1,1)$ and the Newton-Hooke (harmonic oscillator) groups, respectively.

The group $U(1,1)$ is noncompact so, unlike the case of $U(2)$, all its unitary irreducible representations (unirreps) are infinite-dimensional. This group has a number of series of unirreps: principal, discrete and supplementary. We shall consider here only representations of the discrete series where each carrier space $\mathcal{H}_s$ is labelled by the *(conformal) spin* $s = 1/2, 1, 3/2, 2, \ldots$ and is spanned by the orthonormal basis $B(\mathcal{H}_s) = \{|s,n\rangle, n = 0,1,2,\ldots\}$. The action of the operators (17) on this basis vectors is:

$$\begin{aligned} H|s,n\rangle &= n|s,n\rangle, & A_+|s,n\rangle &= \sqrt{(n+1)(2s-\kappa n)}|s,n+1\rangle,\\ \Xi|s,n\rangle &= 2s|s,n\rangle, & A_-|s,n\rangle &= \sqrt{n(2s-\kappa(n-1))}|s,n-1\rangle.\end{aligned} \tag{18}$$

Looking at this representation, we can think of an "exotic" harmonic oscillator with an equispaced energy spectrum $E_n = \epsilon n$, where we have introduced the interlevel energy spacing $\epsilon = \hbar\omega$ to give dimensions. For negative, $\kappa = -1$, and zero, $\kappa = 0$, curvature (i.e., for $U(1,1)$ and the Newton-Hooke groups, respectively), this energy spectrum is unbounded from above, whereas for positive curvature, $\kappa = 1$ (i.e., for $U(2)$), we have a bounded spectrum with $2s+1$ states.

Using the standard Iwasawa decomposition (see e.g. [5]), any group element $U \in G_\kappa$ can be represented as:

$$U(\zeta, \bar{\zeta}, \tau, \varphi) = e^{\zeta A_+ - \bar{\zeta} A_-} e^{i\tau H} e^{i\varphi \Xi}, \tag{19}$$

where $\varphi, \tau \in [0, 2\pi]$ and $\zeta \in \mathbb{C}$. An important ingredient to construct a G_κ invariant quantum field theory, as a second quantization on G_κ, will be the irreducible matrix coefficients of the representation (18) in the orthonormal basis $B(\mathcal{H}_s)$:

$$U^{(s)}_{mn}(\zeta, \bar{\zeta}, \tau, \varphi) \equiv \langle s, m | U(\zeta, \bar{\zeta}, \tau, \varphi) | s, n \rangle. \tag{20}$$

Given the Fourier expansion in modes of a field with (conformal) spin s,

$$|\psi\rangle = \sum_{n=0} a_n |s, n\rangle\,, \tag{21}$$

the Fourier coefficients a_n (resp. $a_n^\dagger$) are promoted to annihilation (resp. creation) operators of energy modes $E_n = \hbar\omega n$ in the second quantized theory, with commutation relations $[a_n, a_m^\dagger] = \delta_{n,m}$. The second quantized version, $\hat{H}, \hat{\Xi}, \hat{A}_\pm$, of the basic operators (17), given by the general expression (4), now reads:

$$\begin{array}{ll} \hat{H} = \sum_n n a_n^\dagger a_n, & \hat{A}_+ = \sum_n \sqrt{(n+1)(2s - \kappa n)}\, a_n^\dagger a_{n+1}, \\ \hat{\Xi} = 2s \sum_n a_n^\dagger a_n, & \hat{A}_- = \sum_n \sqrt{n(2s - \kappa(n-1))}\, a_n^\dagger a_{n-1}, \end{array} \tag{22}$$

where summations start at $n = 0$ and are finite or infinite, depending on the curvature κ. Here we highlight the *total energy* operator, $\hat{E} \equiv \hbar\omega \hat{H}$, and the *total number of particles* operator, $\hat{N} \equiv \frac{1}{2s}\hat{\Xi}$.

The vacuum $|\hat{0}\rangle$ of the second quantized theory is characterized by being stable under the basic isometry group G_κ, i.e.

$$U|0\rangle = |0\rangle, \forall U \in G_\kappa \Rightarrow \{\hat{H}, \hat{\Xi}, \hat{A}_+, \hat{A}_-\}|0\rangle = 0, \tag{23}$$

and annihilated by $a_n|0\rangle = 0, \forall n \geq 0$. Then an orthonormal basis for the Hilbert space of the second quantized theory is constructed as in (7). Note that any second quantized state (7) made up of an arbitrary content of *zero modes*, like (8), has zero total energy, i.e. $\hat{H}|\theta\rangle = 0$. It also verifies $\hat{A}_-|\theta\rangle = 0$ and $a_n|\theta\rangle = 0, \forall n > 0$, so that the state (8) behaves as a (degenerated) vacuum under the (unbroken) subgroup $B \subset G_\kappa$ of affine or similitude transformations, generated by $\mathcal{B} = \{\hat{H}, \hat{A}_-\}$. Moreover, given that a_0 acts as (a multiple of) the identity operator in the broken theory, that is, it commutes with:

$$[a_0, \hat{H}] = [a_0, \hat{A}_-] = [a_0, a_n^\dagger] = 0, \forall n > 0, \tag{24}$$

it is natural to demand a_0 to leave the affine vacuum (8) stable, which implies again (10). Thus, the vacuum of our (spontaneously) broken theory will be a *coherent state of zero modes*. The squared modulus of the complex parameter θ has the physical significance of the vacuum expectation value of the number operator in the new vacuum; indeed, one can verify that:

$$\langle\theta|\hat{N}|\theta\rangle = |\theta|^2. \tag{25}$$

As stated in the Introduction, we can think of zero modes as *virtual* particles without ("bright") energy E, undetectable ("dark") by affine observers. However, we shall show that a general unitary symmetry transformation (20), which incorporates the broken symmetry generator $\hat{A}_+$, produces a "rearrangement" of the the affine vacuum $|\theta\rangle$ and makes it to radiate. In other words, we shall associate a *thermal bath* with the excited (or, let us say, "polarized") vacuum (11) and we shall show in Sections 4 and 5 that the *mean energy per mode*, $\langle\theta'|\hat{E}|\theta'\rangle$, matches the usual BE and FD expressions, for the non-compact $U(1,1)$ and compact $U(2)$ isometry groups ($\kappa = \mp 1$), respectively. The limit of high frequencies (or large interlevel energy spacing $\hbar\omega \gg k_B T$) is regained from a contraction to the Newton-Hooke group (zero curvature, $\kappa = 0$), as explained in Sect. 6. Note however that the average number of particles is conserved under this (unitary) transformation, i.e. $\langle\theta'|\hat{N}|\theta'\rangle = |\theta|^2$, because $U^\dagger U = 1$. We shall relate this quantity to a vacuum energy density, the sign of which depends on the curvature κ, and we shall make some comments on the cosmological constant problem in Sect. 7.

4. Broken $U(1,1)$ symmetry: hyperbolic geometry and BE statistics

The irreducible matrix coefficients of the representation (18) in the orthonormal basis $B(\mathcal{H}_s)$ are given in this case ($\kappa = -1$) by:

$$\begin{aligned} U^{(s)}_{mn}(\zeta,\bar\zeta,\tau,\varphi) &= e^{2si\varphi}(1-z\bar z)^s\sqrt{\frac{C^{(s)}_m}{C^{(s)}_n}}\sum_{q=\max(0,m-n)}^{m}\binom{n}{m-q}\binom{2s+n+q-1}{q} \\ &\quad\times(-1)^{n-m+q}e^{in\tau}z^q\bar z^{n-m+q}, \end{aligned} \tag{26}$$

where $C^{(s)}_n = n!/(2s+n-1)!$, $\zeta = |\zeta|e^{i\phi}$ and $z \equiv e^{i\phi}\tanh|\zeta|$ is restricted to the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, which is the stereographic projection of the upper sheet of the hyperboloid $\mathbb{H}^2 = \{\vec{v} \in \mathbb{R}^3 : \vec{v}^2 = v_0^2 - v_1^2 - v_2^2 = 1, v_0 > 0\}$ onto the complex plane. The correspondence is established through:

$$\vec{v} = (\cosh(2|\zeta|), \sinh(2|\zeta|)\cos\phi, \sinh(2|\zeta|)\sin\phi).$$

The finite action (3) of $U(1,1)$ on annihilation (resp. creation) operators of zero modes is:

$$a_0 \to a'_0 = e^{2si\varphi}(1+z\bar z)^s\sum_{n=0}^{\infty}(-1)^n\binom{2s+n-1}{n}^{1/2}e^{in\tau}\bar z^n a_n, \tag{27}$$

which leads to the excited vacuum:

$$|\theta'\rangle \equiv e^{\theta a'^\dagger_0-\bar\theta a'_0}|0\rangle = e^{-|\theta|^2/2}\sum_{q=0}^{\infty}z^q\sum_{\substack{m_1,\dots,m_q:\\ \sum_{n=1}^q nm_n=q}}\prod_{n=0}^{q}\frac{R_n^{m_n}}{m_n!}\prod_{n=0}^{q}(a_n^\dagger)^{m_n}|0\rangle\,, \tag{28}$$

where $R_n \equiv \theta(-1)^n e^{-2si\varphi-in\tau}(1-z\bar z)^s\binom{2s+n-1}{n}^{1/2}$ and we set $m_0 \equiv 0$. We have used the general identity:

$$\left(\sum_{n=0}^{\infty}\gamma_n z^n\right)^l = \sum_{q=0}^{\infty}\delta_q z^q,\quad \delta_0 = \gamma_0^l,\quad \delta_q = \frac{1}{q\gamma_0}\sum_{s=1}^{q}(sm-q+s)\gamma_s\delta_{q-s}. \tag{29}$$

From (28), we see that the relative probability of observing a state with total energy $E_q = \hbar\omega q$ in the excited vacuum $|\theta'\rangle$ is:

$$\begin{aligned} P_q &= \Omega(E_q)(|z|^2)^q\,, \\ \Omega(E_q) &\equiv \sum_{\substack{m_1,\ldots,m_q:\\ \sum_{n=1}^q nm_n=q}} \prod_{n=0}^{q} \frac{|R_n|^{2m_n}}{m_n!}\,. \end{aligned} \tag{30}$$

We can associate a *thermal bath* with this distribution function by noticing that $\Omega(E_q)$ behaves as a relative weight proportional to the number of states with energy $E_q = \hbar\omega q$; the factor $(|z|^2)^q$ fits this weight properly to a temperature T as:

$$(|z|^2)^q = e^{q\log|z|^2} = e^{-\frac{E_q}{k_BT}}, \quad T \equiv -\frac{\hbar\omega}{k_B\log|z|^2}. \tag{31}$$

We could think of (conformal) $U(1,1) \simeq O(1,2)$ transformations as transitions to a uniformly relativistic accelerated frame (see e.g. [3]), so that $T = \frac{\hbar a}{2\pi c k_B}$ is the temperature associated with a given acceleration $a \equiv -2\pi\omega c/\log|z|^2$.

After some intermediate calculations, the expected value of the total energy $\hat{E}$ in the accelerated vacuum $|\theta'\rangle$ proves to be:

$$\begin{aligned} \langle\theta'|\hat{E}|\theta'\rangle &= |\theta|^2(1-|z|^2)^{2s}\sum_{n=0}^{\infty}\hbar\omega n\binom{2s+n-1}{n}|z|^{2n} \\ &= 2s|\theta|^2\hbar\omega\frac{|z|^2}{1-|z|^2} = 2s|\theta|^2\frac{\hbar\omega e^{-\hbar\omega/k_BT}}{1-e^{-\hbar\omega/k_BT}}\,, \end{aligned} \tag{32}$$

which is proportional to the *mean energy per mode* of the B-E statistics. In D spatial dimensions, the number of states with frequency between ω and $\omega + d\omega$ is proportional to ω^{D-1}. Thus, for $D = 3$, the spectral distribution of the radiation of the accelerated vacuum $|\theta'\rangle$ is *Planckian*, i.e. $|\theta'\rangle$ radiates as a *black body.*

Still, we could have introduced a *chemical potential* μ by setting $|z|^2 = e^{(\mu-\hbar\omega)/k_BT}$ with the restriction $\mu < \hbar\omega$, in order to preserve the condition $|z| < 1$.

5. Broken $U(2)$ symmetry: spherical geometry and FD statistics

The irreducible matrix coefficients of the representation (18) in the orthonormal basis $B(\mathcal{H}_s)$ are given in this case ($\kappa = 1$) by:

$$\begin{aligned} U^{(s)}_{mn}(\zeta,\bar\zeta,\tau,\varphi) &= e^{2si\varphi}(1+z\bar z)^{-s}\sqrt{\frac{C^{(s)}_m}{C^{(s)}_n}}\sum_{q=\max(0,m-n)}^{\min(m,2s-n)}\binom{n}{m-q}\binom{2s-n}{q} \\ &\quad\times(-1)^{n-m+q}e^{in\tau}z^q\bar z^{n-m+q}, \end{aligned} \tag{33}$$

where $C^{(s)}_n = \binom{2s}{n}^{-1}$ and $z \equiv e^{i\phi}\tan|\zeta|$ is now related to the stereographic projection of the two-dimensional sphere $\mathbb{S}^2 = \{\vec v \in \mathbb{R}^3 : \vec v^2 = v_0^2 + v_1^2 + v_2^2 = 1\}$ onto the complex plane, given by:

$$\vec v = (\cos(2|\zeta|), \sin(2|\zeta|)\cos\phi, \sin(2|\zeta|)\sin\phi).$$

The finite action of $U(2)$ on annihilation (resp. creation) operators of zero modes is now:

$$a_0 \to a_0' = e^{2si\varphi}(1+z\bar z)^{-s}\sum_{n=0}^{2s}(-1)^n\binom{2s}{n}^{1/2}e^{in\tau}\bar z^n a_n, \tag{34}$$

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which leads to the polarized vacuum $|\theta'\rangle$ according to the formula (11). As for the hyperbolic case, we can associate a thermal bath with the state $|\theta'\rangle$. The difference now is that the factor $|z|^2$ is unbounded from above, i.e. $|z|^2 < \infty$, which means that we have to introduce a non-zero *chemical potential* μ, such that $|z|^2 = e^{(\mu-\hbar\omega)/k_BT}$, or/and allow for *negative temperatures*.

The expected value of the total energy $\hat{E}$ in the polarized vacuum $|\theta'\rangle$ proves to be:

$$\begin{aligned}\langle\theta'|\hat{E}|\theta'\rangle &= |\theta|^2(1+|z|^2)^{-2s}\sum_{n=0}^{2s}\hbar\omega n\tbinom{2s}{n}|z|^{2n} \\ &= 2s|\theta|^2\hbar\omega\frac{|z|^2}{1+|z|^2} = 2s|\theta|^2\frac{\hbar\omega}{1+e^{(\hbar\omega-\mu)/k_BT}}\,,\end{aligned} \tag{35}$$

which is proportional to the *mean energy per mode* of the FD statistics.

6. Flat geometry and the high frequency limit

Like in the previous two cases, one can compute the finite action of the Newton-Hooke group G_0 on annihilation (resp. creation) operators of zero modes:

$$a_0 \to a'_0 = e^{2si\varphi}e^{-2s|z|^2/2}\sum_{n=0}^{\infty}(-1)^n e^{i\tau}\sqrt{\frac{(2s)^n}{n!}}\bar{z}^n\, a_n, \tag{36}$$

where $z \equiv \zeta$ is unbounded, $|z| < \infty$, like for the spherical geometry. The expected value of the total energy $\hat{E}$ in the excited vacuum $|\theta'\rangle$ is now:

$$\begin{aligned}\langle\theta'|\hat{E}|\theta'\rangle &= |\theta|^2 e^{-2s|z|^2}\sum_{n=0}^{\infty}\hbar\omega n\frac{(2s)^n}{n!}|z|^{2n} \\ &= 2s|\theta|^2\hbar\omega|z|^2 = 2s|\theta|^2\hbar\omega e^{(\mu-\hbar\omega)/k_BT}\,,\end{aligned} \tag{37}$$

which is the $\hbar\omega \gg k_BT$ limit of B-E and F-D statistics. Note that, unlike for the B-E and F-D statistics, here we could "reabsorb" the chemical potential μ into the vacuum expectation value of the total number of particles $|\theta|^2$ by setting $|\theta|^2 = e^{-\mu/k_BT}$. Therefore, the introduction of μ in our scheme is only indispensable for the F-D statistics.

7. Cosmological implications

We have separated standard ("bright") energy $\hat{H}$ from vacuum ("dark") energy $\hat{\Xi}$ in our model. However, we could combine both contributions to define a total Hamiltonian,

$$\hat{H}_{\rm tot} = \hat{H} + \frac{\kappa}{2}\hat{\Xi}, \tag{38}$$

such that the map $\hat{H} \mapsto \hat{H}_{\rm tot}$ renders the original commutation relations (17) into: $[\hat{A}_+, \hat{A}_-] = 2\kappa\hat{H}$, for $\kappa = \pm 1$, and $[\hat{A}_+, \hat{A}_-] = -\hat{\Xi}$, for $\kappa = 0$. Hence, the vacuum energy is now given by:

$$\langle\theta|\hat{H}_{\rm tot}|\theta\rangle = \frac{\kappa}{2}\langle\theta|\hat{\Xi}|\theta\rangle = \kappa s|\theta|^2 \tag{39}$$

Although we are dealing with a simplified (toy) model, we feel tempted to link this vacuum energy to a cosmological constant, as is done in modern cosmology. From this point of view, hyperbolic spatial geometries ($\kappa = -1$), like Anti de Sitter space-time, have positive pressure, which causes the expansion of empty space to slow down. On the contrary, for spherical spatial geometries ($\kappa = 1$), like de Sitter space, the expansion of empty space will tend to speed up. For flat space ($\kappa = 0$) we have zero cosmological constant. Note that we can make the parameter $|\theta|^2$ as small as we like, thus eluding huge vacuum energies for either spherical or hyperbolic geometries. Therefore, our derivation of the cosmological constant from the vacuum energy in quantum field theory is free from the traditional drawbacks of "fine-tuning". Moreover, the three spatial geometries, $\kappa = \pm 1, 0$, considered in this article have different qualitative behaviour under vacuum radiation at low frequencies, something that could be used in addition to experimentally discern between these topologies.

We have presented here a quite simplified model, but we think that it grasps the essentials of more involved instances. A proper discussion of all these vacuum phenomena inside the entire conformal group $SO(4, 2)$ will be developed elsewhere.

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