

OPEN ACCESS

Feynman formula for a diffusion of particles with a variable mass in a domain

To cite this article: Y A Butko *et al* 2008 *J. Phys.: Conf. Ser.* **128** 012050

View the [article online](#) for updates and enhancements.

You may also like

- [The Double Pendulum of Variable Mass: Numerical Study for different cases.](#)
R. Espíndola, G. Del Valle, G. Hernández et al.
- [Remarks on Newton's second law for variable mass systems](#)
K Nakayama
- [Vibrational resonance in a multistable system with position-dependent mass](#)
T O Roy-Layinde, K A Omotoso, O T Kolebaje et al.



ECS
The
Electrochemical
Society
Advancing solid state &
electrochemical science & technology

DISCOVER
how sustainability
intersects with
electrochemistry & solid
state science research

Feynman formula for a diffusion of particles with a variable mass in a domain

Yana A Butko¹, Martin Grothaus² and Oleg G Smolyanov³

¹ Bauman Moscow State Technical University, 105005, Moscow, Russia

² Technische Universität Kaiserslautern, 67653, Kaiserslautern, Deutschland

³ Lomonosov Moscow State University, 119992, Moscow, Russia

E-mail: yanabutko@yandex.ru, grothaus@mathematik.uni-kl.de, smolyanov@yandex.ru

Abstract. In the present note we consider a class of second order parabolic equations with position dependent coefficients; such equations describe a diffusion of (quasi) particles with a variable mass. We represent a solution of Cauchy–Dirichlet problem for such class of equations in a bounded domain in the form of a limit of finite dimensional integrals of elementary functions. Such kind of a representation is usually called Feynman formula and can be used for calculations. Finite dimensional integrals in our Feynman formula give approximations for a functional integral over a probability measure on a set of trajectories in the domain where the solution of the considered problem is investigated; this measure is generated by a diffusion process with variable diffusion coefficient and absorption on the boundary, hence, to get Feynman formula also means to get a representation of the solution of the considered problem with the help of a functional integral (such kind of a representation is usually called Feynman–Kac formula).

1. Introduction

In the present note we consider a class of second order parabolic equations with position dependent coefficients; such equations describe a diffusion of (quasi) particles with a variable mass. Quantum analogues of these (quasi) particles appear in models, describing semiconductors, liquid crystals etc.

We represent a solution of Cauchy–Dirichlet problem for such class of equations in a bounded domain in the form of a limit of finite dimensional integrals. Such kind of representations is usually called Feynman formulas. Following Feynman approach, limits in Feynman formulas can be interpreted as functional (path) integrals over suitable measures or pseudomeasures on the set of trajectories in configuration space. Functional integrals, representing solutions of evolutionary equations, are usually called Feynman–Kac formulas. It is Feynman–Kac formulas that allow to investigate properties of evolutionary equations by methods of stochastic analysis.

Hence, to get Feynman formulas is just another way to get Feynman–Kac formulas. On the other hand, Feynman formulas have their own advantages, since in most cases it is possible to represent solutions of evolutionary equations by Feynman formulas, containing elementary functions only, whereas corresponding Feynman–Kac formulas are actually limits of finite dimensional integrals, containing some transitional probabilities which often can not be expressed by elementary functions. Just this situation takes place in case of boundary value problems.

In the present work we obtain a Feynman formula for a diffusion of particles with variable mass in a bounded domain, applying a method developed in works [3] — [5] for investigation

of diffusion and quantum evolution of particles with constant mass on Riemannian manifolds. This method is based on Chernoff theorem (see [9], [7]) and allows to investigate a wide class of initial and boundary value problems for evolutionary equations with linear and non-linear configuration space (see [2] — [8]).

2. Preliminaries

Let $f(\cdot, \cdot) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{C}$. In papers [1], [10] the evolution of particles with position dependent mass is described by the Hamiltonian $\mathcal{H}$, such that

$$\mathcal{H}f(x) = \mathcal{H}_0f(x) + V(x)f(x)$$

where

$$\mathcal{H}_0 = -\frac{1}{4}\left(m^\alpha(x)\frac{\partial}{\partial x}m^\beta(x)\frac{\partial}{\partial x}m^\gamma(x) + m^\gamma(x)\frac{\partial}{\partial x}m^\beta(x)\frac{\partial}{\partial x}m^\alpha(x)\right)$$

and $\alpha + \beta + \gamma = -1$. We assume that function $V : \mathbb{R} \rightarrow \mathbb{C}$ is continuous and function $m(\cdot) : \mathbb{R} \rightarrow (0, \infty)$ is twice continuously differentiable on $\mathbb{R}$. If $m(x) \equiv \text{const}$ then operator $\mathcal{H}_0$ coincides with a standart quantum mechanics Hamiltonian of a one-dimensional free particle $\mathcal{H}_0 = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}$. Under assumptions above operator $\mathcal{H}_0$ is symmetric and there exist continuous functions $A(\cdot), B(\cdot), C(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ such that $A(x) > 0$ for any $x \in \mathbb{R}$ and

$$\mathcal{H}_0 + V = -\frac{1}{2}A(\cdot)\frac{\partial^2}{\partial x^2} + B(\cdot)\frac{\partial}{\partial x} + C(\cdot).$$

Let $m \in \mathbb{N}$ and A a continuous mapping from $\mathbb{R}^m$ into the space $L(\mathbb{R}^m)$ of linear operators on $\mathbb{R}^m$ such that for any $x \in \mathbb{R}^m$ the operator $A(x)$ is symmetric and positive. Let $\Delta_A(x)$ be the differential operator acting on φ being twice differentiable in $x \in \mathbb{R}^m$ as follows: $(\Delta_A \varphi)(x) := \text{tr}(A(x) \varphi^{(2)}(x))$, where $\varphi^{(2)}(x) := (\text{Hess } \varphi)(x)$ is the Hessian matrix of φ in $x \in \mathbb{R}^m$. Let B be a continuous vector valued function on $\mathbb{R}^m$ and C be a continuous scalar function on $\mathbb{R}^m$. In the sequel we consider the Hamiltonian H defined for φ being twice differentiable in $x \in \mathbb{R}^m$ in the following way:

$$H\varphi(x) := \frac{1}{2}(\Delta_A \varphi)(x) + \langle B(x), \nabla \varphi(x) \rangle + C(x)\varphi(x),$$

where $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product in $\mathbb{R}^m$.

3. Cauchy–Dirichlet problem for a diffusion of particles with a variable mass in a bounded domain

Let $G \subset \mathbb{R}^m$ be a bounded domain. Its closure and boundary we denote by $\overline{G}$ and ∂G , respectively. For $f : [0, \infty) \times \overline{G} \rightarrow \mathbb{R}$ such that $f(t, \cdot)$ is twice differentiable on G for all $t \geq 0$ and $f(\cdot, x)$ is differentiable on $(0, \infty)$ for all $x \in G$ we consider the following Cauchy–Dirichlet problem:

$$\begin{aligned} \frac{\partial f}{\partial t}(t, x) &= Hf(t, x), & t > 0, x \in G, \\ f(0, x) &= f_0(x), & x \in \overline{G}, \\ f(t, x) &= 0, & t \geq 0, x \in \partial G. \end{aligned} \tag{3.1}$$

Here we assume that $f(t, \cdot) \in C_0(\overline{G})$ for any $t \geq 0$, where $C_0(\overline{G})$ is Banach space of continuous functions on $\overline{G}$ vanishing on the boundary. The norm in $C_0(\overline{G})$ is defined by $\|f\|_{C_0(\overline{G})} = \sup_{x \in \overline{G}} |f(x)|$.

We assume that the domain G and the coefficients A , B and C satisfy sufficient conditions for the existence of a strongly continuous semigroup $(T_t)_{t \geq 0}$ on $C_0(\overline{G})$ providing a solution to (3.1). Then we construct a family $(F(t))_{t \geq 0}$ of integral operators, such that (by Chernoff theorem)

$$T(t) = \lim_{n \rightarrow \infty} (F(t/n))^n, \quad (3.2)$$

where the limit is in the strong operator topology in the space $L(C_0(\overline{G}))$ of all continuous linear operators in $C_0(\overline{G})$. Hence, a right hand side of (3.2) gives us a Feynman formula for the solution of the considered Cauchy–Dirichlet problem (3.1).

4. A construction of the family $(F(t))_{t \geq 0}$

Let X be a Banach space, $L(X)$ be the space of all continuous linear operators in X with strong operator topology, $\|\cdot\|$ denotes the operator norm on $L(X)$ and Id the identity operator in X . If $D(T) \subset X$ is a linear subspace and $T : D(T) \rightarrow X$ is linear (operator), then $D(T)$ denotes the domain of T .

Definition 4.1 *The derivative at the origin of a function $F : [0, \varepsilon) \rightarrow L(X)$, $\varepsilon > 0$, is a linear mapping $F'(0) : D(F'(0)) \rightarrow X$ such that*

$$F'(0)g := \lim_{t \searrow 0} t^{-1}(F(t)g - F(0)g),$$

where $D(F'(0))$ is the vector space of all elements $g \in X$ for which the above limit exists.

Theorem 4.2 (Chernoff theorem, see [7], [9]) *Let X be a Banach space, $F : [0, \infty) \rightarrow L(X)$ be a (strongly) continuous mapping such that $F(0) = Id$ and $\|F(t)\| \leq e^{at}$ for some $a \in [0, \infty)$ and all $t \geq 0$. Let D be a linear subspace of $D(F'(0))$ such that the restriction of the operator $F'(0)$ to this subspace is closable. Let $(L, D(L))$ be this closure. If $(L, D(L))$ is the generator of a strongly continuous semigroup $(T_t)_{t \geq 0}$, then for any $t_0 > 0$ the sequence $(F(t/n))^n_{n \in \mathbb{N}}$ converges to $(T_t)_{t \geq 0}$ as $n \rightarrow \infty$ in the strong operator topology, uniformly with respect to $t \in [0, t_0]$, i.e., $T_t = \lim_{n \rightarrow \infty} (F(t/n))^n$ locally uniformly in $L(X)$.*

A family of operators $(F(t))_{t \geq 0}$ is called *Chernoff equivalent* to the semigroup $(T_t)_{t \geq 0}$ if this family satisfies the assertions of the Chernoff theorem with respect to this semigroup.

Our aim is to construct a family of integral operators $(F(t))_{t \geq 0}$, which is Chernoff equivalent to the semigroup $T(t)$, resolving the Cauchy–Dirichlet problem (3.1).

Let fix some $\varepsilon > 0$. Any function $\varphi \in C_0(\overline{G})$ can be extended to a continuous in $\mathbb{R}^m$ function $\tilde{\varphi}$ with compact support in G^ε , where G^ε is ε -neighborhood of G in $\mathbb{R}^m$, and $\|\varphi(x)\|_{C_0(\overline{G})} \geq \sup_{x \in \mathbb{R}^m} |\tilde{\varphi}(x)|$. For each φ a function $\tilde{\varphi}$ can be chosen in such way that for any $a, b \in \mathbb{R}$ if $\varphi = a\varphi_1 + b\varphi_2$ then $\tilde{\varphi} = a\tilde{\varphi}_1 + b\tilde{\varphi}_2$. Such function $\tilde{\varphi}$ with $\varphi = \tilde{\varphi}|_{\overline{G}}$ in sequel we call related to φ . Let's consider a set $D := \left\{ \varphi \in C_0(\overline{G}) \mid \varphi \text{ can be extended to some } \tilde{\varphi} \in C_0^4(G^\varepsilon) \text{ with } \sup_{x \in \mathbb{R}^m} |\tilde{\varphi}(x)| \leq \|\varphi\|_{C_0(\overline{G})} \text{ and } H\varphi \in C_0(\overline{G}) \right\}$, here $C_0^4(G^\varepsilon)$ denotes the space of four times continuously differentiable functions with compact support in G^ε . A set of all $\tilde{\varphi} \in C_0^4(G^\varepsilon)$, such that $\tilde{\varphi}|_{\overline{G}} = \varphi \in D$, we denote as $\tilde{D}$. Hence, $D \subset C_0(\overline{G})$, $\tilde{D} \subset C_0^4(G^\varepsilon) \subset C(\mathbb{R}^m)$. We denote the generator of $(T_t)_{t \geq 0}$ by $(H, D(H))$ and assume that the set D is a core of $(H, D(H))$.

Let $a(x) := \det A(x)$. We introduce a family of operators $(F_1(t))_{t \geq 0}$ in $C(\mathbb{R}^m)$ such that $F_1(0) = Id$ and for $t > 0$ the operator $F_1(t)$ is given by the formula:

$$F_1(t)\tilde{\varphi}(x) = \frac{1}{\sqrt{a(x)(2\pi t)^m}} \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} \tilde{\varphi}(y) dy. \quad (4.1)$$

If A is a constant matrix this family coincides with the semigroup $(e^{\frac{t}{2}\Delta_A})_{t \geq 0}$. If A is not constant it is not so anymore. Nevertheless, the following holds:

Lemma 4.3 For $\tilde{\varphi} \in \tilde{D}$ the derivative of $F_1(t)\tilde{\varphi}(x)$ at zero coincides with $\frac{1}{2}\Delta_A\tilde{\varphi}(x)$ uniformly in $x \in \overline{G^\varepsilon}$, i.e., for any $\tilde{\varphi} \in \tilde{D}$ the following is valid uniformly with respect to $x \in \overline{G^\varepsilon}$ as $t \searrow 0$:

$$F_1(t)\tilde{\varphi}(x) = \tilde{\varphi}(x) + \frac{1}{2}\Delta_A\tilde{\varphi}(x) + o(t).$$

Proof: Let $c(t, x) := (a(x)(2\pi t)^m)^{-1/2}$, $t \geq 0$, $x \in \overline{G^\varepsilon}$. Then

$$\begin{aligned} \frac{1}{t}(F_1(t)\tilde{\varphi} - \tilde{\varphi})(x) &= \frac{c(t, x)}{t} \cdot \left(\int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} \tilde{\varphi}(y) dy - \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} \tilde{\varphi}(x) dy \right) = \\ &= \frac{c(t, x)}{t} \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} (\tilde{\varphi}(y) - \tilde{\varphi}(x)) dy. \end{aligned}$$

Since $\tilde{\varphi} \in C_0^4(G^\varepsilon) \subset C^4(\mathbb{R}^m)$ a Taylor expansion at the point x yields

$$\begin{aligned} \tilde{\varphi}(y) - \tilde{\varphi}(x) &= \tilde{\varphi}^{(1)}(x)(y-x) + \frac{1}{2}\tilde{\varphi}^{(2)}(x)(y-x)^2 \\ &\quad + \frac{1}{3!}\tilde{\varphi}^{(3)}(x)(y-x)^3 + \frac{1}{4!}\tilde{\varphi}^{(4)}(\theta x + (1-\theta)y)(y-x)^4, \end{aligned}$$

where $\tilde{\varphi}^{(k)}(x)$ is understood as a k -linear functional on $(\mathbb{R}^m)^k$ and a symbol $\tilde{\varphi}^{(k)}(x)(y-x)^k$ stands for the result of application of $\tilde{\varphi}^{(k)}(x)$ to the vector $(y-x) \in \mathbb{R}^m$ taken k times, $\theta \in [0, 1]$. Hence,

$$\begin{aligned} \frac{1}{t}(F_1(t)\tilde{\varphi} - \tilde{\varphi})(x) &= \frac{1}{t}c(t, x) \int_{\mathbb{R}^m} e^{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}} (\tilde{\varphi}^{(1)}(x)(y-x) + \dots + \frac{1}{4!}\tilde{\varphi}^{(4)}(\theta x + (1-\theta)y)(y-x)^4) dy = \\ &= \frac{1}{2t}c(t, x) \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} \tilde{\varphi}^{(2)}(x)(y-x)^2 dy + \\ &\quad + t \frac{1}{24t^2}c(t, x) \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} \tilde{\varphi}^{(4)}(\theta x + (1-\theta)y)(y-x)^4 dy. \end{aligned}$$

Since $\tilde{\varphi}^{(4)}$ is bounded, the last term converges to zero as $t \searrow 0$ uniformly in $x \in \overline{G^\varepsilon}$. As in the case of a constant A (now with a parameter), the remaining term converges to $\frac{1}{2}\Delta_A\tilde{\varphi}(x)$ as $t \searrow 0$ uniformly in $x \in \overline{G^\varepsilon}$. ■

Let us now define a family of operators $(F_2(t))_{t \geq 0}$ in $C(\mathbb{R}^m)$ such that $F_2(0) = Id$ and for $t > 0$ the operator $F_2(t)$ is given by the formula:

$$\begin{aligned} F_2(t)\tilde{\varphi}(x) &= \\ &= \frac{1}{\sqrt{a(x)(2\pi t)^m}} \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} \exp\{-\langle A^{-1}B(x), x-y \rangle\} \tilde{\varphi}(y) dy, \end{aligned}$$

where $A^{-1}B(x) := A^{-1}(x)B(x)$.

Lemma 4.4 For any $\tilde{\varphi} \in \tilde{D}$ as $t \searrow 0$ we have uniformly with respect to $x \in \overline{G^\varepsilon}$:

$$F_2(t)\tilde{\varphi}(x) = \tilde{\varphi}(x) + \frac{t}{2}\Delta_A\tilde{\varphi}(x) + t\langle B(x), \nabla\tilde{\varphi}(x) \rangle + \frac{t}{2}\langle A^{-1}B(x), B(x) \rangle\tilde{\varphi}(x) + o(t).$$

Proof: Let us notice that if $\tilde{\varphi} \in \tilde{D}$, then also $\exp\{-\langle A^{-1}B(x), x - \cdot \rangle\}\tilde{\varphi} \in \tilde{D}$ for all $x \in \overline{G^\varepsilon}$ and

$$F_2(t)\tilde{\varphi}(x) = F_1(t)(\exp\{-\langle A^{-1}B(x), x - \cdot \rangle\}\tilde{\varphi})(x).$$

By Lemma 4.3 we get

$$\begin{aligned} F_2(t)\tilde{\varphi}(x) &= \\ &= (\exp\{-\langle A^{-1}B(x), x - \cdot \rangle\}\tilde{\varphi})(x) + \frac{t}{2}\Delta_A(\exp\{-\langle A^{-1}B(x), x - \cdot \rangle\}\tilde{\varphi})(x) + o(t) = \\ &= \tilde{\varphi}(x) + \frac{t}{2}\text{tr}\left(A(\tilde{\varphi}^{(2)} + 2A^{-1}B \otimes \nabla\tilde{\varphi} + (A^{-1}B \otimes A^{-1}B)\tilde{\varphi})\right)(x) + o(t) = \\ &= \tilde{\varphi}(x) + \frac{t}{2}\Delta_A\tilde{\varphi}(x) + t\langle B(x), \nabla\tilde{\varphi}(x) \rangle + \frac{t}{2}\langle A^{-1}B(x), B(x) \rangle\tilde{\varphi}(x) + o(t). \end{aligned}$$

■

Let $s : (0, \infty) \rightarrow (0, \infty)$ be a smooth function which monotonically decreases to 0 as $t \searrow 0$ such that $s(t) = o(t)$. For example, $s = c \arctg$ for some $0 < c < \frac{1}{\pi} \text{diam}(G)$. Let $G_{s(t)} \subset G$ be defined as $G_{s(t)} = \{x \in G \mid \text{dist}(x, \partial G) > s(t)\}$. Let $\psi_{s(t)} : \mathbb{R}^m \rightarrow [0, 1]$, $t > 0$ be a family of smooth functions such that $\psi_{s(t)}(x) = 1$ for $x \in G_{s(t)}$, $\psi_{s(t)}(x) = 0$ for $x \in \mathbb{R}^m \setminus G$. Hence the family $\psi_{s(t)}$ approximates the indicator function of the domain G pointwisely as $t \searrow 0$.

Next let us introduce a family of operators $(F_3(t))_{t>0}$ acting on $\tilde{\varphi} \in C(\mathbb{R}^m)$ as follows:

$$F_3(t)\tilde{\varphi}(x) = \psi_{s(t)}(x)e^{tV(x)}\tilde{\varphi}(x),$$

where $V(x) = C(x) - \frac{1}{2}\langle A^{-1}B(x), B(x) \rangle$, $x \in \mathbb{R}^m$. Finally, we define the family of operators $(F(t))_{t \geq 0}$ on $C_0(\overline{G})$ in the following way: $F(0) = Id$ and, for $t > 0$ and for any $\varphi \in C_0(\overline{G})$, $F(t)\varphi = F_3(t)F_2(t)\tilde{\varphi}$, where $\tilde{\varphi}$ is related to φ as above.

Lemma 4.5 For any $\varphi \in D$ as $t \searrow 0$ we have uniformly with respect to $x \in \overline{G}$:

$$F(t)\varphi(x) = \varphi(x) + \frac{t}{2}\Delta_A\varphi(x) + t\langle B(x), \nabla\varphi(x) \rangle + tC(x)\varphi(x) + o(t).$$

Proof: Since V is a continuous function, by a Taylor expansion we get $e^{tV(x)} = 1 + tV(x) + o(t)$ as $t \searrow 0$ uniformly in $x \in \overline{G}$. Thus, by Lemma 4.4 we get

$$\begin{aligned} F(t)\varphi(x) &= \psi_{s(t)}(x)(1 + tV(x) + o(t)) \\ &\quad \times \left(\tilde{\varphi}(x) + \frac{t}{2}\Delta_A\tilde{\varphi}(x) + t\langle B, \nabla\tilde{\varphi} \rangle(x) + \frac{t}{2}\langle A^{-1}B, B \rangle\tilde{\varphi}(x) + o(t) \right) \\ &= \psi_{s(t)} \left(\tilde{\varphi} + \frac{t}{2}\Delta_A\tilde{\varphi} + t\langle B, \nabla\tilde{\varphi} \rangle + tC\tilde{\varphi} \right)(x) + o(t) = \psi_{s(t)}(\tilde{\varphi} + tH\tilde{\varphi})(x) + o(t) \end{aligned}$$

for any $\tilde{\varphi} \in \tilde{D}$, uniformly in $x \in \overline{G^\varepsilon}$.

Let us show that $F(t)\varphi(x) = \varphi(x) + tH\varphi(x) + o(t)$ as $t \searrow 0$, uniformly with respect to $x \in \overline{G}$, for all $\varphi \in D$. Let firstly $x \in G_{s(t)}$. Then $\psi_{s(t)}(x) = 1$ and $F(t)\varphi(x) = \varphi(x) + tH\varphi(x) + o(t)$ as $t \searrow 0$, uniformly with respect to $x \in G_{s(t)}$. If $x \in \partial G$, then

$\psi_{s(t)}(x) = 0$ and $F(t)\varphi(x) = 0 = \varphi(x) + tH\varphi(x)$. Finally let $x \in G \setminus G_{s(t)}$. Then $|F(t)\varphi(x) - \varphi(x) - tH\varphi(x)| = |(1 - \psi_{s(t)}(x))(\varphi(x) + tH\varphi(x)) + o(t)| \leq |\varphi(x) + tH\varphi(x)| + o(t)$ as $t \searrow 0$. For any $x \in G \setminus G_{s(t)}$ there exist at least one $x_{bd} \in \partial G$ such that $\text{dist}(x, x_{bd}) = \text{dist}(x, \partial G) \leq s(t) = o(t)$ as $t \searrow 0$. Since $\varphi \in D$, we have $\varphi(x) = \varphi(x_{bd}) + \tilde{\varphi}^{(1)}(c_x)(x - x_{bd})$, where $c_x = \theta x + (1 - \theta)x_{bd}$ for some $\theta \in [0, 1]$ and $\tilde{\varphi}$ is related to φ as above. Therefore $|\varphi(x)| \leq s(t) \sup_{y \in \mathbb{R}^m} |\tilde{\varphi}^{(1)}(y)| = o(t)$ as $t \searrow 0$, uniformly with respect to $x \in G \setminus G_{s(t)}$. Since $H\varphi \in C_0(\overline{G})$, also $H\varphi(x) \rightarrow 0$ uniformly with respect to $x \in G \setminus G_{s(t)}$ for $t \searrow 0$. Thus $tH\varphi = o(t)$. Therefore $|F(t)\varphi(x) - \varphi(x) - tH\varphi(x)| \leq o(t)$ as $t \searrow 0$, uniformly with respect to $x \in G \setminus G_{s(t)}$. Summarizing, we get for any $\varphi \in D$ and uniformly with respect to $x \in \overline{G}$: $F(t)\varphi(x) = \varphi(x) + tH\varphi(x) + o(t)$ as $t \searrow 0$. ■

Theorem 4.6 *Let $(T_t)_{t \geq 0}$ be the strongly continuous semigroup on $C_0(\overline{G})$ providing a solution to the Cauchy-Dirichlet problem (3.1). Then the family of operators $(F(t))_{t \geq 0}$ is Chernoff equivalent to this semigroup and therefore*

$$T_t = \lim_{n \rightarrow \infty} [F(t/n)]^n$$

in $L(C_0(\overline{G}))$, locally uniformly with respect to $t \geq 0$.

Proof: We have to prove that $(F(t))_{t \geq 0}$ is Chernoff equivalent to $(T_t)_{t \geq 0}$. Since by Lemma 4.5 the derivative of $F(t)$ on $C_0(\overline{G})$ at zero coincides with the generator of the semigroup $(T_t)_{t \geq 0}$ on its core D , we only need to estimate the norm of the operators $F(t)$ and to show strong continuity of the family $(F(t))_{t \geq 0}$. The norm is estimated as follows (here $X = C_0(\overline{G})$ and $\tilde{\varphi}$ is related to φ as above): $\|F(t)\| =$

$$\begin{aligned} &= \sup_{\|\varphi\|_X \leq 1} \sup_{x \in G} \left| \frac{\psi_{s(t)}(x)e^{tV(x)}}{\sqrt{a(x)(2\pi t)^m}} \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} \exp\{-\langle A^{-1}B(x), x-y \rangle\} \tilde{\varphi}(y) dy \right| = \\ &= \sup_{\|\varphi\|_X \leq 1} \sup_{x \in G} \left| \frac{\psi_{s(t)}(x)e^{tV(x)}}{\sqrt{a(x)(2\pi t)^m}} \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y+tB(x)), x-y+tB(x) \rangle}{2t}\right\} \exp\left\{\frac{t}{2}\langle A^{-1}B, B \rangle(x)\right\} \tilde{\varphi}(y) dy \right| \leq \\ &\leq e^{t \sup_{x \in G} |C(x)|} \sup_{x \in G} \left| \frac{1}{\sqrt{a(x)(2\pi t)^m}} \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y+tB(x)), x-y+tB(x) \rangle}{2t}\right\} d(y - tB(x)) \right| = \\ &= e^{t \sup_{x \in G} |C(x)|}. \end{aligned}$$

Next let us show strong continuity of the family $(F(t))_{t \geq 0}$. For this it is sufficient to show that $\lim_{t \rightarrow t_0} \|F(t)\varphi - F(t_0)\varphi\|_{C_0(\overline{G})} = 0$ for all $\varphi \in D$ and any $t_0 \geq 0$. For $t_0 > 0$ this is true, because all functions in the formula of $F(t)$ are continuous with respect to $t \in (0, +\infty)$, $\tilde{\varphi}$ is continuous with compact support, and $\exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\}$ is a continuous function of the three arguments $(t, x, y) \in [t_1, t_2] \times \overline{G}^\varepsilon \times \overline{G}^\varepsilon$, where $t_0 \in (t_1, t_2)$, $[t_1, t_2] \subset (0, \infty)$.

If $\varphi \in D$, then

$$\lim_{t \searrow 0} \frac{1}{\sqrt{a(x)(2\pi t)^m}} \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} \tilde{\varphi}(y) d(y) = \varphi(x),$$

uniformly in $x \in \overline{G}$, since for any fixed $x \in \overline{G}$

$$f_t^x(y) = \frac{1}{\sqrt{a(x)(2\pi t)^m}} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\}, \quad y \in \mathbb{R}^m,$$

tends to the Dirac function δ_x as $t \searrow 0$ and $\tilde{\varphi}$ is bounded and uniformly continuous. Hence, $\lim_{t \searrow 0} F(t)\varphi = \varphi$ in $C_0(\overline{G})$ for all $\varphi \in D$. Thus, by Chernoff's theorem the statement follows. ■

5. Feynman and Feynman–Kac formulae

By theorem (4.6) the solution $f(t, x)$ of Cauchy–Dirichlet problem (3.1) with initial condition $f(0, x) = f_0(x)$ can be obtained by the formula $f(t, x) = \lim_{n \rightarrow \infty} [F(t/n)]^n f_0(x)$, where for any

$\varphi \in C_0(\bar{G})$

$$F(t)\varphi(x) = \frac{\psi_{s(t)}(x)e^{tV(x)}}{\sqrt{a(x)(2\pi t)^m}} \int_{\mathbb{R}^m} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\} \exp\{-\langle A^{-1}B(x), x-y \rangle\} \tilde{\varphi}(y) dy.$$

Thus,

$$f(t, x) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^m} \dots \int_{\mathbb{R}^m} \left(\prod_{j=1}^n \psi_{s(t/n)}(x_j) \right) \exp\left\{\frac{t}{n} \sum_{j=1}^n V(x_{j-1})\right\} \cdot \exp\left\{\sum_{j=1}^n \langle A^{-1}B(x_{j-1}), x_j - x_{j-1} \rangle\right\} p_A\left(\frac{t}{n}, x_0, x_1\right) \dots p_A\left(\frac{t}{n}, x_{n-1}, x_n\right) \tilde{f}_0(x_n) dx_1 \dots dx_n, \quad (5.1)$$

where $p_A(t, x, y) = \frac{1}{\sqrt{a(x)(2\pi t)^m}} \exp\left\{-\frac{\langle A^{-1}(x)(x-y), x-y \rangle}{2t}\right\}$, $x_0 = x$ and $\tilde{f}_0$ is related to f_0 .

Let a symbol γ_G denote the indicator of the set G , i.e. $\gamma_G(x) = 1$ if $x \in G$ and $\gamma_G(x) = 0$ if $x \notin G$. Neither the speed of convergence of $s(t) \rightarrow 0$ (if it is $o(t)$), nor the choice of a family $\{\psi_{s(t)}\}$ approximating γ_G when $t \rightarrow 0$ can change the limit in the formula (5.1). Since $\psi_{s(t)}$ is a smooth function with a compact support in G and $\tilde{f}_0$ is a continuous function with a compact support in G^ε , we actually integrate a continuous function over a compact $\bar{G}^{n-1} \times \bar{G}^\varepsilon$ in the formula (5.1) and, hence, as $\psi_{s(t)} \rightarrow \gamma_G$ when $t \rightarrow 0$ we can choose $s(t/n)$ such that

$$\left| \int_{\bar{G}^\varepsilon} \int_{\bar{G}^{n-1}} \exp\left\{\frac{t}{n} \sum_{j=1}^n V(x_{j-1})\right\} \exp\left\{\sum_{j=1}^n \langle A^{-1}B(x_{j-1}), x_j - x_{j-1} \rangle\right\} p_A(t/n, x_0, x_1) \cdot \dots \cdot p_A(t/n, x_{n-1}, x_n) \tilde{f}_0(x_n) \left(\prod_{k=1}^n (\psi_{s(t/n)}(x_{k-1}) - \gamma_G(x_{k-1})) \right) dx_1 \dots dx_n \right| < \frac{1}{n},$$

Then the limit in the formula (5.1) coincides with the following limit:

$$f(t, x) = \lim_{n \rightarrow \infty} \int_{\bar{G}^\varepsilon} \int_{\bar{G}^{n-1}} \exp\left\{\frac{t}{n} \sum_{j=1}^n V(x_{j-1})\right\} \exp\left\{\sum_{j=1}^n \langle A^{-1}B(x_{j-1}), x_j - x_{j-1} \rangle\right\} \cdot p_A(t/n, x_0, x_1) \dots p_A(t/n, x_{n-1}, x_n) \tilde{f}_0(x_n) dx_1 \dots dx_n. \quad (5.2)$$

Let $F_n^x(x_n) = \int_{\bar{G}^{n-1}} \exp\left\{\frac{t}{n} \sum_{j=1}^n V(x_{j-1})\right\} \exp\left\{\sum_{j=1}^n \langle A^{-1}B(x_{j-1}), x_j - x_{j-1} \rangle\right\} p_A(t/n, x_0, x_1) \dots$

$\dots p_A(t/n, x_{n-1}, x_n) dx_1 \dots dx_{n-1}$. Then $f(t, x) = \lim_{n \rightarrow \infty} \int_{\bar{G}^\varepsilon} \tilde{f}_0(y) F_n^x(y) dy$. Since for any $\varepsilon > 0$ the

function $f(t, x)$ does not depend on an extension $\tilde{f}_0$ of the initial condition f_0 onto $G^\varepsilon \setminus \bar{G}$, we get $f(t, x) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\bar{G}^\varepsilon} \tilde{f}_0(y) F_n^x(y) dy$ and by Lebesgue dominated convergence theorem

$f(t, x) = \lim_{\varepsilon \rightarrow 0} \int_{\bar{G}^\varepsilon} \tilde{f}_0(y) \lim_{n \rightarrow \infty} F_n^x(y) dy = \int_{\bar{G}} f_0(y) \lim_{n \rightarrow \infty} F_n^x(y) dy = \lim_{n \rightarrow \infty} \int_{\bar{G}} f_0(y) F_n^x(y) dy$. Hence, the

following theorem is proved:

Theorem 5.1 Let $f(t, x)$ be a solution of Cauchy–Dirichlet problem (3.1). Then the following Feynman formula is true:

$$f(t, x) = \lim_{n \rightarrow \infty} \int_G \dots \int_G \exp\left\{\frac{t}{n} \sum_{j=1}^n V(x_{j-1})\right\} \exp\left\{\sum_{j=1}^n \langle A^{-1} B(x_{j-1}), x_j - x_{j-1} \rangle\right\} \cdot p_A(t/n, x_0, x_1) \dots p_A(t/n, x_{n-1}, x_n) f_0(x_n) dx_1 \dots dx_n. \quad (5.3)$$

Remark 5.2. One can show in a similar way that the solution of Cauchy Problem

$$\begin{aligned} \frac{\partial f}{\partial t}(t, x) &= Hf(t, x), \\ f(0, x) &= f_0(x) \end{aligned}$$

in the whole space $\mathbb{R}^m$ can be represented by the same Feynman formula (5.3) where integrals over G are substituted by integrals over $\mathbb{R}^m$.

Remark 5.3. Let now

$$\begin{aligned} \Phi(\xi) &= \exp\left\{-\frac{1}{2} \int_0^t \langle A^{-1} B(\xi(\tau)), B(\xi(\tau)) \rangle d\tau\right\} \cdot \\ &\quad \cdot \exp\left\{\int_0^t \langle A^{-1} B(\xi(\tau)), d\xi(\tau) \rangle\right\} \cdot \exp\left\{\int_0^t C(\xi(\tau)) d\tau\right\} f_0(\xi(t)). \end{aligned}$$

Using a fact that the equation $\frac{\partial f}{\partial t}(t, x) = \frac{1}{2} \Delta_A f(t, x)$ is a Kolmogorov equation for the Ito stochastic differential equation $d\xi(t) = \sqrt{A(\xi(t))} dw(t)$ (where $\sqrt{A(x)}$ means the positive square root), one can show that finite dimensional integrals at right hand side of the Feynman formula (5.3) are finite dimensional approximations of an integral of the function Φ over a measure μ , generated by a diffusion process, in the domain G with absorption on the boundary, governed by this stochastic differential equation. This means that the following Feynman–Kac formula is true:

$$f(t, x) = \int_{C([0, t], G)} \Phi(\xi) \mu(d\xi).$$

It is worth noticing, that just the Feynman formula, containing elementary functions only, gives an effective method to calculate the integral in the Feynman–Kac formula.

Acknowledgments

This work has been supported by RFBR, grant N 06-01-00761.

References

- [1] Gadella M, Kuru S, Negro J 2007 Self adjoint Hamiltonians with a mass jump: General matching conditions *Phys. Lett. A* **362** 265-268
- [2] Gadella M, Smolyanov O G 2008 Feynman Formulas for Particles with Position-Dependent Mass, *Doklady Math.* **77** 120-124
- [3] Weizsacker H v, Smolyanov O G, Wittich O 2000 Diffusion on Compact Riemannian Manifolds and Surface Measures, *Doklady Math.* **61** 230-235
- [4] Smolyanov O G, Weizsäcker H v, Wittich O 2000 Brownian Motion on a Manifold as Limit of Stepwise Conditioned Standard Brownian Motions *Stochastic Processes, Physics and Geometry: New Interplays. II: A Volume in Honor of Sergio Albeverio. Ser. Conference Proceedings. Canadian Math. Society. Providence: AMS*, **29** 589-602

- [5] Smolyanov O G, Weizsaecker H v, Wittich O 2007 Surface Measures and Initial Boundary Value Problems Generated by Diffusions with Drift *Doklady Math.* **76** 606-610
- [6] Smolyanov O G, Weizsaecker H v, Wittich O 2007 Chernoff's Theorem and Discrete Time Approximations of Brownian Motion on Manifolds *Potent. Anal.* **26** 1-29
- [7] Smolyanov O G, Tokarev A G, Truman A 2002 Hamiltonian Feynman path integrals via the Chernoff formula, *J. Math. Phys.* **43** 5161-5171
- [8] Butko Ya A 2008 Feynman formulas and functional integrals for diffusion with drift in a domain on a manifold, *Math. Notes* **83** 301-316
- [9] Engel K-J, Nagel R 2000 One Parameter Semigroups for Linear Evolution Equations. B.: Springer
- [10] Thomsen J, Einevoll G T, Hemmer P C 1989 Operator ordering in effective-mass theory *Phys. Rev. B* **39** 783-788