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Bases in Lie and Quantum Algebras

A. Ballesteros[1], **E. Celeghini**[2] **and M.A. del Olmo**[3]

[1] Departamento de Física, Universidad de Burgos, E-09006, Burgos, Spain
[2] Departimento di Fisica, Università di Firenze and INFN–Sezione di Firenze, I50019 Sesto Fiorentino, Firenze, Italy
[3] Departamento de Física Teórica, Universidad de Valladolid, E-47005, Valladolid, Spain.

E-mail: `angelb@ubu.es, celeghini@fi.infn.it, olmo@fta.uva.es`

Abstract.
Applications of algebras in physics are related to the connection of measurable observables to relevant elements of the algebras, usually the generators. However, in the determination of the generators in Lie algebras there is place for some arbitrary conventions. The situation is much more involved in the context of quantum algebras, where inside the quantum universal enveloping algebra, we have not enough primitive elements that allow for a privileged set of generators and all basic sets are equivalent. In this paper we discuss how the Drinfeld double structure underlying every simple Lie bialgebra characterizes uniquely a particular basis without any freedom, completing the Cartan program on simple algebras. By means of a perturbative construction, a distinguished deformed basis (we call it the analytical basis) is obtained for every quantum group as the analytical prolongation of the above defined Lie basis of the corresponding Lie bialgebra. It turns out that the whole construction is unique, so to each quantum universal enveloping algebra is associated one and only one bialgebra. In this way the problem of the classification of quantum algebras is moved to the classification of bialgebras. In order to make this procedure more clear, we discuss in detail the simple cases of $su(2)$ and $su_q(2)$.

1. Introduction

Algebras are, of course, the same object in mathematics and physics. However, from the mathematical viewpoint the interest is focused on the general description of the structure as abstract as possible, while for physical applications the interest is focused on the basic elements of the algebra to which some precise physical meaning is attributed.

In this "physically" motivated work we summarize a research line devoted to individuate, for both Lie and quantum algebras, these basic objects –the generators– in an intrinsic way eliminating every arbitrariness. In this review we do not describe all the technical details of this approach and the interested reader is invited to look at the specific papers quoted in the bibliography [1]–[5].

In the Lie case the situation is quite simple and almost solved more than half century ago. Inside the universal enveloping algebra of a given Lie algebra the distinguished subspace of the algebra is completely determined by the requirement to be closed under commutation relations or, equivalently, to have a primitive coproduct. Also the decomposition of the space of every simple Lie algebra in the form $n_- \oplus h \oplus n_+$ (where h is the abelian Cartan subalgebra and $n_\pm$ are two nihilpotent equivalent subalgebras) is an intrinsic property of interest for both theory and applications. Then the Cartan analysis of n_+ allows to individuate, always without any arbitrariness, the simple roots and to draw with them the Dynkin diagram. However, some freedom remains in the length of the roots and different conventions have been adopted so far: Weyl and Chevalley canonical bases, etc. (see, for instance, [6]). In this context, a way to "solve" the problem of non-simple roots is just to eliminate them from the game: the Chevalley reformulation is indeed made in terms of the Cartan subalgebra, the simple roots and the Serre relations. However, in Physics this is not the usual way to proceed since, in general, all roots (either simple or not) are considered on the some footing and are related to physical objects. In conclusion, few details remained until now unsolved in the construction of the bases within the Cartan program on simple algebras.

As a first result, we discuss here how these problems can be solved by introducing objects more general than algebras: Lie bialgebras and Drinfeld doubles. It is indeed well known that every simple Lie algebra has associated a Lie bialgebra which is "almost" a Drinfeld double [7]. This "approximation" can be in fact removed because every simple algebra of rank r, $\mathcal{A}_r$, is related to the Drinfeld double $\mathcal{A}_r \oplus t_r$, where t_r is an abelian r-dimensional algebra [2, 3]. It turns out that, up to an overall multiplicative factor implicit in the algebra, this Drinfeld double fixes uniquely the normalization factor for all the generators.

The situation is quite different when we attempt to introduce a quantum deformation. Indeed, a general investigation concerning the uniqueness of this quantization process has not been given yet and only some restrictive results for certain deformations of simple Lie algebras have been obtained. As a consequence, although the existence of the quantization is guaranteed for each Lie bialgebra (see [7]), a classification of quantum groups in the spirit of Cartan cannot be found in the literature [1, 2].

Moreover, the infinitesimal counterpart of a quantum group is not well defined. So, we are not dealing with a deformation of a Lie algebra but with a quantum universal enveloping algebra $U_q(g)$, which is a Hopf algebra deformation of the universal enveloping algebra $U(g)$ of the Lie algebra g, i.e. a deformation of an infinite-dimensional object. A possible basis of $U(g)$ is the Poincaré-Birkhoff-Witt (PBW) basis on g (i.e. the set of all ordered monomials of powers of the generators of g). Of course, there are infinitely many basic sets different from g (all related among them by invertible transformations) that originate suitable PBW bases. However g is a special subspace of $U(g)$ and, for this reason the only one considered. But in quantum algebras it is not easy to find a prescription that permits the choice of a privileged bases. Indeed, contrarily to the non deformed case, where inside all the sets of basic elements the vector space of generators is univocally defined, in $U_q(g)$ there is an infinitude of basic sets coexisting on the same footing. This implies that quantum deformations lead us locally (therefore, geometrically) to structures which are quite different from Lie algebras. In particular, when we consider a deformation of an n-dimensional Lie algebra, only the infinite dimensional $U_q(g)$ makes sense in despite that its PBW basis is constructed in terms of a basic set of n elements. This problem of the basis underlies many difficulties encountered when a precise physical/geometrical meaning has to be assigned to $U_q(g)$ as, for instance, in the context of quantum deformations of space-time symmetries. In that case, it is well known that the models so obtained depend on the choice of different bases [8], where different possibilities are related through nonlinear transformations.

The second result that we present here is the solution of the above mentioned problem, providing a universal prescription for the characterization and the explicit construction of a n-dimensional basis, that can be properly called, the quantum analogue of the Lie basis.

To begin with, we analyze the role and properties of the Lie algebra generators within $U(g)$. Among the infinite possible PBW bases, all related by nonlinear invertible transformations, the generators determine the only one that is closed under commutation rules and whose tensor product representations are constructed additively. The latter property can be stated in Hopf-algebraic terms as the Friedrichs theorem [9], which states that the only primitive elements in $U(g)$ (*i.e.* the elements such that $\Delta(X) = \Delta_{(0)}(X) := 1 \otimes X + X \otimes 1$) are just the generators of g as a Lie algebra. In this way, the generators of g become distinguished elements of $U(g)$. However, the situation changes drastically in $U_q(g)$, where the law for the construction of tensor product representations (coproduct) uses nonlinear functions and no primitive bases exist. We show that, among the infinitely many possible bases, there is just one where the coproducts are "as primitive as possible", since all its inessential terms, related to nonlinear transformations in the enveloping algebra, have been removed. Thus, the only changes from the Lie primitive coproducts are those imposed by the consistency between the bialgebra cocommutator δ and the Hopf algebra postulates. This basis (that has, of course, the same dimension than the corresponding Lie algebra) is proposed as the true deformation of the Lie algebra and called "quantum algebra", to be distinguished from the (much larger) quantum universal enveloping algebra.

To develop our approach we recall that Lie group theory is based on analyticity with respect to group parameters. In the same way, analyticity in the deformation parameter(s) will give us the keystone for the identification of the proper quantum algebra, that will be defined as the n-dimensional vector space $(g_q, \Delta) \subset U_q(g)$ obtained as analytical prolongation of the Lie bialgebra $(g, \Delta_{(0)})$. Note that analyticity in the deformation has already played a useful role in quantum algebras, for instance in their contractions [10].

In this analytical prolongation, the cocommutator map of the Lie bialgebra plays a fundamental role since it describes the first order deformation and can be considered as the derivative at the origin of the quantum coproduct. Thus, for a given Lie-Hopf algebra $(g, \Delta_{(0)})$, there are as many different quantizations as inequivalent cocommutators δ. The Hopf algebra postulates together with the assumption that all the results are analytical functions of both the deformation parameter(s), introduced by the bialgebra, and of generators determine uniquely the "analytical" generators.

As a third result we will discuss how the Friedrichs theorem can be extended to quantum algebras. Like in the Lie case, where to a $U(g)$ corresponds one and only one Lie algebra $(g, \Delta_{(0)})$, to a $U_q(g)$ corresponds one and only one quantum algebra (g_q, Δ). Since we prove that (g_q, Δ) is in one-to-one correspondence with (g, δ), then to each bialgebra corresponds one and only one quantization: the classification of quantum groups is reduced to the classification of Lie bialgebras.

Summarizing, we close the commutative diagram by adding the vertex (g_q, Δ)

$$\begin{array}{ccc}
(g, \Delta_{(0)}) & \xleftarrow{\text{Friedrichs theorem}} & U(g) \\
\Big\downarrow q & & \Big\downarrow q \\
(g_q, \Delta) & \xleftarrow{\text{Generalized Friedrichs th.}} & U_q(g)
\end{array}$$

The vertical lines represent the quantization procedure, and the horizontal ones are related to the definition of the basic set of generators of the universal enveloping algebra and of its quantum analogue.

The paper is organized as follows. Section 2 is devoted to Lie bialgebras and Drinfeld doubles, and after a short description of these objects we explicitly discuss the simplest case of $su(2)$, where the length of the root vector turns out to be fixed by the self-dual Drinfeld double structure. The generalization to any simple Lie algebra is briefly discussed. In Section 3 we describe the analytical approach to quantization, which allows to relate, in a perturbative way, the Lie bialgebra (g, δ) and the quantum algebra (g_q, Δ). In this way, we show that quantization not only relates $U_q(g)$ to $U(g)$ but also quite simpler n-dimensional objects like (g_q, Δ) to (g, δ). Again the discussion will be illustrated through the standard deformation of $su(2)$. Starting from the standard bialgebra of $su(2)$ we derive a well defined form of the quantum algebra $su_q(2)$. Section 4 is devoted to revisit the first horizontal line of the diagram. Its generalization, that corresponds to the lower line, is discussed in Section 5. Finally, some conclusions close the paper.

2. Lie bialgebras and Drinfeld doubles

A Lie bialgebra (g, η) is a Lie algebra $g := \{Z_p\}$ endowed with a skew-symmetric linear map (cocommutator) $\eta : g \to g \otimes g$ such that

$$\eta([Z_p, Z_q]) = (\mathrm{ad}_{Z_p} \otimes 1 + 1 \otimes \mathrm{ad}_{Z_p})\eta(Z_q) - (\mathrm{ad}_{Z_q} \otimes 1 + 1 \otimes \mathrm{ad}_{Z_q})\eta(Z_p), \qquad \forall Z_p, Z_q \in g.$$

Thus, besides the commutation relations

$$[Z_p, Z_q] = f^r_{p,q}\, Z_r\,, \tag{1}$$

we have another set of relations

$$\eta(Z_p) = c^{q,r}_p\, Z_q \otimes Z_r. \tag{2}$$

The two sets of structure constants $f^r_{p,q}$ and $c^{r,q}_p$ look quite symmetric and, indeed, on the dual space $g^* := \{z^p\}$ they play an inverted role

$$[z^p, z^q] = c^{p,q}_r\, z^r, \qquad \eta(z^p) = f^p_{q,r}\, z^q \otimes z^r\,. \tag{3}$$

The compatibility relations between the structure constants

$$c^{p,q}_r f^r_{s,t} = c^{p,r}_s f^q_{r,t} + c^{r,q}_s f^p_{r,t} + c^{p,r}_t f^q_{s,r} + c^{r,q}_t f^p_{s,r} \tag{4}$$

allow to combine g and g^* in an unique algebra called a Drinfeld double [7]. This Drinfeld double is, indeed, defined as a Lie algebra $\bar{g}$ such that, $\bar{g} = g \oplus g^*$ as vector space, where g and g^* are disjoint sub-bialgebras, and the crossed commutation rules between g and g^* are defined in terms of the structure tensors $f^p_{q,r}$ and $c^{q,r}_p$ by eq. (5). The Jacobi identities for the algebra $\bar{g}$ are just the compatibility relations (4).

The crossed commutation relations

$$[z^p, Z_q] = f^p_{q,r} z^r - c^{p,r}_q Z_r. \tag{5}$$

are related to the invariance of the pairing

$$\langle Z_p, Z_q\rangle = 0, \qquad \langle Z_p, z^q\rangle = \delta^q_p, \qquad \langle z^p, z^q\rangle = 0. \tag{6}$$

Moreover, the double Lie algebra $\bar{g}$ has a (quasitriangular) double Lie bialgebra structure $(\bar{g}, \delta)$, where its cocommutators are determined by its sub-bialgebras (g, η) and (g^*, η) as

$$\delta(Z_p) = -\eta(Z_p) = -c^{q,r}_p Z_q \otimes Z_r, \qquad \delta(z^p) = \eta(z^p) = f^p_{q,r} z^q \otimes z^r. \tag{7}$$

In the particular case that $c_p^{q,r} = -f_{q,r}^p$ [11] the Drinfeld double gets an additional symmetry, and we call it a self-dual Drinfeld double. It is worthy noticing that the positive and negative Borel subalgebras $b_\pm$ of any simple Lie algebra g have this last property, but, as they have the Cartan subalgebra in common, they cannot be identified **as** g and g^*. Hence, the classical Lie algebras are "almost" Drinfeld doubles [12, 7].

In fact, this problem has been solved with a slight extension of $\bar{g}$ considering the direct sum of $\bar{g}$ with an abelian algebra t_n, where n is the rank of $\bar{g}$ [2]. The most elementary example of self-dual Drinfeld double structure is that of the algebra $su(2) \oplus t_1$. We write $su(2)$ as

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = \alpha J_3, \tag{8}$$

where the most common physical convention is $\alpha = 2$, while in mathematical contexts different conventions and a different overall factor are used to write $su(2) := \{h, f, g\}$ relations as $[h, f] = 2f,\ [h, g] = -2g,\ [f, g] = h$. In disagreement with both of them, from $su(2) \oplus t_1$, we will obtain $\alpha = 1$.

The two $su(2)$ Borel subalgebras $s^\pm := \{J_3, J_\pm\}$ are not disjoint (as it should be required by (6)) but, if we introduce a new abelian algebra $t_1 \equiv \mathbb{R}$ with generator I, that commutes with $su(2)$ (i.e. $[I, \cdot] = 0$), we can define two disjoint algebras $b_+ := \{Z_1, Z_2\}$ and $b_- := \{z^1, z^2\}$ that allow us to define a true self-dual Drinfeld double. Let us indeed define

$$Z_1 = \frac{1}{\sqrt{2}}(J_3 + iI), \qquad Z_2 = J_+, \qquad z^1 = \frac{1}{\sqrt{2}}(J_3 - iI), \qquad z^2 = J_-.$$

The commutation relations imposed by $su(2) \oplus t_1$ inside b_+ and b_- are, respectively

$$[Z_1, Z_2] = \frac{1}{\sqrt{2}} Z_2, \qquad [z^1, z^2] = -\frac{1}{\sqrt{2}} z^2,$$

such that, the structure constants are

$$f_{1,2}^2 = -f_{2,1}^2 = -c_2^{1,2} = c_2^{2,1} = \frac{1}{\sqrt{2}}.$$

Compatibility conditions (4) are easily checked. Hence, because $c_k^{i,j} = -f_{i,j}^k$, we have a self-dual Drinfel double. From (5) we get the crossed commutation relations as

$$[Z_1, z^1] = 0, \qquad [Z_1, z^2] = -\frac{1}{\sqrt{2}} z^2, \qquad [Z_2, z^1] = -\frac{1}{\sqrt{2}} Z_2, \qquad [Z_2, z^2] = \frac{1}{\sqrt{2}}(Z_1 + z^1).$$

The cocommutators are

$$\delta(Z_1) = 0, \qquad \delta(Z_2) = \frac{1}{\sqrt{2}} Z_2 \wedge Z_1, \qquad \delta(z^1) = 0, \qquad \delta(z^2) = \frac{1}{\sqrt{2}} z^2 \wedge z^1.$$

Now, we can come back to our original basis. The commutation relations inside $b_\pm$ have been, of course, assumed but the value of α is determined by the Drinfeld double to be exactly $\alpha = 1$. This is equivalent to impose the following relation on the $so(3)$ generators

$$J_\pm \equiv \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$$

as, from a different point of view, is required by the Killing form.

In this way the full double Lie bialgebra is obtained. Since we are interested in the subalgebra $su(2)$, we consider the trivial representation of t_1 (where $I = 0$)

$$\delta(I) = 0, \qquad \delta(J_3) = 0, \qquad \delta(J_+) = \frac{1}{2} J_+ \wedge J_3, \qquad \delta(J_-) = \frac{1}{2} J_- \wedge J_3. \tag{9}$$

The generalization to larger algebras is straightforward. We display here the results for the A_n series [3]. Let us consider the Lie algebra $gl(n+1) = A_n \oplus h := \{H_i,\ F_{ij}\}$, where h is the Lie algebra generated by $\sum H_i$, we define:

$$\begin{aligned}
&[H_i, H_j] = 0,\\
&[H_i, F_{jk}] = (\delta_{ij} - \delta_{ik}) F_{jk},\\
&[F_{ij}, F_{kl}] = (\delta_{jk} F_{il} - \delta_{il} F_{kj}) + \delta_{jk}\delta_{il}(H_i - H_j).
\end{aligned} \tag{10}$$

As before, we introduce an abelian algebra $t_{n+1} := \{I_i\}$ with $n+1$ central generators I_i and we construct the algebra $gl(n+1) \oplus t_{n+1}$ that we show to be a Drinfeld double. With a change of basis similar to the $su(2)$ case, the new generators are

$$Z_i := \frac{1}{\sqrt{2}}(H_i + \mathbf{i} I_i), \quad Z_{ij} := F_{ij}, \qquad z^i := \frac{1}{\sqrt{2}}(H_i - \mathbf{i} I_i), \quad z^{ij} := F_{ji} \quad (i < j). \tag{11}$$

We obtain thus two equivalent soluble Lie algebras $b_\pm$ each of dimension $(n+1)(n+2)/2$

$$b_+ := \{Z_i, Z_{ij}\}, \qquad b_- := \{z^i, z^{ij}\}, \qquad i, j = 1, \dots, n+1, \quad i < j,$$

such that $b_+ + b_- = gl(n+1) \oplus t_{n+1}$ as vector spaces.

The commutation rules imposed by (10) inside b_+ and b_- are

$$\begin{aligned}
&[Z_i, Z_j] = 0, \qquad [Z_i, Z_{jk}] = \tfrac{1}{\sqrt{2}}(\delta_{ij} - \delta_{ik})\, Z_{jk}, \qquad [Z_{ij}, Z_{kl}] = (\delta_{jk} Z_{il} - \delta_{il} Z_{kj}),\\
&[z^i, z^j] = 0, \qquad [z^i, z^{jk}] = -\tfrac{1}{\sqrt{2}}(\delta_{ij} - \delta_{ik})\, z^{jk}, \qquad [z^{ij}, z^{kl}] = -(\delta_{jk} z^{il} - \delta_{il} z^{kj}).
\end{aligned} \tag{12}$$

The two algebras b_+ and b_- can be paired by

$$\langle z^i, Z_j \rangle = \delta^i_j, \qquad \langle z^{ij}, F_{kl} \rangle = \delta^i_k \delta^j_l, \tag{13}$$

defining a bilinear form such that both $b_\pm$ are isotropic.

The compatibility relations (4) can be easily checked and we are now able to came back from (12) and (13) to all the formulas (10) but this is strictly related to our definition of the root vectors. A change of scale like $F_{ij} \to k F_{ij}$ will introduce into (10) changes that are is incompatible with the Drinfeld double structure.

This procedure can be easily extended to any arbitrary semi-simple Lie algebra g_n by considering $g_n \oplus t_n$, which can be equipped with a self-dual Drinfeld double structure. In Ref. [3, 4] the explicit form for the classical series A_n, B_n, C_n and D_n can be found.

In conclusion: there is a Drinfeld double behind every simple Lie algebra but only if it is written in the correct basis.

The Cartan program on semi-simple Lie algebras has been thus concluded by fixing, without any freedom, the length of the vectors associated to all the roots. This is the main result coming from the Drinfeld double perspective at a “classical” level.

It is worthy to note that the Drinfeld double fixes everything not only at the Lie algebraic level but also at the Lie bialgebra one: the canonical Lie bialgebra structure for $gl(n+1)\oplus t_{n+1}$ is determined by the cocommutator δ and reads

$$\begin{aligned}
&\delta(I_i) = 0, \\
&\delta(H_i) = 0, \\
&\delta(F_{ij}) = -\tfrac{1}{2}F_{ij} \wedge (H_i - H_j) - \tfrac{\mathbf{i}}{2}F_{ij} \wedge (I_i - I_j) + \textstyle\sum_{k=i+1}^{j-1} F_{ik} \wedge F_{kj}, \qquad i < j, \\
&\delta(F_{ij}) = \tfrac{1}{2}F_{ij} \wedge (H_i - H_j) - \tfrac{\mathbf{i}}{2}F_{ij} \wedge (I_i - I_j) - \textstyle\sum_{k=j+1}^{i-1} F_{ik} \wedge F_{kj}, \qquad i > j.
\end{aligned} \tag{14}$$

We have to observe that the chain of Drinfeld doubles $g_n \oplus t_n \subset g_{n+1} \oplus t_{n+1}$ is preserved at the level of Lie bialgebras. However, although g_n is a subalgebra of $g_n \oplus t_n$, the cocommutator $\delta(g_n)$ does not define a sub-bialgebra since it depends on the extra t_n sector.

3. Analytical deformation

The problem of the generalization of the previous results to quantum algebras is quite more difficult. Strictly speaking, quantum algebras are not deformed Lie algebras in the sense that the initial object to be deformed is $U(g_n)$, the (infinite-dimensional) universal enveloping algebra built on the Lie algebra g_n, and the final object, with similar properties, is the quantum universal enveloping algebra $U_q(g_n)$ (also infinite-dimensional).

The universal enveloping algebra $U(g_n)$ has, as a possible basis, the Poincaré-Birkhoff-Witt (PBW) basis, i.e. the set of all ordered monomials built up on the powers of the generators of g_n. Moreover, the PBW basis is only a particular one among the possible bases all related between them by invertible transformations. The crucial point is that, when considering $U_q(g_n)$, the algebra subset looses all its privileges in a structure where linearity does not make sense. On the contrary, when considering $U(g_n)$, we have the chance that the Lie subset is the only subset that has two relevant properties: first, it is the only n-dimensional set closed under commutation and, second, only the elements of the Lie algebra are primitive. This last property is –at least in physical terms– the most relevant as it implies the additivity of physical observables related to the algebra.

Indeed, $U(g_n)$ and $U_q(g_n)$ are equipped with Hopf algebra structures. The main point is the existence of an isomorphism, called coproduct,

$$\Delta : U(g_n) \to U(g_n) \otimes U(g_n), \qquad \Delta_q : U_q(g_n) \to U_q(g_n) \otimes U_q(g_n).$$

Those elements of $U(g_n)$ such that $\Delta(X) = 1 \otimes X + X \otimes 1$ are said to be primitive and the Friedrichs theorem establishes that only the elements of g_n are primitive. Moreover, as the Lie basis elements are cocommutative (i.e. $\Delta(X) = \sigma \circ \Delta(X)$ with $\sigma(A \otimes B) = B \otimes A$), this property is also valid for all $U(g_n)$.

Unfortunately, deformation breaks both properties, primitivity and cocommutativity. So, we looses the canonical procedure to individuate among all the possible basic sets a privileged one. At this point we are compelled to considerer at the same time the whole $U_q(g_n)$ where all the bases are equivalent. Let us recall that three basic sets are usually considered in the literature: Lie basis, crystal basis [13, 14] and the most common one, which for $su_q(2) := \{J_3, J_\pm\}$ [7] is

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = \frac{\sinh(zJ_3)}{\sinh(z/2)}; \tag{15}$$

$$\Delta(J_3) = 1 \otimes J_3 + J_3 \otimes 1, \qquad \Delta(J_\pm) = e^{\frac{z}{2}J_3} \otimes J_\pm + J_\pm \otimes e^{-\frac{z}{2}J_3}. \tag{16}$$

The Lie basis of $U_q(g_n)$ describes a situation where both the basic set and its coproduct are closed under commutation relations but the relation between them is different (and quite more complex) from the Lie one. For a physicist this means that we have the same symmetry of the Lie situation but we loose the usual definition of composed system. It could, perhaps, be the appropriate scheme to describe systems of particles with an interaction among them. The crystal basis (often called canonical basis) is, in some sense, the opposite of the Lie basis: while in the Lie basis the algebra is written for $z \to 0$ and all deformation is transferred to the coproduct, in the crystal basis the algebra is obtained for $z \to \infty$ while, as in the Lie basis, the problem of consistency is left to the coproduct. Also in the crystal basis the representations look quite simple and have been applied in statistical mechanics [15] and in genetics [16].

However, it is not an accident that everybody knows $U_q(su(2))$ in terms of the basic set displayed in (15), (16), since this basis is, up to few details, the analytical prolongation to the $su(2)$ generators defined in Sect. 2. More information can be found in Ref. [5], where the discussion is more general and technical.

Analyticity in Lie theory is analyticity in the parameter (dual space of the generators), but here we have to extend the analyticity to the deformation parameters z in such a way that in a well precise limit (usually $z \to 0$) the quantum algebra reduces to the corresponding Lie algebra (without loss of generality, we shall restrict here to the one-parameter case). We also recall that analyticity in quantum algebras is not new since, for instance, contraction theory relies upon this concept [10].

The point is that from the theory of analytical functions we know that there exist infinitely many analytical functions that have the same value in a point and the same derivative in the same point. Consistently, from a Lie bialgebra we can obtain, by analytic prolongation, infinitely many basic sets all of them describing the same $U_q(g)$ and expressed by means of infinitely many free parameters, independent from z. This is coherent with the fact that $U_q(g)$ (as well as $U(g)$) can be described by infinitely many basic sets. But, as discussed in the following, all these free parameters remain present in the limit $z \to 0$: each one of the basic sets of $U_q(g)$ is related by analyticity to a basic set of $U(g)$ and, as $U(g)$ has a privileged basic set in the Lie generators, the basic set related by analyticity to Lie generators is a privileged basic set for $U_q(g)$.

For instance $U(su(2))$ can be defined, as usual, from the $su(2)$ generators, but –equivalently– starting from another basic set like, for instance,

$$Y_\pm = J_\pm + J_3^2 J_\pm, \qquad Y_3 = J_3 + J_3^2. \tag{17}$$

As these relations are (formally) invertible $U(su(2))$ –as well as its deformation– does not change, but $\{Y_\pm, Y_3\}$ is clearly not the best choice for applications. This is exactly the freedom we remove in the definition of the analytical basis by imposing that the deformation starts from the Lie generators and that only the contributions necessary to save the consistency between the bialgebra (g, δ) and the Hopf postulates are introduced. All contribution related to nonlinear transformations in $U(g)$, like (17), are in this way eliminated.

As it is well-known Δ is related to δ by

$$\delta = \lim_{z \to 0} \frac{\Delta - \sigma \circ \Delta}{2z} : \tag{18}$$

the cocommutator δ can be seen as the derivative at the origin of the quantization and $U_q(g)$ is sometimes called a "quantization of $U(g)$ in the direction of δ".

The analytical deformation is introduced in three steps:

(i) We find order-by-order the changes imposed by δ in $\Delta_{(0)}$ to be consistent with coassociativity (21) and we determine in this way the full coproduct Δ.

(ii) By using analyticity and the homomorphism property of Δ (23), we obtain the commutation rules for g_q starting from the known ones of g. Thus, the n-dimensional (g_q, Δ) is constructed.

(iii) From g_q a PBW basis in $U_q(g)$ is built.

In this way we construct an unique connection $(g, \delta) \to (g_q, \Delta) \to U_q(g)$. Since (g, δ) is the limit of (g_q, Δ) and, as it will be show in Sect. 5, every $U_q(g)$ admits one and only one basis g_q, the arrows can be inverted and a one-to-one correspondence is found between (g, δ) and $U_q(g)$. So, equivalences between q-deformations of $U_(g)$ imply equivalences among their associated bialgebras and the classification of the $U_q(g)$ is carried to the quite simpler classification of Lie bialgebras.

Analyticity means that the commutation relations of any basic set $\{Y_j\}$ $(j = 1, 2, \ldots, n)$ of $U_q(g)$ (as well as of $U(g)$) are analytical functions of the Y_j and that the quantum coproduct Δ of the Y_j can be written as a formal series

$$\Delta(Y_i) = \sum_{k=0}^{\infty} \Delta_{(k)}(Y_i) = \Delta_{(0)}(Y_i) + \Delta_{(1)}(Y_i) + \ldots \tag{19}$$

where $\Delta_{(k)}(Y_i)$ is a homogeneous polynomial of degree $k+1$ in $1 \otimes Y_j$ and $Y_j \otimes 1$.

There are two properties of Hopf algebras which are relevant in this analytical approach: the coassociativity condition

$$(\Delta \otimes 1 - 1 \otimes \Delta) \circ \Delta(Y_i) = 0, \tag{20}$$

that in a perturbative form will be rewritten as

$$\sum_{j=0}^{k} \left(\Delta_{(j)} \otimes 1 - 1 \otimes \Delta_{(j)}\right) \circ \Delta_{(k-j)}(Y_i) = 0, \qquad \forall k; \tag{21}$$

and the homomorphism property

$$\Delta([Y_i, Y_j]) = [\Delta(Y_i), \Delta(Y_j)]\,, \tag{22}$$

which can be rewritten as

$$\Delta_{(k)}([Y_i, Y_j]) = \sum_{l=0}^{k} \left[\Delta_{(l)}(Y_i), \Delta_{(k-l)}(Y_j)\right], \qquad \forall k\,. \tag{23}$$

To enlighten the construction we discuss explicitly the quantization of the Lie bialgebra $(su(2), \delta)$ described by (8) with $\alpha = 1$ and (9). More details can be found in [5]. The results of this Section show that analyticity chooses the coproduct given by formula (16) among all possible ones [1] and the usual expressions (15) must be replaced by other slightly different ones.

As stated before, let us start by getting the quantum coproduct. The zero-approximation of the quantum coproduct is $\Delta_{(0)}$, while $\Delta_{(1)}$ is essentially δ and the $\Delta_{(k)}$ are obtained adding order by order the contributions imposed by consistency with eq. (21).

The case of J_3 is simple: $\Delta_{(0)}(J_3) = J_3 \otimes 1 + 1 \otimes J_3$ and $\delta(J_3) = 0$, that implies that the anti-cocommutative part of $\Delta_{(1)}(J_3)$ is zero. Thus $\Delta_{(1)}(J_3)$ must be cocommutative. But cocommutative contributions are related to non linear bases like the one of eq. (17) in the Lie algebra and can be removed by coming back, with an appropriate change of basis, to the Lie generators. So (see eq.(31) and Ref. [5] for an exhaustive discussion) we write $\Delta_{(1)}(J_3) = 0$. Then, we have

$$\Delta(J_3) = \Delta_{(0)}(J_3) + \mathcal{O}_{(2)}(J_3),$$

where we have defined $\mathcal{O}_{(m)}(Y_i)$ as a polynomial or degree $> m$ in $1 \otimes Y_j$ and $Y_j \otimes 1$. From eq.(19), we can write

$$\Delta(J_3) = \Delta_{(0)}(J_3) + \Delta_{(2)}(J_3) + \mathcal{O}_{(3)}(J_3).$$

Again, as eq. (21) for $k = 2$ is consistent with $\Delta_{(2)}(J_3) = 0$, we perform a (cubic) change of basis and we write

$$\Delta(J_3) = \Delta_{(0)}(J_3) + \Delta_{(3)}(J_3) + \mathcal{O}_{(4)}(J_3).$$

As this procedure can be iterated for all k, the analytical prescription for J_3 imposes $\Delta_{(k)}(J_3) = 0, \ \forall k > 0$. Hence

$$\Delta(J_3) = \Delta_{(0)}(J_3) = J_3 \otimes 1 + 1 \otimes J_3, \tag{24}$$

i.e., the analytical procedure associates a primitive coproduct to any null δ. This could be considered trivial, as $\{J_3\}$ closes an $u(1)$ algebra, but this is not the case since it has been obtained from a precise prescription and it is not, like in [7], an arbitrary choice inside $U(g)$.

For $\Delta(J_+)$ we have

$$\Delta(J_+) = \Delta_{(0)}(J_+) + \Delta_{(1)}(J_+) + \mathcal{O}_{(2)}(J_+).$$

From (18) and the coassociativity condition (21) for $k = 1$, we have that $\Delta_{(1)}(J_+) = \frac{z}{2}\,\delta(J_+)$ as, like in the J_3 case, the possible cocommutative contribution –independent from δ– can be put to zero by a change of basis. We have thus

$$\Delta(J_+) = \Delta_{(0)}(J_+) + \frac{z}{2}\,\delta(J_+) + \Delta_{(2)}(J_+) + \mathcal{O}_{(3)}(J_+).$$

The coassociativity condition (21) for $k = 2$ solved in the unknown $\Delta_{(2)}(J_+)$ gives (again disregarding arbitrary contributions that can be removed by a nonlinear change of basis)

$$\Delta_{(2)}(J_+) = \frac{z^2}{8}({J_3}^2 \otimes J_+ + J_+ \otimes {J_3}^2). \tag{25}$$

Considering next order, we can now write

$$\Delta(J_+) = \Delta_{(0)}(J_+) + \frac{z}{2}\,\delta(J_+) + \Delta_{(2)}(J_+) + \Delta_{(3)}(J_+) + \mathcal{O}_{(4)}(J_+),$$

where $\Delta_{(2)}(J_+)$ is given by eq. (25) and $\Delta_{(3)}$ is the new unknown. Solving the coassociativity condition (21) for $k = 3$ we have

$$\Delta_{(3)}(J_+) = \frac{z^3}{48}({J_3}^3 \otimes J_+ - J_+ \otimes {J_3}^3) \tag{26}$$

and the general formula obtained by iteration, always putting to zero the cocommutative contributions related to nonlinearity in $U(su(2))$ and independent from δ, is

$$\Delta_{(k)}(J_+) = \frac{1}{k!}\left(\frac{z}{2}\right)^k (J_3^{\,k} \otimes J_+ + (-1)^k J_+ \otimes J_3^{\,k}), \qquad \forall k.$$

The $\Delta_{(k)}$ are easily summed to

$$\Delta(J_+) = e^{\frac{z}{2}J_3} \otimes J_+ + J_+ \otimes e^{-\frac{z}{2}J_3}.$$

The approach is exactly the same for J_- and gives a similar result. Thus, we obtain the analytical quantum coproduct associated to $(su(2), \delta)$

$$\begin{aligned} \Delta(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\ \Delta(J_+) &= e^{\frac{z}{2}\,J_3} \otimes J_+ + J_+ \otimes e^{-\frac{z}{2}\,J_3}, \\ \Delta(J_-) &= e^{\frac{z}{2}\,J_3} \otimes J_- + J_- \otimes e^{-\frac{z}{2}\,J_3}. \end{aligned} \tag{27}$$

that it is exactly the coproduct displayed in expression (16).

Now we have simply to start from the commutators (8) with $\alpha = 1$ and to impose order by order the homomorphism condition for the deformed commutation rules. We thus find

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = \frac{1}{z}\,\sinh(z\,J_3). \tag{28}$$

Note that $\Delta_{(0)}\left([J_+, J_-]\right) = \Delta_{(0)}(J_3)$ forbids the introduction of z-dependent normalizations, like $\sinh(z/2)$ of expressions (15), and a factor 2 has been removed at the classical level from the Drinfeld double structure. Expressions (27) and (28) define uniquely the standard analytical deformation of the Cartan basis of $su(2)$ such that the q-generators J_3, J_+, J_- could be called the q-Cartan basis of $su_q(2)$.

By inspection, all the symmetries (for example, $\{J_3, J_+, J_-\} \leftrightarrow \{J_3, -J_+, -J_-\}$) and the embedding conditions (for instance, $su(2) \supset borel(J_3, J_+) \supset u(1) := \{J_3\}$) of the bialgebra (g, δ) are automatically preserved in the quantization (g_q, Δ). Like for the bialgebra only the $u(1)$ generated by J_3 is a sub-quantum-algebra of $su_q(2)$.

4. Perturbative Friedrichs theorem

We give now a constructive proof of the Friedrichs theorem, building explicitly the primitive generators $\{X_j\}$ in terms of an arbitrary set $\{Y_j\}$ on which a PWB basis for the whole $U(g)$ can be built. The elements Y_j are cocommutative but, in principle, non primitive. The machinery consists in repeated changes of bases that allow to obtain each time a better approximation to primitivity where –this is the essential point– the problem is reformulated at each step in terms of the preceding basis. One infinite iteration of the procedure allows to find, among the infinitely many possible bases of $U(g)$, the Lie generators.

In more detail, let us consider $X_i \equiv \lim_{k\to\infty} X_i^k$ were $\{X_i^k\}$ is a basic set that approximates the Lie-Hopf coproducts up to order k. The terms $\mathcal{O}_{(m)}(X_i^k)$ are, in this Section, cocommutative since we are working in $U(g)$. Now any original basic set $\{Y_i\}$ is a zero approximation to $\{X_i\}$, i.e. $X_i^0 := Y_i$. Indeed

$$\Delta(X_i^0) = \Delta_{(0)}(X_i^0) + \mathcal{O}_{(1)}(X_i^0) = X_i^0 \otimes 1 + 1 \otimes X_i^0 + \mathcal{O}_{(1)}(X_i^0). \tag{29}$$

The explicit form consistent with formula (21) of $\mathcal{O}_{(1)}(X_i^0)$ in expression (29) is

$$\mathcal{O}_{(1)}(X_i^0) = \sum d_i^{jl}\ \left(X_j^0 \otimes X_l^0 + X_l^0 \otimes X_j^0\right)\ +\ \mathcal{O}_{(2)}(X_i^0),$$

where d_i^{lj} are arbitrary constants. From expression (19) we get

$$\mathcal{O}_{(1)}(X_i^0) = \Delta_{(1)}(X_i^0) + \mathcal{O}_{(2)}(X_i^0), \tag{30}$$

but if we define (see [5] for details) the next approximation

$$X_i^1 := X_i^0 - \sum d_i^{jl} \{X_j^0, X_l^0\} \tag{31}$$

we obtain a coproduct for X_i^1 with vanishing first order contributions, i.e.

$$\Delta(X_i^1) = \Delta_{(0)}(X_i^1) + \mathcal{O}_{(2)}(X_i^0).$$

Eq. (31) allows to rewrite $\mathcal{O}_{(2)}(X_i^0)$ in terms of X_i^1 as $\mathcal{O}_{(2)}(X_i^1)$

$$\Delta(X_i^1) = \Delta_{(0)}(X_i^1) + \mathcal{O}_{(2)}(X_i^1).$$

Now, defining a new change of basis X_i^2 we obtain, in terms of the same X_i^2, the coproduct of the second approximation X_i^2 to the generators as

$$\Delta(X_i^2) = \Delta_{(0)}(X_i^2) + \mathcal{O}_{(3)}(X_i^2),$$

which is free form both first and second order contributions. The procedure can now be iterated and the $\Delta_m(X_i^{m-1})$ contribution eliminated through a new change of basis that affects the higher orders only. The residual term becomes $\mathcal{O}_{(m+1)}(X_i^m)$ and we get the m-order approximation to the Lie generators

$$\Delta(X_i^m) = \Delta_{(0)}(X_i^m) + \mathcal{O}_{(m+1)}(X_i^m).$$

The true generators of the Lie algebra g are (formally) recovered in the limit

$$X_i := \lim_{m\to\infty} X_i^m$$

and, in agreement with the Friedrichs theorem, their coproduct is the primitive one

$$\lim_{m\to\infty} \Delta(X_i^m) = \lim_{m\to\infty} \Delta_{(0)}(X_i^m) = \Delta_{(0)}(X_i) = \Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i.$$

This coproduct is an algebra homomorphism with respect to the Lie commutation rules

$$\Delta[X_i, X_j] = [X_i, X_j] \otimes 1 + 1 \otimes [X_i, X_j],$$

and the n generators of the Lie algebra are univocally identified in a constructive manner within $U(g)$, pushing away order by order the corrections to a primitive coproduct.

The central point of this perturbative approach to Friedrichs theorem (as well as to its following extension to $U_q(g)$) is that at each order all the relations can be rewritten in terms of the corresponding approximations of the generators.

5. Unicity of deformation

This novel proof of Friedrich's Theorem has been described here because the procedure that allows to individuate the generators of the quantum algebra g_q, among the infinitely many possible bases within $U_q(g)$, is exactly the same that allows to individuate the generators of the Lie algebra g among the infinitely many possible bases of $U(g)$. The perturbative approach allows thus a generalization of the Friedrichs theorem to quantum algebras.

Indeed, the construction introduced in Sect. 4 works also when $\delta \neq 0$. Hence, it provides a prescription for the construction of the almost primitive generators starting from an arbitrary set of basic elements of any $U_q(g)$. In this way we obtain (g_q, Δ) from $U_q(g)$ and we close the diagram displayed in the Introduction.

Let us sketch the proof of such result. As in the undeformed case, let $\{Y_j\}$ be an arbitrary set of basic elements that determine $U_q(g)$, whose classical limit is a Lie bialgebra with $\delta \neq 0$. Eqs. (29) and (30) are still valid, but now we have to add to the cocommutative part of $\Delta_{(1)}$ the anti-cocommutative contribution given by δ

$$\Delta(X_i^0) = \Delta_{(0)}(X_i^0) + z\,\delta(X_i^0) + \sum d_i^{jl}\ \left(X_j^0 \otimes X_l^0 + X_l^0 \otimes X_j^0\right)\ + \mathcal{O}_{(2)}(X_i^0).$$

Like in Sect. 4 we define $X_i^1 := X_i^0 - \sum d_i^{jl}\ \{X_j^0, X_l^0\}$. As this change of variables does affect the δ contribution only to higher orders, the differences between $\delta(X_i^0)$ and $\delta(X_i^1)$ can be included in $\mathcal{O}_{(2)}(X_i^0)$ (or, equivalently, $\mathcal{O}_{(2)}(X_i^1)$). So,

$$\Delta(X_i^1) = \Delta_{(0)}(X_i^1) + z\,\delta(X_i^1) + \mathcal{O}_{(2)}(X_i^1).$$

From eq. (19), we introduce $\Delta_{(2)}(X_i^1)$

$$\mathcal{O}_{(2)}(X_i^1) = \Delta_{(2)}(X_i^1) + \mathcal{O}_{(3)}(X_i^1).$$

Similarly to $\Delta_{(1)}$, the second order $\Delta_{(2)}$ contains two contributions: the first one – characteristic of the deformations– is proportional to z^2, and is constrained by the consistency between δ and the coassociativity condition. The second one is only related to the definition of the basis in $U(g)$ and can be removed by another change of basis that does not modify the form of the z-dependent terms since the modifications can be included in $\mathcal{O}_{(3)}(X_i^2)$.

This procedure can be iterated. Thus, once the problem is solved for $\Delta_{(m-1)}$, a contribution to $\Delta_{(m)}$ proportional to z^m (cocommutative for m even and anti-cocommutative for m odd) is found. Indeed the unessential z-independent terms are removed, exactly as in the case $\delta = 0$, with a change of basis that does not affect the form of the previous z-depending terms because the introduced changes are always of orders higher of z^m and thus pushed out in $\mathcal{O}_{(m+1)}$. For $m \to \infty$ the same coproducts derived from $(g, \Delta_{(0)})$ in Sect. 3 are found. The deformed commutation rules are then imposed by the homomorphism (23). Hence, we have closed the lower row of the diagram, finding that $U_q(g)$ has as one of its basic sets the analytical basis (g_q, Δ) obtained as the analytical continuation in z of the Lie generators.

6. Concluding remarks

The main result of this paper is the definition and the operative prescription for the complete construction of the basis in semi-simple algebras, both Lie and quantum. In the Lie case our result end the Cartan program by fixing the few normalizations adjusted before by convention. The situation is completely different in quantum algebras where also the basic sub-space of $U_q(g)$, corresponding to the Lie algebra, has to be found. We solve the problem by means of analyticity: by starting from the cocommutator δ given by the Lie bialgebra, a perturbative method based on cocommutativity and followed by the appropriate sum of the series allows us to determine the full expressions for the deformed coproducts. In this way, each primitive coproduct is deformed to its quantum counterpart. A perturbative application of the homomorphims property of the coproduct map allows to start from the Lie form of the commutators and to arrive to their deformations. In this way the full analytical quantization –including non simple roots– is obtained.

As an important consequence, this quantization procedure imposes that the deformation of the commutators remain antisymmetric, i.e. in the analytical basis there is no room for q-commutators, objects without a precise symmetry. This is relevant for physical applications since the commutators are essential elements in Quantum Mechanics as well as in Poisson-Lie structures (whose q-counterparts are not well-defined).

The realization of one and only one quantum analog of the generators has also a relevant byproduct. As $(g, \delta) \leftrightarrows (g_q, \Delta) \leftrightarrows U_q(g)$, a one-to-one correspondence between $U_q(g)$ and (g, δ) is established. In other words, each Lie bialgebra admits one (as it was known) and only one (as stated here) quantization and the problem of classification of semisimple quantum groups can be fully solved by restoring to the classification of the underlying Lie bialgebras.

Finally let us note that the proposed method is constructive and it could be implemented to build by computer the quantum algebra deformation of every Lie bialgebra (g, δ).

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