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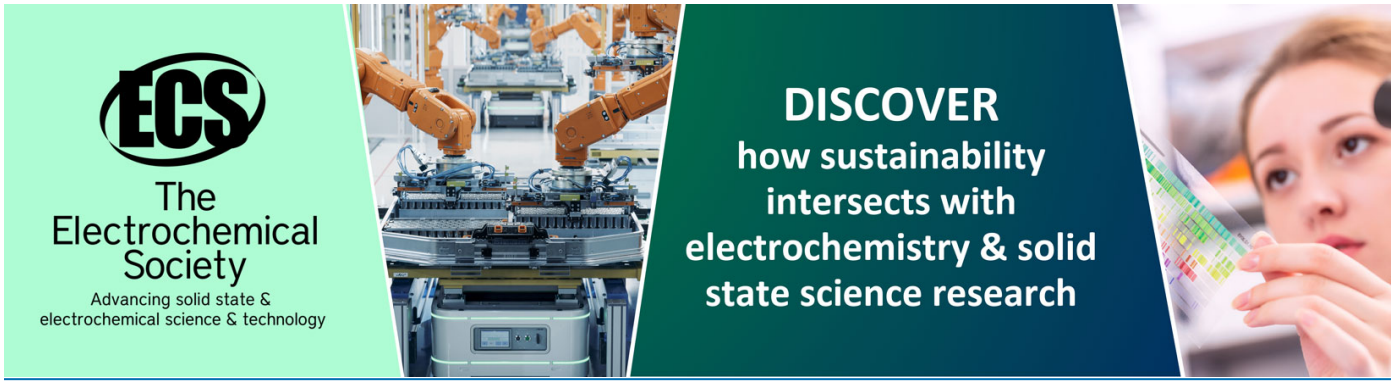

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Solvable $\mathcal{PT}$-symmetric potentials in 2 and 3 dimensions

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Abstract. The solution of non-central $\mathcal{PT}$-symmetric potentials is discussed by the separation of the variables in polar and angular coordinates. Conditions are formulated to guarantee the separation of the variables and the $\mathcal{PT}$ symmetry of the potential. The original eigenvalue equation is separated into one-dimensional Schrödinger-type differential equations. The importance of the boundary conditions, especially that of the periodic boundary condition of the azimuthal equation is pointed out. Further conditions leading to exact solutions of the whole problem are also formulated. An example combining the harmonic oscillator and the Scarf I potential in the radial and polar equation is discussed in detail, and the bound-state wave functions and the energy eigenvalues are derived. The spectrum exhibits partial degeneracies similar to those observed in the spectrum of the isotropic harmonic oscillator.

1. Introduction

Since the introduction of $\mathcal{PT}$-symmetric quantum mechanics [1] much effort has been devoted to exploring the physical significance of this theory and its connection with ordinary Hermitian quantum mechanics. It was found that $\mathcal{PT}$-symmetric quantum mechanics is a special case of pseudo-Hermiticity [2] and this can explain some of the unusual features of $\mathcal{PT}$-symmetric systems, such as the partly or fully real energy spectrum and the conservation of norm. The probabilistic interpretation of $\mathcal{PT}$-symmetric quantum mechanics, and in general pseudo-Hermitian models also received much attention, and positive definit metric operators have been cosntructed [3]. Although non-Hermitian theories that make use of modified metric operators have been known previously under various names, *e.g.* pseudo- [4], quasi- [5] and crypto-Hermiticity [6], the role of $\mathcal{PT}$-symmetric quantum mechanics is indisputable in reviving interest in these theories.

Exactly solvable quantum mechanical models are indispensable in gaining insight in the physical background of any model, and this is also the case with $\mathcal{PT}$-symmetric quantum mechanics. The $\mathcal{PT}$-symmetric versions of real potentials have been constructed, and have been used to explore various aspects of $\mathcal{PT}$ symmetry, such as identifing potentials with real and complex eigenvalues [7, 8, 9, 10, 11, 12, 13], describing the mechanism of the spontaneous breakdown of $\mathcal{PT}$ symmetry [14, 15], combining $\mathcal{PT}$-symmetry with supersymmetric [10, 16] and algebraic [17, 18, 19] techniques, determining the normalization constants and pseudo-norm of $\mathcal{PT}$-symmetric potentials [15, 20, 21], *etc.*

The scope of investigations has been extended from one-dimensional non-relativistic problems, in various directions, such as scattering solutions [18, 22], periodic structures [23], coupled

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channels [24], more particles [25] and relativistic wave equations [26]. After some first efforts [27, 28, 29] the systematic extension of $\mathcal{PT}$-symmetric potentials in higher dimensions has also started [30, 31]. In these studies the separation of the variables has been applied and the multi-dimensional problem has been reduced to one-dimensional eigenvalue problems resembling the Schrödinger equation. The importance of the angular sector in introducing the spontaneous breakdown of $\mathcal{PT}$ symmetry has been pointed out in the case of both two- [30] and three-dimensional [31] models. Similar studies have been performed earlier on real non-central potentials [32], nevertheless, the $\mathcal{PT}$ symmetry requirement makes these studies rather more complicated. The previous works focused more on the formal solution of $\mathcal{PT}$-symmetric potentials in two and three dimensions, and actual solvable examples with complete analytical solutions have not been given. The aim of the present work is to fill this gap by presenting explicit examples for solvable models.

In section 2 the general formalism is presented for both two- and three-dimensional potentials and conditions are formulated for the implementation of $\mathcal{PT}$ symmetry. Section 3 deals with further conditions allowing for the exact solution of the separated eigenvalue equations in terms of known solvable potentials, and it also contains the explicit wave functions and energy eigenvalues of a concrete example. Finally in section 4 a summary of the results is given.

2. Separation of the variables and conditions for $\mathcal{PT}$ symmetry

Let us consider the Schrödinger equation with a general non-central potential and constant mass

$$\left(\frac{\mathbf{p}^2}{2m} + V(\mathbf{r})\right)\psi(\mathbf{r}) = -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}) \ . \tag{1}$$

In what follows we specify (1) for two and three spatial dimensions and investigate the conditions under which its formal solution is possible by the separation of the angular and radial variables. We also formulate the conditions for the $\mathcal{PT}$ symmetry of the potential, which also implies the $\mathcal{PT}$ symmetry of the corresponding Hamiltonian, as the kinetic term is always $\mathcal{PT}$-symmetric. Without the loss of generality we can use the units $2m = \hbar = 1$.

2.1. $\mathcal{PT}$-symmetric Hamiltonians in 2 dimensions

The Schrödinger equation (1) expressed in terms of polar coordinates in two dimensions is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} - V(\rho,\varphi)\psi + E\psi = 0 \ . \tag{2}$$

With the substitution

$$\psi(\rho,\varphi) = \rho^{-1/2}\phi(\rho)\tau(\varphi) \tag{3}$$

the equation

$$\phi''\tau + \frac{1}{\rho^2}\phi\tau'' - \left(V(\rho,\varphi) - \frac{1}{4\rho^2} - E\right)\phi\tau = 0 \tag{4}$$

is obtained, where prime denotes derivation with respect to the corresponding single variable and $\rho \in [0,\infty)$ and $\varphi \in [0, 2\pi]$. The separation of the radial and angular variables is possible, if $\tau(\varphi)$ satisfies the equation

$$\tau'' = (K(\varphi) - k)\tau \ . \tag{5}$$

In this case the potential can be written as the sum of a central and a non-central term as

$$V(\rho,\varphi) = V_0(\rho) + \frac{1}{\rho^2}K(\varphi) \ . \tag{6}$$

The condition for $\mathcal{PT}$ symmetry is then

$$V(\rho,\varphi) = V^*(\rho,\varphi+\pi) \ , \tag{7}$$

which means that the central potential component $V_0(\rho)$ has to be real and $K(\varphi)$ has to satisfy

$$K^*(\varphi+\pi) = K(\varphi) \ . \tag{8}$$

Equations (5) and (8) indicate that the $\mathcal{PT}$ symmetry of the full two-dimensional potential (6) generates a $\mathcal{PT}$-symmetric Hamiltonian in the angular variable too, although the space reflection operator assigned to this problem has to be defined as

$$\mathcal{P}_\varphi \varphi = \varphi + \pi \ . \tag{9}$$

Furthermore, the problem has to be defined on the $\varphi \in [0, 2\pi]$ domain with periodic boundary conditions $\tau(\varphi) = \tau(\varphi + 2\pi)$ and $\tau'(\varphi) = \tau'(\varphi + 2\pi)$. Note that $\tau(\varphi)$ need not vanish at the boundaries, so the solutions of (5) are less restricted than those of a proper Schrödinger equation defined on a finite domain.

The eigenvalue k of equation (5) can be real or complex, depending on whether the $\mathcal{PT}$ symmetry of this angular problem is unbroken or spontaneously broken. It is also important to note that (8) implies *different* constraints on the real and imaginary component of $K(\varphi)$: $K_R(\varphi+\pi) = K_R(\varphi)$ and $K_I(\varphi+\pi) = -K_I(\varphi)$. In the case of a periodic $K(\varphi)$ this means that the periodicity of the two components have to be different, with even and odd number of periods for $K_R(\varphi)$ and $K_I(\varphi)$. This is analogous to the usual $V^*(-x) = V(x)$ $\mathcal{PT}$ symmetry requirement in one dimension, where the real and imaginary potential components have to be even and odd functions of the coordinate, respectively.

Separating the angular parts, we are left with the equation

$$-\phi'' + \left[V_0(\rho) + \left(k - \frac{1}{4}\right)\frac{1}{\rho^2}\right]\phi - E\phi = 0 \ , \tag{10}$$

in which the eigenvalue k of the angular equation (5) appears explicitly. Equation (10) is similar to the radial Schrödinger equation obtained in the case of real central potentials, and the boundary conditions also have to be chosen similarly. It is notable though, that the "angular momentum" can now take on complex values too if the $\mathcal{PT}$ symmetry of equation (5) is spontaneously broken. In this case the energy eigenvalues E will also be complex, which means that the $\mathcal{PT}$ symmetry of the *whole* two-dimensional system will also be spontaneously broken. Taking the complex conjugate of (10) it is straightforward to see that there will be two complex conjugate energy eigenvalues originating from the complex conjugate values k and k^*. This is similar to the situation in the case of the spontaneous breakdown of the $\mathcal{PT}$ symmetry in one-dimensional problems.

2.2. $\mathcal{PT}$-symmetric Hamiltonians in 3 dimensions

Using polar coordinates equation (1) takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} - V(r,\theta,\varphi)\psi + E\psi = 0 \ . \tag{11}$$

Factorizing the solutions as

$$\psi(r,\theta,\varphi) = r^{-1}\phi(r)\sin^{-1/2}\omega(\theta)\tau(\varphi) \ , \tag{12}$$

where $r \in [0, \infty)$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ we obtain the three-dimensional analogue of equation (4):

$$\phi''\omega\tau + \frac{1}{r^2}\phi\omega''\tau + \frac{1}{r^2\sin^2\theta}\phi\omega\tau'' - \left(V(r,\theta,\varphi) - \frac{1}{4r^2} - \frac{1}{4r^2\sin^2\theta} - E\right)\phi\omega\tau = 0 \ . \tag{13}$$

The separation of the variables is possible, if $\tau(\varphi)$ satisfies (5) as in the two-dimensional case, while for $\omega(\theta)$ a similar second-order differential equation holds:

$$\omega'' = (P(\theta) - p)\omega \ , \tag{14}$$

with the difference that the boundary conditions need not be periodic. Similarly to $\tau(\varphi)$, $\omega(\theta)$ need not vanish at the boundaries either, so, again solutions beyond the physical wave functions of one-dimensional confined potentials are allowed.

Making use of the angular functions $K(\varphi)$ and $P(\theta)$, the non-central potential can be written as

$$V(r,\theta,\varphi) = V_0(r) + \frac{K(\varphi)}{r^2\sin^2\theta} + \frac{P(\theta)}{r^2} + \frac{1}{r^2\sin^2\theta}\left(\frac{1}{4} - k\right) \ , \tag{15}$$

Here $V_0(r)$ is a spherical potential, which appears in a radial equation

$$-\phi'' + \left[V_0(r) + \frac{1}{r^2}\left(p - \frac{1}{4}\right)\right]\phi - E\phi = 0 \ . \tag{16}$$

Similarly to the two-dimensional case, equation (16) is formally identical with a radial Schrödinger equation in which the $l(l+1)$ angular momentum term is replaced by $p - 1/4$, where p can be real or imaginary, depending on the solution of the angular equation (14). It is notable that p is related to the "angular momentum" via $p = (l + 1/2)^2$.

The conditions under which the non-central potential (15) is $\mathcal{PT}$-symmetric are determined by the $\mathcal{PT}$ symmetry of the angular equations (5) and (14). Considering that the $\mathcal{P}$ operator acts like $\mathcal{P} : \mathbf{r} \to -\mathbf{r}$, the $\mathcal{PT}$ operation can be factorized into angular terms as

$$\mathcal{P} = \mathcal{P}_\theta\mathcal{P}_\varphi \ , \tag{17}$$

$$\mathcal{P}_\theta\theta = \pi - \theta \ , \tag{18}$$

where $\mathcal{P}_\varphi$ is the same as in the two-dimensional case: (9). From equations (17), (18) and (9) it is seen that the $\mathcal{PT}$ transform of (15) is

$$V(r,\theta,\varphi) = V^*(r,\pi-\theta,\varphi+\pi) \tag{19}$$

$$= V_0^*(r) + \frac{K^*(\varphi+\pi)}{r^2\sin^2\theta} + \frac{P^*(\pi-\theta)}{r^2} + \frac{1}{r^2\sin^2\theta}\left(\frac{1}{4} - k^*\right) \ . \tag{20}$$

This implies the following restrictions of the quantities appearing in the expression (15):

$$V_0(r) = V_0^*(r) \ , \tag{21}$$

$$P^*(\pi-\theta) = P(\theta) \ , \tag{22}$$

$$K^*(\varphi+\pi) = K(\varphi) \ , \tag{23}$$

$$k^* = k \ . \tag{24}$$

This means that the $\mathcal{PT}$ symmetry of the non-central potential (15) hinges on the $\mathcal{PT}$ symmetry of the angular equations, while further conditions require the reality of the central potential component and that of k, *i.e.* the eigenvalue of the equation (5). It is important to note that the eigenvalue p of the polar equation (14) need not be real. Note that if p is real, then (16)

becomes an ordinary radial Schrödinger equation with real angular momenta and real energy eigenvalue E. On the contrary, if p is complex, then the energy eigenvalue of (16) and thus that of the *whole* Hamiltonian (1) E will be complex, *i.e.* the $\mathcal{PT}$ symmetry of (1) will be spontaneously broken. Furthermore, in this case E^* will also be an eigenvalue belonging to p^*, as can be seen from the complex conjugation of (16). This was also the case in two dimensions, where k played the role of p.

3. Conditions for exact solvability

The results of section 2 indicate that the general solution of the Schrödinger equation is possible by solving the angular and radial equations such that the energy eigenvalues E of the Schrödinger equation (1) will be supplied by the energy eigenvalues of the radial equation, in which the role of the "angular momentum" is played by the eigenvalue of a radial equation (that of (5) for $d = 2$ and that of (14) for $d = 3$). The energy eigenvalues E will be real or complex depending on whether the "angular momentum" is real or complex. Furthermore in the three-dimensional case the eigenvalue of the azimuthal equation (5) appear explicitly in the non-central component of the potential function (15), so further considerations have to be made if we expect the potential to be state-independent. In this section we discuss the conditions under which the exact discrete solutions of the non-central $\mathcal{PT}$-symmetric potentials can be given, and illustrate the results with some examples. It turns out that the solutions of equation (5) play a crucial role in both the two- and the three-dimensional cases.

3.1. Solving the radial equations

Due to the analogous structure of equations (10) and (16) it is reasonable to discuss their solution jointly. Exact solution of this type of equation with arbitrary value of the "angular momentum" and the principal quantum number is possible for the harmonic oscillator, Coulomb and square well potentials. Usually these potentials are discussed with integer number of the angular momentum l, although the results are easily generalized to non-integer real values too, as is the case with the Kratzer potential, for example [33]. Table 1 contains the quantities relevant to the harmonic oscillator and Coulomb potentials, together with the appropriate substitutions that lead to the exact solution of equations (10) and (16). Note that the formulae allow complex values of the "angular momentum" too, which occur as complex conjugate pairs in the case of the spontaneous breakdown of $\mathcal{PT}$ symmetry. The wave functions are expressed in terms of generalized Laguerre polynomials [34] in both cases. Note that in the substitution for l the indefinite sign comes from the square root taken from $k = (l + \frac{1}{2})^2$ and $p = (l + \frac{1}{2})^2$.

When the radial potential is the infinite square well, the solutions can be expressed in terms of Bessel functions the order of which is related to l [33]. However, the energy eigenvalues cannot be expressed in closed form in this case, because they are determined by the zeros of the Bessel functions. Nevertheless, the square well potential can be used to construct non-central $\mathcal{PT}$-symmetric potentials with exact solutions.

Some solutions can also be obtained for arbitrary k or p for quasi-exactly solvable (QES) potentials [37] in the sense that the first few solutions (up to a given principal quantum number) can be determined exactly then. There are many more solvable potentials for the special value of $k = 1/4$ or $p = 1/4$ (these are the ones solvable in the usual framework for s waves only), but these are unimportant from the point of view of constructing $\mathcal{PT}$-symmetric potentials.

3.2. Solving the angular equation in $d = 2$

The most important prescriptions here are the unusual $\mathcal{PT}$ symmetry requirement (23) of the "potential" $K(\varphi)$ and the periodic boundary conditions of the solutions.

The simplest choice is applying the *real* infinite square well as a special $\mathcal{PT}$-symmetric potential. Then $K(\varphi) = 0$ so (6) reduces to the real central potential $V_0(\rho)$. The general

Table 1. The energy eigenvalues and wave functions of the harmonic oscillator and Coulomb potentials with the substitutions necessary to obtain the exact solutions of the radial Schrödinger equations (10) and (16) in two and three dimensions. The notations of [35] and [36] are followed.

	Harmonic oscillator	Coulomb potential
Potential	$\frac{\omega^2}{4}x^2 + \frac{l(l+1)}{x^2}$	$-\frac{\mathrm{e}^2}{x} + \frac{l(l+1)}{x^2}$
E_n	$(2n + l + \frac{3}{2})\omega$	$-\frac{\mathrm{e}^4}{4(n+l+1)^2}$
$\phi_n(x)$	$C^{HO}x^{l+1}\exp(-\omega x^2/4)L_n^{(l+\frac{1}{2})}(\frac{\omega}{2}x^2)$	$C^C x^{l+1}\exp(-\frac{\mathrm{e}^2 x}{2(n+l+1)})L_n^{(2l+1)}(\frac{\mathrm{e}^2 x}{n+l+1})$
x for $d = 2$	ρ	ρ
l for $d = 2$	$\pm k^{1/2} + \frac{1}{2}$	$\pm k^{1/2} + \frac{1}{2}$
x for $d = 3$	r	r
l for $d = 2$	$\pm p^{1/2} + \frac{1}{2}$	$\pm p^{1/2} + \frac{1}{2}$

solutions can be written in terms of the exponential functions $\mathrm{e}^{\pm \mathrm{i}m\varphi}$, but in order to construct $\tau(\varphi)$ functions that are the eigenfunctions of the $\mathcal{P}_\varphi\mathcal{T}$ operator (see (9)) with unit eigenvalue, a special combination of them has to be taken:

$$\tau_m(\varphi) = \frac{\mathrm{i}^m}{(2\pi)^{1/2}}\cos(m\varphi) \ . \tag{25}$$

These wave functions are $\mathcal{PT}$-normalized as $\langle\tau_n|\mathcal{P}_\varphi|\tau_m\rangle = \delta_{nm}(-1)^m$ and the corresponding energy eigenvalues are $k = k_m = m^2$. Nothing changes essentially if $K(\varphi)$ is chosen to be a real constant: this case corresponds to a redefinition of the "angular momentum" as the relation of k and m is changed.

Further solutions can be obtained by appling exactly solvable $\mathcal{PT}$-symmetric potentials defined on a finite domain, such as the $\mathcal{PT}$-symmetric Scarf I [20] or Rosen–Morse I [21] potentials. Due to the requirement (23), however, these potentials have to be defined in two (or even number of) separate domains $\varphi \in [0, 2\pi]$, with periodic boundary conditions. One problem with these potentials is that they possess inverse-square-type singularity at the boundaries, which will thus separate the segments from each other by an impenetrable wall. This can be avoided by using parameters that result in a weakly attractive singularity at the boundaries, which would, in principle allow communication between the individual segments. It is notable that considering these potentials under the usual $\mathcal{PT}$-symmetry requirements and with solutions vanishing at the boundaries, only real energy eigenvalues can be obtained [20, 21].

Let us now turn to semi-analytically solvable potentials. Examples for this are the combination of imaginary step potentials on a ring [28, 29]. Due to the periodic boundary conditions the energy eigenvalues of these systems asymptotically go to those of the infinite square well, but the number and arrangement of the steps are also reflected in the local relative position of the levels. The solutions up to 20 or so can be easily determined for these potentials by graphical and analytical methods. It was also found that by increasing non-Hermiticity (*i.e.* the height of the steps), complex-energy solutions appear at a certain point [29]. With the use of these potentials in (5) therefore at least a finite set of the solutions of non-central 2-dimensional $\mathcal{PT}$-symmetric potentials can be handled. It has to be noted that the potential in [28] does not obey (23), because it has even number of imaginary steps on the ring, nevertheless, the results are instructive to the case considered here too.

Further possibility could be applying $\mathcal{PT}$-symmetric arrangements of Dirac delta potentials [23] or attempting to construct the Lamé-type potentials [38] with $\mathcal{PT}$ symmetry.

3.3. Solving the angular equations in $d = 3$

Equation (5) appears in the three-dimensional case as the azimuthal equation, and since the boundary conditions are the same as in the two-dimensional case, the solutions outlined in subsection 3.2 can be used here too. An important difference is, however, that the eigenvalue k of equation (5) appears now in the potential function (15), rather than in the radial equation (16), where its role is taken over by p, the eigenvalue of equation (14).

In what follows, therefore we discuss the exact solutions of the polar equation (14). When discussing the exact solution of this equation, it is worthwhile to modify the formalism in order to handle the θ-dependent terms in (15) more consistently. Let us modify (14) such that we introduce in it a term that depends on the eigenvalue k of equation (5):

$$\omega'' = (P(\theta) - p)\omega \equiv \left(\widetilde{P}(\theta) + \frac{k}{\sin^2\theta} - p\right)\omega \ . \tag{26}$$

Then (15) is formally simplified to a form which does not contain the constant k explicitly:

$$V(r,\theta,\varphi) = V_0(r) + \frac{K(\varphi)}{r^2\sin^2\theta} + \frac{\widetilde{P}(\theta)}{r^2} + \frac{1}{4r^2\sin^2\theta} \ . \tag{27}$$

The state-independence of $V(r,\theta,\varphi)$ is thus achieved at the price of transferring k-dependence into (26). However, when our objective is finding solvable potentials in 3 dimensions, the closed solutions of (26) also have to be determined. Unfortunately, the range of exactly solvable potentials containing a $\sin^{-2}\theta$ term is rather limited. In fact, considering potentials that are non-singular within the $\theta \in (0,\pi)$ domain, it is resticted to the Scarf I and Rosen–Morse I potentials (see *e.g.* [7]).

As a further aspect of this modification we note that prescribing the $\mathcal{PT}$-invariance of $V(r,\theta,\varphi)$ leads to the (21) and (23) conditions and the $\widetilde{P}^*(\pi-\theta) = \widetilde{P}(\theta)$ requirement. This is equivalent with the already established conditions, *i.e.* (22) and (24), of which the latter one can be dropped if $P(\theta)$ is a potential that contains a $\sin^{-2}\theta$ term. In summary, the modification of the formalism by (26) has no effect on the results as long as we are interested in exactly solvable $\mathcal{PT}$-symmetric potentials in 3 dimensions.

Considering the importance of the $\mathcal{PT}$-symmetric Scarf I [20] and Rosen–Morse I [21] potentials from the point of view of constructing exactly solvable $\mathcal{PT}$-symmetric potentials in three dimensions, it seems worthwhile to present the most important results adapted to the present situation.

In order to use the results of [20], first a coordinate shift $\theta = x + \frac{\pi}{2}$ is necessary to define the Scarf I potential in the right domain $\theta \in [0,\pi]$. Obviously, this does not influence the results in any ways. Applying the $\mathcal{PT}$-symmetric Scarf I potential in (14) one then gets

$$P(\theta) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)\frac{1}{\sin^2\theta} - \frac{\alpha^2-\beta^2}{2}\frac{\cos\theta}{\sin^2\theta} \tag{28}$$

and

$$p_j = \left(j + \frac{\alpha+\beta+1}{2}\right)^2 \ , \tag{29}$$

while the bound-state solutions are expressed in terms of Jacobi polynomials as

$$\omega_j(\theta) = C_j^{SI}(1+\cos\theta)^{\alpha/2+1/4}(1-\cos\theta)^{\beta/2+1/4}P_j^{(\beta,\alpha)}(\cos\theta) \ . \tag{30}$$

The normalization constants have been determined in [7] such that $\mathcal{P}_\theta\omega_j(\theta) = \omega_j(\theta)$. The corresponding expressions for the $\mathcal{PT}$-symmetric Rosen–Morse I potential [21] are

$$P(\theta) = \left(j + \frac{\alpha+\beta}{2}\right)\left(j + \frac{\alpha+\beta}{2} + 1\right)\frac{1}{\sin^2\theta} + \mathrm{i}\frac{\alpha^2-\beta^2}{2}\cot\theta \ , \tag{31}$$

Table 2. Expressions for the potential (15), the α and β parameters and the relation of the "angular momentum" l of the radial equations with p, when the $\mathcal{PT}$-symmetric Scarf I and Rosen–Morse I potentials are used in the polar equation (14).

	Scarf I	Rosen–Morse I
$V(r,\theta,\varphi)$	$V_0(r)+\frac{K(\varphi)}{r^2\sin^2\theta}+\frac{A}{r^2\sin^2\theta}-\frac{B\cos\theta}{r^2\sin^2\theta}$	$V_0(r)+\frac{K(\varphi)}{r^2\sin^2\theta}+\frac{A}{r^2\sin^2\theta}+\frac{\mathrm{i}B}{r^2}\cot\theta$
A	$\frac{\alpha_m^2+\beta_m^2}{2}-k_m$, real	$\left(j+\frac{\alpha_{jm}+\beta_{jm}+1}{2}\right)^2-k_m$, real
B	$\frac{\alpha_m^2-\beta_m^2}{2}$, imaginary	$\frac{\alpha_{jm}^2-\beta_{jm}^2}{2}$, real
α	$\alpha_m=\pm(A+B+k_m)^{1/2}$	$\alpha_{jm}=\pm\left[(A+k_m)^{1/2}-j-\frac{1}{2}\right]$ $\pm\frac{B}{2}\left[(A+k_m)^{1/2}-j-\frac{1}{2}\right]^{-1}$
β	$\beta_m=\pm(A-B+k_m)^{1/2}$	$\beta_{jm}=\pm\left[(A+k_m)^{1/2}-j-\frac{1}{2}\right]$ $\mp\frac{B}{2}\left[(A+k_m)^{1/2}-j-\frac{1}{2}\right]^{-1}$
$(l+\frac{1}{2})^2$	$p_{jm}=\left(j+\frac{\alpha_m+\beta_m+1}{2}\right)^2$	$p_{jm}=\left(\frac{\alpha_{jm}+\beta_{jm}}{2}\right)^2+\left(\frac{\alpha_{jm}-\beta_{jm}}{2}\right)^2$

$$p_j=\left(\frac{\alpha_j+\beta_j}{2}\right)^2+\left(\frac{\alpha_j-\beta_j}{2}\right)^2 \tag{32}$$

and

$$\omega_j(\theta)=C_j^{RMI}(1+\mathrm{i}\cot\theta)^{\alpha_j/2}(1-\mathrm{i}\cot\theta)^{\beta_j/2+1/4}P_j^{(\alpha_j,\beta_j)}(\mathrm{i}\cot\theta)\ . \tag{33}$$

Note that in (31) we used the original form of the potential, which contains the principal quantum number j. Obviously, the j-dependence of the potential has to be cancelled by the appropriate choice of the parameters (hence the indices in α_j and β_j), however, the above expression is more suited to the application of the Rosen–Morse I potential in the present context.

Finally, we note that in principle the Pöschl–Teller I potential could also be applied in the polar equation as another trigonometric potential containing the $\sin^{-2}\theta$ term [35, 7], however, in this case the other term is $\cos^{-2}\theta$, which would introduce a singularity at $\theta=\pi/2$. Defining it only in one of the domains and then extending the length scale as $\theta\to 2\theta$ would only simply recover the Scarf I potential.

3.4. Illustration

As an example, let us consider the potential

$$V(r,\theta,\varphi)=\frac{\omega^2}{4}r^2+\frac{A}{r^2\sin^2\theta}-\frac{B\cos\theta}{r^2\sin^2\theta} \tag{34}$$

which is nothing but (15) with the harmonic oscillator, the Scarf I potential and the free motion in the r, θ and φ variables, respectively. Since $K(\varphi)=0$, the solution of the azimuthal equation (5) is given by (25) with $k_m=m^2$, so this latter quantity has to be substituted into α_m and β_m in Table 2.

Making use of the results from Table 1 and equations (30) and (25), the solution of the potential (34) are given by

$$\begin{aligned}\psi(r,\theta,\varphi) &= \frac{\mathrm{i}^m}{(2\pi)^{1/2}}C_n^{HO}C_j^{SI}(1+\cos\theta)^{\alpha/2+1/4}(1-\cos\theta)^{\beta/2+1/4}P^{(\beta,\alpha)}(\cos\theta)_j\\ &\quad\times r^{l_m+1}\exp(-\omega r^2/4)L_n^{(l_m+\frac{1}{2})}(\frac{\omega}{2}r^2)\cos(m\varphi)\ ,\end{aligned} \tag{35}$$

where

$$\alpha_m = \pm(A + B + k_m)^{1/2} \tag{36}$$
$$\beta_m = \pm(A - B + k_m)^{1/2} \tag{37}$$
$$l_m = \pm\left(j + \frac{\alpha_m + \beta_m + 1}{2}\right) - \frac{1}{2} \tag{38}$$

and B is imaginary. It is seen that there are several possible choices for choosing the signs in the formulae. It can be shown that the solutions of the ordinary harmonic oscillator (or any central potential) can be obtained as a special case if opposite signs are chosen in equations (36) and (37), while in (38) the positive sign has to taken.

The energy eigenvalues of this potential will then be

$$E_{njm} = \left(2n + j + \frac{\alpha_m + \beta_m}{2} + \frac{3}{2}\right)\omega \ . \tag{39}$$

The spectrum will be real or complex depending on whether the $\alpha_m^* = \beta_m$ or the $\alpha_m^* = -\beta_m$ relation holds.

Due to the somewhat complicated dependence of α_m and β_m on m, the spectrum is considerably less simple than in the case of the ordinary isotropic harmonic oscillator. Nevertheless, there are degeneracy patterns in it, as when m is fixed, states with the same value of $2n + j$ have degenerate eigenvalues. This is similar to the degeneracy observed in the energy spectrum of the isotropic harmonic oscillator. Operators increasing/decreasing n with one unit and decreasing/increasing j by two units can be constructed, so in any fixed-m subspace similar operators can act as in the case of the isotropic harmonic oscillator (see *e.g.* [39]).

Similar results could be obtained applying the Coulomb potential as $V_0(r)$, and both radial potentials could be combined with the Rosen–Morse I potential in the polar angle variable. The degeneracy pattern of the energy spectrum would be less rich in that case, because the "angular momentum" depends on j in a more complicated form (see Table 2), so finding combinations of n and j that lead to the same energy are less probable.

Finally, the free motion in the azimuthal variable can also be replaced with more general potentials, as has been discussed in subsection (3.2). The dependence on the results on m becomes more complicated then (including the case when even there is no closed formula for k_m in terms of m), so the complexity of the situation certainly increases.

4. Summary and outlook

We discussed non-central $\mathcal{PT}$-symmetric quantum potentials in 2 and 3 dimensions by separating the variables in radial and angular variables. The original problem can then be reduced to two or three ordinary Schrödinger equation eigenvalue problem with various boundary conditions. The $\mathcal{PT}$ symmetry of the whole potential also introduces conditions for the "potentials" and eigenvalues appearing in the separated eigenvalue equations. The role of the boundary conditions, and in particular, that of the periodic boundary condition characterizing the azimuthal equation has been discussed. It was shown that although usual one-dimensional potentials can be used to formulate the angular equations, some care has to be taken in adapting these systems to the actual situation.

We also analyzed the possible exact solutions of the separate eigenvalue equations. The radial equation turned out to be similar to the ordinary radial Schrödinger equation, with the difference that the "angular momentum" appearing in it can take on complex, as well as real values. In the former case the energy eigenvalues appear in complex conjugate pairs, as is the case also in one-dimensional $\mathcal{PT}$-symmetric potentials. Exact solutions of the radial equation can be obtained using the harmonic oscillator and the Coulomb potentials. It is notable that

in one-dimensional $\mathcal{PT}$-symmetric quantum mechanics the Coulomb potential can be discussed only on trajectories outside the real x axis, because otherwise the boundary conditions cannot be enforced [40, 7].

The analysis of the polar equation showed that its exact solutions are possible with potentials containing the $\sin^{-2}\theta$ term, and this indicated the important role of the Scarf I and Rosen–Morse I potentials.

Finally, we constructed an exactly solvable $\mathcal{PT}$-symmetric potential using the harmonic oscillator, the Scarf I potential and the the free particle in the radial, polar and azimuthal coordinates and determined its bound-state wave functions and energy eigenvalues. The energy spectrum exhibited degeneracies similar to those observed in the case of the (real) isotropic harmonic oscillator, although the spectrum is separated to subspaces defined by the quantum number m.

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