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M Gadella

Departamento de Física Teórica, Atómica y Optica. Universidad de Valladolid
Facultad de Ciencias, c. Real de Burgos, s.n.
47011 Valladolid, Spain.

E-mail: manuelgadella@yahoo.com.ar

Abstract. In this note, we discuss some mathematical properties of Gamow vectors. We start with the definition and properties of the approximate Gamow vectors that are Hilbert space vectors with similar properties to Gamow vectors and are defined by integration over the Hamiltonian spectrum and not over the whole real line, as is the case with ordinary Gamow vectors. We also discuss some properties of Gamow dyads, which are density operators constructed with Gamow vectors.

1. Introduction and motivation

The aim of this communication is to discuss some properties of Gamow vectors for quantum resonances. Here, we limit ourselves to nonrelativistic resonances following a model that has been extensively discussed in [1, 2, 3], where the definition, construction and basic properties of Gamow vectors are discussed in detail. This model for resonances requires of some basic assumptions that we can briefly summarize as follows: i.) Resonances are produced in a process of quantum resonance scattering with "free" Hamiltonian H_0 and total Hamiltonian $H = H_0+V$. ii.) We assume asymptotic completeness and the existence of Møller wave operators. iii.) Concerning the S operator, we are assuming that in the energy representation, it is given by a two valued meromorphic function $S(E)$ on a complex variable E. In the language of Riemann surfaces, $S(E)$ is a meromorphic function defined on a two sheeted Riemann surface. iv) Then, resonances are given by pairs of complex poles located in the analytic continuation of $S(E)$ on the second sheet. Each pair of conjugate poles have the form $z_R = E_R - i\Gamma/2$ and $z_R^* = E_R + i\Gamma/2$, with $E_R > 0$ and $\Gamma > 0$.

Further discussions on resonance scattering can be found in [4, 5, 6, 7].

For a resonance with resonance poles at z_R and z_R^*, we have respective Gamow vectors. To the resonance pole z_R corresponds the decaying Gamow vector, which can be defined [8, 9] as a state vector that decays exponentially with decay rate Γ or as an eigenvector of the total Hamiltonian H with eigenvalue z_R. Both notions are equivalent. Similarly, the growing Gamow vector grows exponentially with rate Γ and is an eigenvector of the total Hamiltonian H with eigenvalue z_R^*. One is the time reversal of the other. From this notion, it is clear that neither of the Gamow vectors can be represented by a vector in a Hilbert space.

In the sequel, we shall assume that we have only one resonance with resonance poles at z_R and z_R^* as above. This idea can bring all the properties of the resonance scattering with a maximum of simplicity [10] and can be implemented with the use of Friedrichs model [11] that has the important property of being exactly solvable.

In resonance scattering, a state $\psi^{\rm in}$ is prepared in the remote past and enters in the interaction region, where a resonance is created. Pictorially, we can imagine that the state represents a particle and the idea that a resonance is created by assuming that the particle stays within the interaction region a time large compared with the time it would have spent if no interaction existed. As the particle abandons the interaction region it has a state represented by the vector $\psi^{\rm out}$, where $S\psi^{\rm in} = \psi^{\rm out}$. Both, $\psi^{\rm in}$ and $\psi^{\rm out}$ evolve freely, which means that their time evolution is governed by the free Hamiltonian H_0.

However, we cannot detect in general the whole outgoing wave, but instead its projection into a region where a detector was placed. This projection gives a state that, when properly normalized, has the form $\varphi^{\rm out}$. The amplitude between the output and the state we are really measuring is given by $\langle\varphi^{\rm out}|\psi^{\rm out}\rangle = \langle\varphi^{\rm out}|S\psi^{\rm in}\rangle$. We shall consider this amplitude as the main tool in the construction of the Gamow vectors [1, 10].

As we assume that the Møller operators exists, we can write: $\psi^- := \mathbf{\Omega}_{\rm IN}\psi^{\rm in}$ and $\varphi^+ = \mathbf{\Omega}_{\rm OUT}\varphi^{\rm out}$. Since $S = \mathbf{\Omega}^\dagger_{\rm OUT}\mathbf{\Omega}_{\rm IN}$, where the dagger represents the adjoint, we have the following formula for our amplitude:

$$\langle\varphi^{\rm out}|S\psi^{\rm in}\rangle = \langle\varphi^+|\psi^-\rangle \, . \tag{1}$$

In terms of the energy, (1) reads

$$\int_0^\infty \langle\varphi^{\rm out}|E\rangle\, S(E)\, \langle E|\psi^{\rm in}\rangle\, dE = \langle\varphi^{\rm out}|S\psi^{\rm in}\rangle = \langle\varphi^+|\psi^-\rangle \, , \tag{2}$$

where $|E\rangle$ are the generalized eigenvectors of the free Hamiltonian, $H_0|E\rangle = E|E\rangle$, $E \in [0,\infty)$ [1].

The use of generalized eigenvectors a la Dirac [12] and the notion of Gamow vectors shows the need of having a structure bigger than the Hilbert space. This notion is the rigged Hilbert space, which is, as is well known, a triplet $\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^\times$, where $\mathbf{\Phi}$ is a locally convex topological vector space, dense in the Hilbert space $\mathcal{H}$ and endowed with a topology strictly finer (with more open sets and therefore with less convergent sequences) that the topology on $\mathcal{H}$. The space $\mathbf{\Phi}^\times$ is the antidual of $\mathbf{\Phi}$ or space of all the continuous antilinear mappings from $\mathbf{\Phi}$ into the set of complex numbers.

Thus, we can construct [2, 3] two rigged Hilbert spaces $\mathbf{\Phi}^\pm \subset \mathcal{H} \subset (\mathbf{\Phi}^\pm)^\times$ so that we can assume that $\varphi^+ \in \mathbf{\Phi}^+$ and $\psi^- \in \mathbf{\Phi}^-$, where $\mathcal{H}$ is the Hilbert space of scattering states of the total Hamiltonian H (i.e., the absolutely continuous space of H). The spaces $\mathbf{\Phi}^\pm$ are constructed so that the following conditions are satisfied:

i.) The total Hamiltonian H reduces them, i.e., $H\mathbf{\Phi}^\pm \subset \mathbf{\Phi}^\pm$.

ii.) The total Hamiltonian H is a continuous operator on $\mathbf{\Phi}^\pm$.

iii.) In the energy representation, in which the total Hamiltonian is the multiplication operator (for simplicity we assume no degeneration), the functions in $\mathbf{\Phi}^+$ and in $\mathbf{\Phi}^-$ are represented by restrictions to the real axis of Schwartz functions which are also Hardy functions [2, 3] on the upper and on the lower half planes respectively[1]. These are the so called very well behaved functions [18]. Their values on the positive semiaxis determine both their values on the negative semiaxis and also on the corresponding half plane [19].

iv.) For each real number E, there exists a vector $|E^\pm\rangle \in (\mathbf{\Phi}^\pm)^\times$ such that $H|E^\pm\rangle = E|E^\pm\rangle$, i.e., $|E^\pm\rangle$ is a generalized eigenvector[2] of H. If w^+ and w^- are in the open upper and lower half

[1] The use of Hardy functions has lately received some criticism in the sense that they are incompatible with quantum mechanics [13]. This criticism is however nonsense [14, 15, 16, 17].

[2] The extension of H into the antiduals is done via the duality formula

$$\langle H\varphi^\pm|F^\pm\rangle = \langle\varphi^\pm|HF^\pm\rangle \, , \qquad \forall\, \varphi \in \mathbf{\Phi}^\pm \, , \qquad \forall\, F^\pm \in (\mathbf{\Phi}^\pm)^\times \, .$$

of the complex plane, there are respective vectors $|w^\pm\rangle \in (\mathbf{\Phi}^\pm)^\times$ such that $H|w^\pm\rangle = w^\pm|w^\pm\rangle$. If $E \in [0,\infty)$, i.e., E is in the absolutely continuous spectrum of H, we have the Lippmann-Schwiger equations, $|E^+\rangle = \mathbf{\Omega}_{\rm OUT}|E\rangle$ and $|E^-\rangle = \mathbf{\Omega}_{\rm IN}|E\rangle$. These expressions can be generalized to similar expressions where E runs into the whole real line [2, 3]. This generalization plays no role in our discussion here.

v.) We have the following duality formulas:

$$\langle\varphi^{\rm out}|E\rangle = \langle\mathbf{\Omega}_{\rm OUT}\varphi^{\rm out}|\mathbf{\Omega}_{\rm OUT}|E\rangle = \langle\varphi^+|E^+\rangle$$

$$\langle\psi^{\rm in}|E\rangle = \langle\mathbf{\Omega}_{\rm IN}\psi^{\rm in}|\mathbf{\Omega}_{\rm IN}|E\rangle = \langle\psi^-|E^-\rangle\,. \tag{3}$$

Note that $\langle E|\psi^{\rm in}\rangle = \langle\psi^{\rm in}|E\rangle^* = \langle E^-|\psi^-\rangle = \langle\psi^-|E^-\rangle^*$, where the star denotes complex conjugation. These formulas can be extended to all real values of E [2].

After all these considerations, (2) can be written as [2, 3]

$$\langle\varphi^+|\psi^-\rangle = \int_0^\infty \langle\varphi^{\rm out}|E\rangle\, S(E)\, \langle E|\psi^{\rm in}\rangle\, dE = \int_0^\infty \langle\varphi^+|E^+\rangle\, S(E)\, \langle E^-|\psi^-\rangle\, dE$$

$$= 2\pi i s_R\langle\varphi^+|z_R^+\rangle\langle z_R^{*-}|\psi^-\rangle - \int_{-\infty}^0 \langle\varphi^+|E^+\rangle\, S_{II}(E)\, \langle E^-|\psi^-\rangle\, dE\,, \tag{4}$$

where:

i.) The complex number s_R is the residue of $S(E)$ at the pole z_R.

ii.) The functionals $|z_R^+\rangle$ and $|z_R^{*-}\rangle$ belong to the antiduals $(\mathbf{\Phi}^+)^\times$ and $(\mathbf{\Phi}^-)^\times$ respectively. They have the properties: $H|z_R^+\rangle = z_R|z_R^+\rangle$ and $H|z_R^{*-}\rangle = z_R^*|z_R^{*-}\rangle$. In addition,

$$e^{-itH}|z_R^+\rangle = e^{-itE_R}\, e^{-t\Gamma/2}|z_R^+\rangle\,, \text{if } \; t>0\,; \quad e^{-itH}|z_R^{*-}\rangle = e^{-itE_R}\, e^{+t\Gamma/2}|z_R^{*-}\rangle\,, \text{if } \; t<0\,,$$

so that $|z_R^+\rangle$ and $|z_R^{*-}\rangle$ are the decaying and growing Gamow vector respectively. These vectors can be written in integral form as[3] [1, 10]

$$|z_R^+\rangle = \sqrt{\frac{\Gamma}{2\pi}}\int_{-\infty}^\infty \frac{|E^+\rangle\, dE}{(E-E_R)+i\Gamma/2}\,, \qquad |z_R^{*-}\rangle = \sqrt{\frac{\Gamma}{2\pi}}\int_{-\infty}^\infty \frac{|E^-\rangle\, dE}{(E-E_R)-i\Gamma/2}\,. \tag{5}$$

As Gamow vectors are functionals, equations (5) should be understood as

$$\langle\varphi^+|z_R^+\rangle = \sqrt{\frac{\Gamma}{2\pi}}\int_{-\infty}^\infty \frac{\langle\varphi^+|E^+\rangle\, dE}{(E-E_R)+i\Gamma/2}\,, \qquad \langle\psi^-|z_R^{*-}\rangle = \sqrt{\frac{\Gamma}{2\pi}}\int_{-\infty}^\infty \frac{\langle\psi^-|E^-\rangle\, dE}{(E-E_R)-i\Gamma/2}\,, \tag{6}$$

for all $\varphi^+ \in \mathbf{\Phi}^+$ and $\psi^- \in \mathbf{\Phi}^-$.

Note that the integration goes from $-\infty$ to ∞ and this is possible because $\langle\varphi^+|E^+\rangle$ and $\langle\psi^-|E^-\rangle$ are Hardy functions [2, 3]. Gamow vectors $|z_R^+\rangle$ and $|z_R^{*-}\rangle$ are **not** in the Hilbert space $\mathcal{H}$. The complex conjugate of (4) gives:

[3] However, we are not strictly using the notation in [1, 10, 2] here. The signs for the vectors are changed while the signs for the spaces remain the same. Then, we use the same signs for spaces and their vectors and, with our choice for these signs, we endorse the intuitive idea of using sign plus for the future and sign minus for the past.

$$\langle\psi^-|\varphi^+\rangle = -2\pi i s_R^* \langle\psi^-|z_R^{*-}\rangle\langle z_R^+|\varphi^+\rangle - \int_{-\infty}^0 \langle\psi^-|E^-\rangle\, S_{II}^*(E)\, \langle E^+|\varphi^+\rangle\, dE\,. \tag{7}$$

Finally, if we omit in (4) the arbitrary vectors φ^+ and ψ^-, we obtain:

$$I = 2\pi i s_R |z_R^+\rangle\langle z_R^{*-}| - \int_{-\infty}^0 |E^+\rangle\, S_{II}(E)\, \langle E^-|\, dE\,. \tag{8}$$

This is the identity operator from $\mathbf{\Phi}^-$ into $(\mathbf{\Phi}^+)^\times$. We call $\mathcal{L}(\mathbf{\Phi}^-,(\mathbf{\Phi}^+)^\times)$ the space of linear continuous operators from $\mathbf{\Phi}^-$ into $(\mathbf{\Phi}^+)^\times$. Then, one can show [20, 21] that both terms in the right hand side of (8) belong to $\mathcal{L}(\mathbf{\Phi}^-,(\mathbf{\Phi}^+)^\times)$. In Section 3, we present a proof that the first term in the right hand side of (8) is continuous (it is obviously linear). Then, the second one must be continuous as the identity always is.

Analogously, from (7) one can get

$$I = -2\pi i s_R^* |z_R^{*-}\rangle\langle z_R^+| - \int_{-\infty}^0 |E^-\rangle\, S_{II}^*(E)\, \langle E^+|\, dE\,, \tag{9}$$

which is in $\mathcal{L}(\mathbf{\Phi}^+,(\mathbf{\Phi}^-)^\times)$ as well as both terms in the right hand side of (9). Note that both identities in (8) and (9) are different.

Operators of the form $|z_R^+\rangle\langle z_R^{*-}|$ and $|z_R^{*-}\rangle\langle z_R^+|$ are called **Gamow dyads** and were considered in [21] as a first attempt to construct a Liouvillian formalism for resonances.

In addition to (8) and (9), we have the following expressions for all integer n [20]:

$$H^n = 2\pi i s_R z_R^n\, |z_R^+\rangle\langle z_R^{*-}| - \int_{-\infty}^0 E^n\, |E^+\rangle\, S_{II}(E)\, \langle E^-|\, dE\,, \tag{10}$$

with $H^n \in \mathcal{L}(\mathbf{\Phi}^-,(\mathbf{\Phi}^+)^\times)$ and

$$H^n = -2\pi i s_R^* (z_R^*)^n\, |z_R^{*-}\rangle\langle z_R^+| - \int_{-\infty}^0 E^n\, |E^-\rangle\, S_{II}^*(E)\, \langle E^+|\, dE\,, \tag{11}$$

with $H^n \in \mathcal{L}(\mathbf{\Phi}^+,(\mathbf{\Phi}^-)^\times)$. See Section 3.

So far, the notion of Gamow dyad for simple pole resonances. This notion can be extended to multiple pole resonances [20, 21] and the generalization of (10) and (11) will be given in Section 3. The integral term in the above formulas is called the background term. This term is responsible for the non exponential decay of state vectors in Hilbert space [22] and it is always present.

This communication is organized as follows: In the next section, we shall define approximate Gamow vectors and give their properties and their relation with Gamow vectors. In Section 3, we discuss some properties of Gamow dyads. Finally, we give some concluding remarks.

2. Approximate Gamow vectors

Although the absolutely continuous spectrum of H is the positive semiaxis, the construction of Gamow vector given formula (5) implies the integration on the whole real line. This is possible because the space of test functions here considered is given by a space of Hardy functions. This integration over the whole real axis is necessary to preserve the exponential decay, which is equivalent to the Breit-Wigner energy distribution. However, as the continuous spectrum of H coincides with $[0,\infty)$, we propose to study the effect that a truncated Breit-Wigner energy distribution may have in the definition of the Gamow vector. This motivates the use of the notion of approximate Gamow vector, which can be introduced by means of the following formula:

$$f^D := \sqrt{\frac{\Gamma}{2\pi}} \int_0^\infty \frac{|E^+\rangle}{E - E_R + (i\Gamma)/2}\, dE\,, \tag{12}$$

where $H|E^+\rangle = E|E^+\rangle$. In fact, f^D is a functional on $\mathbf{\Phi}^+$ defined by its action on an arbitrary $\phi^+ \in \mathbf{\Phi}^+$ as

$$\langle \phi^+|f^D\rangle := \sqrt{\frac{\Gamma}{2\pi}} \int_0^\infty \frac{\langle \phi^+|E^+\rangle}{E - E_R + (i\Gamma)/2}\, dE\,. \tag{13}$$

The integral in (13) is well defined because the numerator is a Schwartz function and $(E - E_R + (i\Gamma)/2)^{-1}$ is a square integrable function in the variable E. The use of the Schwarz inequality shows that f^D is a continuous functional. In fact, it is obviously antilinear and

$$\begin{aligned} |\langle \phi^+|f^D\rangle| &\le \sqrt{\frac{\Gamma}{2\pi}} \left\{ \int_0^\infty \frac{1}{|E - E_R + (i\Gamma)/2|^2}\, dE \right\}^{1/2} \left\{ \int_0^\infty |\langle \phi^+|E^+\rangle|^2\, dE \right\}^{1/2} \\ &= C \left\{ \int_{-\infty}^\infty |\langle \phi^+|E^+\rangle|^2\, dE \right\}^{1/2} = C\, ||\langle \phi^+|E^+\rangle||_{L^2(\mathbb{R})}\,. \end{aligned} \tag{14}$$

Since the norm in $L^2(\mathbb{R})$ is one of the seminorms that define the topology on $\mathbf{\Phi}^+$ [2, 3], we conclude that f^D is a continuous antilinear functional on $\mathbf{\Phi}^+$ and hence it belongs to the antidual, $f^D \in (\mathbf{\Phi}^+)^\times$.

Gamow vectors are not normalizable and therefore they do not belong to Hilbert space. However, our approximate decaying Gamow vectors are normalizable and henceforth vectors in the Hilbert space as we can readily show:

$$\begin{aligned} ||f^D||^2 = \langle f^D|f^D\rangle &= \frac{\Gamma}{2\pi} \int_0^\infty \int_0^\infty \frac{\langle E'|E\rangle\, dE'\, dE}{(E' - E_R - (i\Gamma)/2)(E - E_R + (i\Gamma)/2)} \\ &= \frac{\Gamma}{2\pi} \int_0^\infty \int_0^\infty \frac{\delta(E - E')\, dE'\, dE}{(E' - E_R - (i\Gamma)/2)(E - E_R + (i\Gamma)/2)} \\ &= \frac{\Gamma}{2\pi} \int_0^\infty \frac{dE}{(E - E_R)^2 + \Gamma^2/4} = \frac{\Gamma}{2\pi} \int_0^\infty \frac{dE/(\Gamma/2)^2}{\left[\frac{E - E_R}{\Gamma/2}\right]^2 + 1}\,. \end{aligned} \tag{15}$$

With the change of variable

$$x = \frac{2(E - E_R)}{\Gamma} \Longrightarrow dx = \frac{2dE}{\Gamma}\,, \tag{16}$$

the last integral in (15) becomes

$$\frac{1}{\pi} \int_{-2E_R/\Gamma}^\infty \frac{dx}{x^2 + 1} = \frac{1}{\pi} \left(\frac{\pi}{2} + \arctan\left(\frac{2E_R}{\Gamma} \right) \right)\,, \tag{17}$$

which is a positive number smaller than one. Note that

$$\lim_{2E_R/\Gamma \mapsto \infty} ||f^D|| = 1\,. \tag{18}$$

In the sequel, we study some interesting properties of the approximate decaying Gamow vector f^D.

Proposition.- In the weak topology on $(\mathbf{\Phi}^+)^\times$, we have that:

$$\lim_{2E_R/\Gamma \mapsto \infty} f^D = i(2\pi\Gamma)^{1/2}\, |z_R^+\rangle\,, \tag{19}$$

where $|z_R^+\rangle$ is the ordinary decaying Gamow vector (5).

Proof.- For brevity, let us call $\psi^D := i(2\pi\Gamma)^{1/2}\,|z_R^+\rangle$. According to the definition of the weak topology, proving (19) is equivalent to show that

$$p_\varphi(f^D - \psi^D) := |\langle\varphi^+|f^D\rangle - \langle\varphi^+|\psi^G\rangle| \longmapsto 0 \qquad \text{as} \qquad \frac{2E_R}{\Gamma} \longmapsto \infty\,, \tag{20}$$

for any $\varphi^+ \in \mathbf{\Phi}^+$. Thus, ($z_R = E_R - i\Gamma/2$ is the position of the resonance pole)

$$p_\varphi(f^D - \psi^D) = \left| \frac{\Gamma}{\sqrt{2\pi}} \int_0^\infty \frac{\langle\varphi^+|E^+\rangle\, dE}{E - z_R} - \frac{i(2\pi)^{1/2}\Gamma}{2\pi i} \int_{-\infty}^\infty \frac{\langle\varphi^+|E^+\rangle\, dE}{E - z_R} \right|$$

$$= \frac{\Gamma}{\sqrt{2\pi}} \left| \int_{-\infty}^0 \frac{\langle\varphi^+|E^+\rangle\, dE}{E - z_R} \right| \le \frac{\Gamma}{\sqrt{2\pi}} \left\{ \int_{-\infty}^0 |\langle\varphi^+|E^+\rangle|^2\, dE \right\}^{1/2} \left\{ \int_{-\infty}^0 \frac{dE}{|E - z_R|^2} \right\}^{1/2} . \tag{21}$$

This last integral can be easily evaluated:

$$\int_{-\infty}^0 \frac{dE}{|E - z_R|^2} = \int_{-\infty}^0 \frac{dE}{(E - E_R)^2 + (\Gamma/2)^2} = \int_{-\infty}^0 \frac{dE/(\Gamma/2)^2}{\left[\frac{E - E_R}{\Gamma/2}\right]^2 + 1} . \tag{22}$$

With the change of variable given by (2.5), the latter integral yields to

$$\frac{2}{\Gamma} \int_{-\infty}^{-2E_R/\Gamma} \frac{dx}{x^2 + 1} . \tag{23}$$

Thus,

$$p_\varphi(f^D - \psi^D) \le \frac{1}{\pi} \left\{ \int_{-\infty}^0 |\langle\varphi^+|E^+\rangle|^2\, dE \right\}^{1/2} \int_{-\infty}^{-2E_R/\Gamma} \frac{dx}{x^2 + 1} . \tag{24}$$

The integral between brackets is constant for fixed φ^+. The second integral goes to zero as $2E_R/\Gamma \longmapsto \infty$. This proves our result.

We conclude that the approximate decaying Gamow vector, that is a regular Hilbert space vector, is a reasonable approximation to the decaying Gamow vector either when the resonance energy is high or when the width is very narrow. In this cases, the decaying Gamow vector can be very well, within a reasonable accuracy, replaced by the approximate Gamow vector. These are the good news.

Now, they come the bad news. One of the problems that Gamow vectors have to represent truly state vectors is the difficulty to define a mean value of the energy. It was Berggren who first tried to give a solution to this problem [23], which is also discussed in [24, 3]. The bad news are that we find the same difficulty when dealing with approximate decaying Gamow vectors, notwithstanding these are Hilbert space vectors. The reason is that f^D does not belong to the domain of the total Hamiltonian H. The proof of this statement is quite simple:

$$Hf^D = \sqrt{\frac{\Gamma}{2\pi}} \int_0^\infty \frac{H|E^+\rangle}{E - E_R + (i\Gamma)/2}\, dE = \sqrt{\frac{\Gamma}{2\pi}} \int_0^\infty \frac{E|E^+\rangle}{E - E_R + (i\Gamma)/2}\, dE\,, \tag{25}$$

Then, taking the square norm, we get:

$$||Hf^D||^2 = \frac{\Gamma}{2\pi} \int_0^\infty \frac{E^2\, dE}{(E - E_R)^2 + \Gamma^2/4} = \infty\,, \tag{26}$$

so that Hf^D is not a vector in Hilbert space. In this circumstances, we cannot have a clear definition of $\langle f^D|Hf^D\rangle$. Even if we pose:

$$\langle f^D|Hf^D\rangle := \frac{\Gamma}{2\pi} \int_0^\infty \int_0^\infty \frac{E\langle E'|E\rangle\, dE\, dE'}{(E' - E_R - i\Gamma/2)(E - E_R + i\Gamma/2)} = \frac{\Gamma}{2\pi} \int_0^\infty \frac{E\, dE}{(E - E_R)^2 + \Gamma^2/4} \tag{27}$$

and perform the change of variable given by (16), we get:

$$\langle f^D|Hf^D\rangle = \frac{E_R}{\pi} \int_{-2E_R/\Gamma}^\infty \frac{dx}{x^2+1} + \frac{\Gamma}{2\pi} \int_{-2E_R/\Gamma}^\infty \frac{x\, dx}{x^2+1}\,. \tag{28}$$

The second integral in (28) diverges. In the limit $2E_R/\Gamma \longmapsto \infty$, this second integral has a vanishing Cauchy principal value. In this limit, the first integral is equal to π and this gives $\langle f^D|Hf^D\rangle \longmapsto E_R$, a result already obtained in [24].

Needless to say that a similar definition can be given for the approximate growing Gamow vector f^G. Its properties are similar to f^D. In addition, if T is the time reversal operator, one can show that $T|E^\pm\rangle = |E^\mp\rangle$ [25] and therefore, $Tf^D = f^G$ and $Tf^G = f^D$.

Finally, note that f^D cannot decay exponentially as is a vector in Hilbert space. The decaying formula for f^D is

$$\langle f^D|e^{-itH} f^D\rangle = \frac{\Gamma}{2\pi} \int_0^\infty \frac{e^{-itE_R}\, dE}{(E - E_R)^2 + \Gamma^2/4}\,. \tag{29}$$

Again, if we use the change of variables (16) and take the limit $2E_R/\Gamma \longmapsto \infty$ in (29), we obtain $e^{-itE_R}\, e^{-t\Gamma/2}$, by using the properties of the Fourier transform. This decaying behavior coincides with the decaying mode of $|z_R^+\rangle$, an expected result.

3. Gamow dyads: some properties

The use of Gamow dyads for multiple pole resonances suggest us the convenience of changing the notation used so far for Gamow vectors and that has its origin in the work of Arno Bohm [10]. Although this notation can be generalized for multiple pole resonances [27], we believe that a new one is simpler and even more practical. Thus, we write:

$$\sqrt{2\pi i s_R}\, |z_R^+\rangle = |f_0\rangle\,, \qquad \sqrt{-2\pi i s_R^*}\, |z_R^{*+}\rangle = |\widetilde{f}_0\rangle\,. \tag{30}$$

This notation has been already used in a certain number of publications, see for instance [20, 21, 26, 3]. If the resonance is described by a pole of order n, then, there are n decaying Gamow vectors, $|f_0\rangle, |f_1\rangle, \dots |f_{n-1}\rangle$ and n growing Gamow vectors, $|\widetilde{f}_0\rangle, |\widetilde{f}_1\rangle, \dots |\widetilde{f}_{n-1}\rangle$. Only $|f_0\rangle$ decays exponentially and only $|\widetilde{f}_0\rangle$ grows exponentially. The exponential decay of $|\widetilde{f}_1\rangle, \dots |\widetilde{f}_{n-1}\rangle$ as well as the exponential grow of $|\widetilde{f}_1\rangle, \dots |\widetilde{f}_{n-1}\rangle$ is modified by a multiplicative polynomial on time [27, 20, 26, 3].

The study of properties of multiple pole resonances is mostly due to A. Mondragón and his group [28]. They use the name of *degenerate resonances* instead of multiple pole resonances.

In the case of the presence of a multiple pole resonance, with resonance poles at z_R and z_R^* with multiplicity equal to p (both complex conjugate resonance poles must have the same multiplicity), formulas (10) and (11) can be easily generalized

$$H^n = \sum_{k=0}^{p-1} z_R^n |f_k\rangle\langle\widetilde{f}_k| + \sum_{s=1}^{p-2} |f_s\rangle\langle\widetilde{f}_{s+1}| - \int_{-\infty}^{0} E^n S_{II}(E)|E^+\rangle\langle E^-|\, dE \tag{31}$$

and

$$H^n = \sum_{k=0}^{p-1} (z_R^*)^n |\widetilde{f}_k\rangle\langle f_k| + \sum_{s=1}^{p-2} |\widetilde{f}_s\rangle\langle f_{s+1}| - \int_{-\infty}^{0} E^n S^*_{II}(E)|E^-\rangle\langle E^+|\, dE\,, \tag{32}$$

respectively, where n is a natural number. Exactly as in the case of a simple pole resonance, the operator in (31) is in $\mathcal{L}(\mathbf{\Phi}^-, (\mathbf{\Phi}^+)^\times)$ and the operator in (32) is in $\mathcal{L}(\mathbf{\Phi}^+, (\mathbf{\Phi}^-)^\times)$, as we shall show in the last theorem of this section. Note that in both cases the background integral remains the same.

In order to abbreviate the notation, we shall use

$$|w\rangle := \sqrt{-S_{II}(w)}\, |w^+\rangle\,, \qquad |\widetilde{w}\rangle := \sqrt{-S^*_{II}(w)}\, |w^-\rangle\,, \qquad w \equiv E\,, \tag{33}$$

so that the background terms are written as

$$\int_{-\infty}^{0} w\, |w\rangle\langle\widetilde{w}|\, dw\,, \qquad \int_{-\infty}^{0} w\, |\widetilde{w}\rangle\langle w|\, dw\,, \tag{34}$$

for (31) and (32) respectively.

The background term cannot be eliminated and this is a consequence of our next result.

Theorem.- We cannot find $f(w) \in L^p(\mathbb{R}^-)$ with $1 \leq p \leq \infty$, $\mathbb{R}^- \equiv (-\infty, 0]$, nor $f(w)$ equal to a polynomial such that

$$|f_0\rangle\langle\widetilde{f}_0| = \int_{-\infty}^{0} f(w)\, |w\rangle\langle\widetilde{w}|\, dw\,. \tag{35}$$

Proof.- Pick a $\psi^- \in \mathbf{\Phi}^-$ such that $\langle\psi^-|\widetilde{f}_0\rangle \neq 0$. Let φ^+ arbitrary in $\mathbf{\Phi}^+$. Then, using (33) and omitting the irrelevant sign minus, we have

$$\langle\varphi^+|f_0\rangle\langle\widetilde{f}_0|\psi^-\rangle = \int_{-\infty}^{0} f(w) S_{II}(w)\, \langle\varphi^+|w^+\rangle\langle w^-|\psi^-\rangle\, dw\,. \tag{36}$$

The functions $\langle\varphi^+|w^+\rangle$ and $\langle w^-|\psi^-\rangle$ are in the Schwartz space by hypothesis [2, 3]. Also, by hypothesis, the function $S_{II}(w)$ is polynomially bounded at the infinite. This means that for large $|w|$, there exists a polynomial $P(w)$ such that $|S_{II}(w)| \leq |P(w)|$ [2, 3]. Then, if $f(w)$ is either in $L^p(\mathbb{R}^-)$ or is a polynomial, the function $g(w) := f(w)S_{II}(w)\langle w^-|\psi^-\rangle \in L^2(\mathbb{R}^-)$, i.e. it is square integrable in the negative semiaxis. For simplicity, let us write $\varphi^+(w) := \langle w^+|\varphi^+\rangle = \langle\varphi^+|w^+\rangle^*$. Then, (36) and the Schwarz inequality give

$$|\langle\varphi^+|f_0\rangle|\, |\langle\widetilde{f}_0|\psi^-\rangle| \leq \left| \int_{-\infty}^{0} [\varphi^+(w)]^*\, g(w)\, dw \right| \leq ||g(w)||_-\, ||\varphi^+(w)||_-\,, \tag{37}$$

where $||-||_-$ denotes the norm in $L^2(\mathbb{R}^-)$.

We recall that $\langle\varphi^+|f_0\rangle$ is defined as follows: if $\varphi^{\#}(w) := [\varphi^+(w)]^*$ for real w, then $\langle\varphi^+|f_0\rangle = \varphi^{\#}(z_R)$. Thus, (37) shows that for any $\varphi^+ \in \mathbf{\Phi}^+$, there exists a positive constant such that

$$|\varphi^{\#}(z_R)| = |\langle\varphi^+|f_0\rangle| \leq C\, ||\varphi^+(w)||_- \, . \tag{38}$$

According to our hypothesis, $\varphi^+(w) \in \mathcal{H}^2_+ \cap \mathcal{S}$, where $\mathcal{H}^2_+$ is the space of Hardy functions in the lower half plane [29] and $\mathcal{S}$ is the Schwartz space [2, 3]. This means in particular that $\varphi^+(w)$ is defined for all real values of w. We have shown in [2] that the Gamow vector $|f_0\rangle$ cannot be a continuous functional on $L^2(\mathbb{R}^+)$, so that the inequality $|\varphi^{\#}(z_R)| = |\langle\varphi^+|f_0\rangle| \leq C\,||\varphi^+(w)||_+$ for all $\varphi^+ \in \mathcal{H}^2_+ \cap \mathcal{S}$ is false, where $|| - ||_+$ denotes the norm on $L^2(\mathbb{R}^+)$. A similar argument shows that (38) is also false and this contradicts the hypothesis (35). Our result is proven. ♠

Corollary 1.- In the case of multiple pole resonances, one can use a similar argument to show that under the hypothesis of the theorem, the identity

$$|f_k\rangle\langle\widetilde{f}_k| = \int_{-\infty}^0 f(w)\,|w\rangle\langle\widetilde{w}|\,dw \tag{39}$$

is also false for natural k.

Corollary 2.- This result can also be extended to any linear combination of dyads of the form $|f_k\rangle\langle\widetilde{f}_k|$.

For simplicity, let us prove this result for the simplest case $|f_0\rangle\langle\widetilde{f}_0| + |f_1\rangle\langle\widetilde{f}_1|$. The proof for any other linear combinations of dyads is similar. We need to show that the relation

$$\langle\varphi^+|f_0\rangle\langle\widetilde{f}_0|\psi^-\rangle + \langle\varphi^+|f_1\rangle\langle\widetilde{f}_1|\psi^-\rangle = \int_{-\infty}^0 f(w)S_{II}(w)\,\langle\varphi^+|w^+\rangle\langle w^-|\psi^-\rangle\,dw \tag{40}$$

is impossible for arbitrary $\varphi^+ \in \mathbf{\Phi}^+$ and $\psi^- \in \mathbf{\Phi}^-$. This case can be obviously reduced to the previous one, provided that we find one $\psi^- \in \mathbf{\Phi}^-$ such that $\langle\widetilde{f}_0|\psi^-\rangle = 0$ and $\langle\widetilde{f}_1|\psi^-\rangle \neq 0$ and another with $\langle\widetilde{f}_0|\psi^-\rangle \neq 0$ and $\langle\widetilde{f}_1|\psi^-\rangle = 0$.

Note that for real w, $\psi^{\#}(w) := [\psi^-(w)]^*$ with $\psi^{\#}(w) \in \mathcal{H}^2_+ \cap \mathcal{S}$ and $\langle\varphi^+|f_0\rangle = \psi^{\#}(z_R^*)$ and $\langle\widetilde{f}_1|\psi^-\rangle = (\psi^{\#})'(z_R^*)$, where the prime means derivative. Thus,

i.) we have to find a $\psi^{\#}(w) \in \mathcal{H}^2_+ \cap \mathcal{S}$ such that $\psi^{\#}(z_R^*) = 0$ and $(\psi^{\#})'(z_R^*) \neq 0$ and

ii.) we have to find a $\psi^{\#}(w) \in \mathcal{H}^2_+ \cap \mathcal{S}$ such that $\psi^{\#}(z_R^*) \neq 0$ and $(\psi^{\#})'(z_R^*) = 0$.

Note that there are always functions $\varphi^+(w) \in \mathcal{H}^2_+ \cap \mathcal{S}$ such that $\varphi^+(z_R^*) \neq 0$. If this were false and all $\varphi^+(w) \in \mathcal{H}^2_+ \cap \mathcal{S}$ fulfilled $\varphi^+(z_R^*) = 0$, by the Titchmarsh theorem we would have:

$$\varphi^+(z_R^*) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi^+(w)\,dw}{E - z_R^*} = 0\, . \tag{41}$$

This means that the arbitrary function $\varphi^{\#}(w) = [\varphi^+(w)]^* \in \mathcal{H}^2_- \cap \mathcal{S}$ is orthogonal in $L^2(\mathbb{R})$ to $(E - z_R^*)^{-1}$. Since $\mathcal{H}^2_- \cap \mathcal{S}$ is dense in $\mathcal{H}^2_-$ and $L^2(\mathbb{R}) = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$ [29] this implies that $(E - z_R^*)^{-1} \in \mathcal{H}^2_+$, which is false because $(E - z_R^*)^{-1}$ is not analytic on the open upper half plane. Then,

i.) Take $\eta(w) \in \mathcal{H}^2_+ \cap \mathcal{S}$ with $\eta(z_R^*) \neq 0$ and define $\psi^{\#}(w) := (w - z_R^*)\,\eta(w)$. Clearly, $\psi^{\#}(z_R^*) = 0$ and $(\psi^{\#})'(z_R^*) = \eta(z_R^*) \neq 0$.

ii.) Take now $\psi^{\#}(w) := (w - z_R^*)\,\eta'(w) - \eta(w)$, with the same $\eta(w)$ as in i.). Then, $\psi^{\#}(z_R^*) = -\eta(z_R^*) \neq 0$ and $(\psi^{\#})'(z_R^*) = 0$.

Thus, corollary 2 is proven.

Remark.- The proof of the above theorem applies also to those $f(w)$ for which $f(w)\,S_{II}(w)\,\langle w^-|\psi^-\rangle \in L^2(\mathbb{R}^-)$. This include, in particular, an $f(w)$ continuous and polynomially bounded at the infinite. In addition, if $f(w)=\delta(w-\alpha)$ with α real, our result is also true. A similar result can be obtained for dyads of the form $|\widetilde{f_k}\rangle\langle f_k|$.

The conclusion of the above results is clear, we cannot avoid the background term, which can not be reduced to a Gamow of a linear combination of Gamows. This shows that the integral term cannot be eliminated in the expressions for H^n in (31) and (32). This remark includes also the integral term in the evolution operator. The explicit for for this operator in the decaying case ($t>0$) and multiple pole resonance is

$$U_t=\int_{-\infty}^{0} e^{-iwt}\,|w\rangle\langle\widetilde{w}|\,dw+e^{-itz_R}\left[\sum_{k=0}^{n-1}|f_k\rangle\langle\widetilde{f_k}|\right.$$

$$\left.-it|f_1\rangle\langle\widetilde{f_2}|+\frac{(-it)^2}{2}\,|f_2\rangle\langle\widetilde{f_3}|+\dots\frac{(-it)^{n-2}}{(n-2)!}\,|f_{n-2}\rangle\langle\widetilde{f}_{n-1}|\right]. \tag{42}$$

This operator belongs to $\mathcal{L}(\mathbf{\Phi}^-,(\mathbf{\Phi}^+)^\times)$ for $t>0$. In the case of a simple pole resonance, only the term with $k=0$ survives from (42). For the growing part a similar formula holds, which is time reversal of (42) [30] and is an operator in $\mathcal{L}(\mathbf{\Phi}^+,(\mathbf{\Phi}^-)^\times)$. Note that the function $f(w)$ in the integral term is of the type mentioned in the remark as it has modulus equal to one.

Next, we are giving a result concerning some mathematical properties of the Gamow dyads.

Theorem.- **1**.- The operator $\left\{\begin{array}{c}|f_k\rangle\langle\widetilde{f_k}|\\ |\widetilde{f_k}\rangle\langle f_k|\end{array}\right\}$ belongs to $\left\{\begin{array}{c}\mathcal{L}(\mathbf{\Phi}^-,(\mathbf{\Phi}^+)^\times)\\ \mathcal{L}(\mathbf{\Phi}^+,(\mathbf{\Phi}^-)^\times)\end{array}\right\}$, for $k=0,1,\dots,n-1$ (assuming that the resonance pole is of order n), although it cannot be extended to an operator in $\left\{\begin{array}{c}\mathcal{L}(\mathbf{\Phi}^+)^\times\\ \mathcal{L}(\mathbf{\Phi}^-)^\times\end{array}\right\}$, which are the spaces of linear operators on $\left\{\begin{array}{c}(\mathbf{\Phi}^+)^\times\\ (\mathbf{\Phi}^-)^\times\end{array}\right\}$ continuous with the weak topology.

2.- The operator $\int_{-\infty}^0 e^{-itw}\,|w\rangle\langle\widetilde{w}|\,dw$ is in $\mathcal{L}(\mathbf{\Phi}^-,(\mathbf{\Phi}^+)^\times)$ for **all** values of time t. Same is true for $\int_{-\infty}^0 e^{-itw}\,|\widetilde{w}\rangle\langle\ w|\,dw$ and $\mathcal{L}(\mathbf{\Phi}^+,(\mathbf{\Phi}^-)^\times)$.

3.- The operators H^n in (10) and (31) belong to $\mathcal{L}(\mathbf{\Phi}^-,(\mathbf{\Phi}^+)^\times)$ and the operators H^n in (11) and (32) belong to $\mathcal{L}(\mathbf{\Phi}^+,(\mathbf{\Phi}^-)^\times)$ for all $n=0,1,2,\dots$.

Proof.- **1**.- We present the proof for $|f_0\rangle\langle\widetilde{f_0}|$ only, because the proof for other dyads is similar. We recall that a linear operator $B:\mathbf{\Phi}^-\longrightarrow(\mathbf{\Phi}^+)^\times$ is continuous if and only if for any continuous seminorm p on $(\mathbf{\Phi}^+)^\times$, there exists a constant C and n seminorms on $\mathbf{\Phi}^-$, $p_1,p_2,\dots,p_n$ such that

$$p(B\psi^-)\le C\{p_1(\psi^-)+p_2(\psi^-)+\dots+p_n(\psi^-)\}\,. \tag{43}$$

Take an arbitrary $\psi^-\in\mathbf{\Phi}^-$ and consider its image by $|f_0\rangle\langle\widetilde{f_0}|$, $|f_0\rangle\langle\widetilde{f_0}|\psi^-\rangle\in(\mathbf{\Phi}^+)^\times$. Then, (43) reads here as

$$p(|f_0\rangle\langle\widetilde{f_0}|\psi^-\rangle)\le C\{p_1(\psi^-)+p_2(\psi^-)+\dots+p_n(\psi^-)\}\,. \tag{44}$$

We have to show that the inequality (44) holds. Since $(\mathbf{\Phi}^-)^\times$ is endowed with the weak topology with respect to the dual pair $(\mathbf{\Phi}^-,(\mathbf{\Phi}^-)^\times)$, which in our case coincides with the strong topology because $(\mathbf{\Phi}^-)^\times$ is the dual of a nuclear Fréchet space, there is a $\eta^-\in\mathbf{\Phi}^-$ such that $p(\psi^-)=|\langle\eta^-|\psi^-\rangle|$, for all $\psi^-\in\mathbf{\Phi}^-$. Thus,

$$p(|f_0\rangle\langle\widetilde{f_0}|\psi^-\rangle)=|\langle\eta^-|f_0\rangle|\,|\langle\widetilde{f_0}|\psi^-\rangle|\,. \tag{45}$$

Since $|\langle\eta^-|f_0\rangle| = |\eta^\#(z_R^*)|$ is independent of ψ^-, (44) and (45) implies that $|f_0\rangle\langle\widetilde{f_0}|$ is continuous if and only if

$$|\langle\widetilde{f_0}|\psi^-\rangle| \leq C'\{p_1(\psi^-)+p_2(\psi^-)+\ldots+p_n(\psi^-)\}\,, \tag{46}$$

where C' is a constant. Then, carry (46) into (45) and (44) is proven. Thus, the continuity of $|\widetilde{f_0}\rangle$ on $\mathbf{\Phi}^-$ shows the existence of the constant C' and the seminorms $p_1, p_2, \ldots, p_n$, so that the continuity of $|f_0\rangle\langle\widetilde{f_0}|$ is proven.

To show that $|f_0\rangle\langle\widetilde{f_0}|$ cannot be in $\mathcal{L}(\mathbf{\Phi}^-)^\times$, we assume the contrary. Then, (44) and hence (46) would be true being this time $p_1, p_2, \ldots, p_n$ continuous seminorms on $(\mathbf{\Phi}^-)^\times$ (note that $\mathbf{\Phi}^- \subset (\mathbf{\Phi}^-)^\times$). If this were the case, the mapping $\psi^- \longmapsto \langle\widetilde{f_0}|\psi^-\rangle$ would have been continuous even when $\mathbf{\Phi}^-$ has the topology inherited from $(\mathbf{\Phi}^-)^\times$, which is false, since this mapping is not continuous when $\mathbf{\Phi}^-$ has the stronger Hilbert space topology [2].

Remark.- Note that this means in particular that objects like $\langle f_0|f_0\rangle$ and $\langle\widetilde{f_0}|\widetilde{f_0}\rangle$ cannot be defined in principle. At least, no by means of continuous extensions. These objects have been proposed in the literature (see [31] and references therein), although they have a doubtful sense.

2.- The technique for proving this second part is essentially the same of what we have applied in the former. Only that we have to take into account that the function $S_{II}(w)$ is bounded on the negative semiaxis of the second Riemann sheet [5]. Then, if p is a seminorm on $(\mathbf{\Phi}^+)^\times$ and $\eta^+ \in \mathbf{\Phi}^+$ with $p(\varphi^+) = |\langle\eta^+|\varphi^+\rangle|$, for all $\varphi^+ \in \mathbf{\Phi}^+$, we have

$$p\left(\int_{-\infty}^0 e^{-itw}\,|w\rangle\langle\widetilde{w}|\psi^-\rangle\,dw\right) = \left|\int_{-\infty}^0 e^{-itw}\,\langle\eta^+|w\rangle\langle\widetilde{w}|\psi^-\rangle\,dw\right|$$

$$\leq \int_{-\infty}^0 |\langle\eta^+|w\rangle|\,|\langle\widetilde{w}|\psi^-\rangle|\,dw = \int_{-\infty}^0 |S_{II}(w)|\,|\langle\eta^+|w^+\rangle|\,|\langle w^-|\psi^-\rangle|\,dw$$

$$\leq C\int_{-\infty}^0 |\langle\eta^+|w^+\rangle|\,|\langle w^-|\psi^-\rangle|\,dw \leq C\int_{-\infty}^\infty |\langle\eta^+|w^+\rangle|\,|\langle w^-|\psi^-\rangle|\,dw$$

$$\leq C\,||\eta^+||\,||\psi^-||\,, \tag{47}$$

where $||-||$ is the $L^2(\mathbb{R})$ norm. This proves the second part. We want to stress that this functional is well defined for *all* values of t and not only for $t > 0$.

3.- For $n = 0$, $H^0 = I$ the identity, which is continuous because the topology in the antiduals is weaker than the topology on the Hilbert space, which is weaker than the topology on the test spaces $\mathbf{\Phi}^\pm$ (note that $\mathbf{\Phi}^\pm \subset \mathcal{H} \subset (\mathbf{\Phi}^\mp)^\times$, which are similar to rigged Hilbert spaces). Then, we discuss the nontrivial case $n = 1, 2, \ldots$. If we note that $f(w)S_{II}(w)\langle w^-|\psi^-\rangle \in L^2(\mathbb{R}^-)$, using a similar argument as in the second part of the present theorem, the result follows. The continuity of the operators H^n is proven in all cases.

Concluding remarks

This communication deals with two problems in relation with Gamow vectors. In the first case, we define the notion of approximate Gamow vectors by restricting the integration in the usual definition of Gamow vectors to the positive real axis. This makes sense from the point of view that the positive semiaxis is the continuous spectrum of the Hamiltonian and integration over the whole real line needs analytic continuation of wave functions in the energy representation. Then, when we integrate over the interval $[0, \infty)$, we obtain vectors in the Hilbert space, which in some sense approximate the Gamow vectors and for this reason are called the approximate Gamow vectors. Approximate and exact Gamow vectors coincide in the weak limit when E_R/Γ goes to $+\infty$, where E_R is the resonant energy and Γ the width.

In the second half, we study some mathematical properties of the so called Gamow dyads. In particular, we study the continuity of these objects as operators on a space of test vectors into a space of functionals. We also discuss the nontriviality of the background in the sense that it cannot be described as a Gamow dyad or a linear combination of Gamow dyads (in the case of multiple pole resonances) and consequently, it is something different from a resonance. This fact is well known, but this point of view is new.

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