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Semigroups of tomographic probabilities and quantum correlations

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Abstract. Semigroups of stochastic and bistochastic matrices constructed by means of spin tomograms or tomographic probabilities and their relations to the problem of Bell's inequalities and entanglement are reviewed. The probability determining the quantum state of spins and the probability densities determining the quantum states of particles with continuous variables are considered. Entropies for semigroups of stochastic and bistochastic matrices are studied, in view of both the Shannon information entropy and its generalization like Rényi entropy. Qubit portraits of qudit states are discussed in the connection with the problem of Bell's inequality violation for entangled states.

1. Introduction

The quantum state of a particle with spin j is determined by the probability distribution function $w(m, u)$, where $-j \leq m \leq j$ is the spin projection and u is unitary group $(2j + 1) \times (2j + 1)$ -matrix which can be also considered as matrix of irreducible representation of $SU(2)$ group with matrix elements depending on two Euler angles φ and θ identified with coordinates on a sphere S^2 [1, 2].

For two spins j_1 and j_2 , the quantum state is determined by the joint probability distribution $w(m_1, m_2, u)$ of random spin projections m_1 and m_2 depending also on the unitary $N \otimes N$ -matrix u , where $N = (2j_1 + 1)(2j_2 + 1)$. The matrix u can be taken as tensor product $u = u_1 \times u_2$ of two unitary matrices which can be considered as generic unitary transform matrices or as matrices of two irreducible representations of the $SU(2)$ group depending on two points on two spheres $S^2 \otimes S^2$ or two directions given by unit vectors \vec{n}_1 and \vec{n}_2 [3].

The physical meaning of the probability distribution $w(m, u)$ consists in the fact that it provides the probability to get the spin projection on the direction \vec{n} to be equal to m . The function $w(m_1, m_2, u)$ provides the probability to get the spin projections j_1 and j_2 on the directions \vec{n}_1 and \vec{n}_2 to be equal, respectively, to m_1 and m_2 .

The joint probability $w(m_1, m_2, \dots, m_N, u)$ of N qudit states, where m_1, m_2, \dots, m_N are spin j_k projections on corresponding directions \vec{n}_k ($k = 1, 2, \dots, N$) determines the multiparticle quantum state of the system. The approach using probabilities to determine quantum states is called the probability representation of quantum mechanics. It is equivalent to all other representations of quantum mechanics reviewed, for example, in [4].

Since in the probability representation the quantum states are identified with probability distributions, such representation provides natural consideration of the aspects related to such

probability characteristics of quantum correlations as the entanglement phenomenon [5] and Bell's inequalities [6–8]. The application of the approach in this context was suggested in [9–15]. One should point out that some probabilities associated to quantum states [16–18] were applied to study Bell's inequalities [6]. Understanding that Bell's inequalities are characteristics not only of quantum picture but also characteristics of standard classical probability distributions was emphasized in [12, 19, 20].

The stochastic and bistochastic matrices have columns which can be considered as probability vectors with nonnegative vector components equal to a probability of some random event. The properties of such matrices which form semigroup are given, for example, in [21]. In view of this, the probability representation of quantum states is naturally connected with stochastic maps realized by semigroups of stochastic matrices. It is worthy noting that semigroups are well-known ingredients in quantum theory of states of decaying particles [22].

The aim of this work is to review the probability approach to qudit states, both separable and entangled, and to relate Bell's inequalities [7, 8] to the properties of stochastic matrix semigroup following [11–13].

The paper is organized as follows.

In section 2, the probabilities determining the quantum states of qudits called tomographic probabilities (or tomograms) are constructed. In section 3, the properties of semigroups of stochastic matrices, including Shannon [23] and Rényi [24] entropies of the matrices, are presented. In section 4, Bell's inequalities [7, 8] are discussed and their connection with separability and entanglement of multipartite qudit states is considered. Conclusions and perspectives are given in section 5.

2. Spin-tomogram construction

The tomographic probability density $w(X, \mu, \nu) \geq 0$ determining the quantum state of a particle with density operator $\hat{\rho}$ reads [25]

$$w(X, \mu, \nu) = \text{Tr} \left[\hat{\rho} (X - \mu \hat{q} - \nu \hat{p}) \right], \quad (1)$$

where X is a random position, μ and ν are real parameters, and \hat{q} and \hat{p} are position and momentum operators. The tomographic probability of one qudit pure state $|\psi\rangle$ reads [1, 2, 26]

$$w(m, u) = \langle m | u^\dagger | \psi \rangle \langle \psi | u | m \rangle = \text{Tr} \hat{\rho}_\psi u | m \rangle \langle m | u^\dagger. \quad (2)$$

The tomographic probability of the pure state of two qudits is constructed analogously

$$w(m_1, m_2, u) = \left| \langle m_1 m_2 | u^\dagger | \psi \rangle \right|^2, \quad (3)$$

where matrix u belonging to unitary group $U(N)$ can be taken as matrix of the subgroup of unitary group given as tensor product of local transforms

$$U(N) = U(N_1) \otimes U(N_2), \quad (4)$$

with $N_1 = 2j_1 + 1$ and $N_2 = 2j_2 + 1$.

For arbitrary number \mathcal{K} of qudits, i.e., $N = \prod_{k=1}^{\mathcal{K}} N_k$ with $N_k = 2j_k + 1$, the state tomogram reads

$$w_\rho(m_1, m_2, \dots, m_{\mathcal{K}}, u) = \langle m_1, m_2, \dots, m_{\mathcal{K}} | u^\dagger \rho u | m_1, m_2, \dots, m_{\mathcal{K}} \rangle, \quad (5)$$

which for the pure state gives an analog of (3)

$$w_\psi(\vec{m}, u) = \left| \langle \vec{m} | u^\dagger | \psi \rangle \right|^2, \quad (6)$$

where we introduced the vector $\vec{m} = (m_1, m_2, \dots, m_K)$.

The probability is normalized for any matrix u

$$\sum_{\vec{m}} w_{\rho}(\vec{m}, u) = 1. \quad (7)$$

For example, the qubit state $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has the tomogram, which we associate with the probability vector

$$\vec{w}_+(u) = \begin{pmatrix} |u_{11}|^2 \\ |u_{22}|^2 \end{pmatrix}. \quad (8)$$

This means that the probability of the spin projection on the direction \vec{n} reads

$$w(+1/2, \vec{n}) = \cos^2 \theta/2, \quad w(-1/2, \vec{n}) = \sin^2 \theta/2. \quad (9)$$

The two qubit entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|+1/2, +1/2\rangle + |-1/2, -1/2\rangle) \quad (10)$$

is determined by tomogram which can be presented as the probability four-vector depending on matrix elements of 4×4 -unitary matrix

$$\vec{w}_{\psi}(u) = \frac{1}{2} \begin{pmatrix} |u_{11} + u_{41}|^2 \\ |u_{12} + u_{42}|^2 \\ |u_{13} + u_{43}|^2 \\ |u_{14} + u_{44}|^2 \end{pmatrix}. \quad (11)$$

For separable state, in the case of $u = u_1 \otimes u_2$, the tomographic probability vector can be presented in the form of convex sum of tensor-product vectors

$$\vec{w} = \sum_k p_k \vec{w}^{(k)}(u_1) \otimes \vec{w}^{(k)}(u_2). \quad (12)$$

If one takes the matrix u in the form of tensor-product of two 2×2 -unitary matrices u_1 and u_2 , vector (11) will be expressed in terms of probability of spin projections on the directions \vec{n}_1 and \vec{n}_2 given as follows:

$$\begin{aligned} w(+1/2, +1/2, u_1, u_2) &= \frac{1}{2} |u_{11} + u_{41}|^2, & w(+1/2, -1/2, u_1, u_2) &= \frac{1}{2} |u_{12} + u_{42}|^2, \\ w(-1/2, +1/2, u_1, u_2) &= \frac{1}{2} |u_{13} + u_{43}|^2, & w(-1/2, -1/2, u_1, u_2) &= \frac{1}{2} |u_{14} + u_{44}|^2, \end{aligned} \quad (13)$$

In the case of tensor product of 2×2 -matrices $u^{(1)}$ and $u^{(2)}$, one has the matrix elements of 4×4 -matrix u as follows:

$$\begin{aligned} u_{11} &= u_{11}^{(1)} u_{11}^{(2)}, & u_{12} &= u_{11}^{(1)} u_{12}^{(2)}, & u_{13} &= u_{12}^{(1)} u_{11}^{(2)}, & u_{14} &= u_{12}^{(1)} u_{12}^{(2)}, \\ u_{41} &= u_{21}^{(1)} u_{11}^{(2)}, & u_{42} &= u_{21}^{(1)} u_{12}^{(2)}, & u_{43} &= u_{22}^{(1)} u_{11}^{(2)}, & u_{44} &= u_{22}^{(1)} u_{12}^{(2)}. \end{aligned} \quad (14)$$

The tomographic-probability vectors can be transformed by means of matrices. For example, the qubit state (9) can evolve due to some interaction corresponding to a Hamiltonian which is Hermite 2×2 -matrix H providing the evolution of the state vector $|\psi\rangle$ as follows:

$$|\psi, t\rangle = \exp(-iHt) |\psi, 0\rangle. \quad (15)$$

The unitary matrix $u_H(t) = \exp(-iHt)$ gives the change of the tomographic-probability vector

$$w(m, u, 0) = \left| \langle m | u^\dagger | \psi, 0 \rangle \right|^2 \longrightarrow \left| \langle m | \left(u^\dagger u_H(t) \right) | \psi, 0 \rangle \right|^2 = w(m, u, t). \quad (16)$$

The time evolution of a quantum state provides a change of the tomographic probability vector of the quantum state. If the initial density matrix is diagonal, the evolution is realized by the matrix

$$M_{jk} = \left| \left(e^{-itH} \right)_{jk} \right|^2. \quad (17)$$

3. Stochastic and bistochastic matrices

Let us consider matrices M_{jk} with nonnegative matrix elements satisfying the condition

$$\sum_{j=1}^N M_{jk} = 1. \quad (18)$$

Such stochastic matrices form semigroup. The matrices M_{jk} with the property

$$\sum_{j=1}^N B_{jk} = \sum_{k=1}^N B_{jk} = 1 \quad (19)$$

also form semigroup. The matrix elements in columns of stochastic matrices provide probability distributions. The matrix elements of bistochastic matrices B_{jk} give the probability distributions in both columns and rows.

In infinite-dimensional Hilbert space with discrete basis like Fock states $|n\rangle$, the definitions (18) and (19) hold with the replacement $N \rightarrow \infty$. For example, in the oscillator case, one can use the formalism of semigroups transforming the tomograms of oscillator states. If one has a probability vector \vec{w} , the stochastic and bistochastic matrices give the linear transform of the vector providing new probability vectors

$$\vec{w}_M = M\vec{w}, \quad \vec{w}_B = B\vec{w}. \quad (20)$$

One can consider entropy of stochastic and bistochastic matrices

$$H_M = - \sum_{j,k=1}^N M_{jk} \ln M_{jk}, \quad H_B = - \sum_{j,k=1}^N B_{jk} \ln B_{jk}. \quad (21)$$

The entropies are larger than zero. The minima of the entropies are zero and the maxima of the entropies are $N \ln N$. The qudit states are also characterized by entropies. For example, the state with tomogram $w(m, u)$ has entropy, which is a function on unitary group [27, 14]

$$H(u) = - \sum_m w(m, u) \ln w(m, u). \quad (22)$$

This is the Shannon entropy of quantum qudit state. The von Neumann entropy of the state

$$S = -\text{Tr } \rho \ln \rho \quad (23)$$

is the minimum of the tomographic Shannon entropy [27, 14]

$$S = \min H(u). \quad (24)$$

One can construct the stochastic matrices using as their columns the tomographic-probability vectors taken at different unitary group elements u . For example, the qubit state $w(m, u)$ creates 2×2 stochastic matrix

$$M_q(u_1, u_2) = \begin{pmatrix} w(+1/2, u_1) & w(+1/2, u_2) \\ w(-1/2, u_1) & w(-1/2, u_2) \end{pmatrix}, \quad (25)$$

which depends on two unitary group 2×2 -matrices u_1 and u_2 .

For $u_1 = u_2 = u$, the matrix constructed has the property

$$M_q(u, u)M = M_q(u, u), \quad (26)$$

where M is an arbitrary stochastic matrix. Any tomogram under the action of matrix M_q becomes the tomogram which is the vector in column of the matrix $M_q(u, u)$.

For given tomogram of two qubits $w(m_1, m_2, u)$, one can construct some probability distributions. For example, one can consider the probability to find spin projection of the first particle to be equal to $+1/2$ ($-1/2$). One has for these probabilities

$$w(+1/2, u) = \sum_{m_2} w(+1/2, m_2, u), \quad w(-1/2, u) = \sum_{m_2} w(-1/2, m_2, u). \quad (27)$$

Analogous probabilities can be found for the second particle. These probabilities are the marginals of joint probability distributions

$$w(m_1, u) = \sum_{m_2} w(m_1, m_2, u), \quad w(m_2, u) = \sum_{m_1} w(m_1, m_2, u). \quad (28)$$

Other probabilities can be found answering questions like “what is the probability that both spin projections are equal? and “what is the probability that both spin projections are different?

The answer to these questions provides the probability distributions

$$P_{\text{eq}}(u) = \sum_m w(m, m, u), \quad P_{\text{d}}(u) = \sum_m w(m, m+1, u), \quad (29)$$

where the sum $(m+1)$ means summation modulo 1. This means that here $-1/2 + 1 = 1/2$ and $1/2 + 1 = -1/2$. The marginals (29) provide the qubit portrait of the initial probability distribution [13, 10].

Let us study a qubit state with density matrix ρ_1 and the corresponding tomographic-probability vector

$$\vec{w}_1(u_1) = \begin{pmatrix} w_1(+1/2, u_1) \\ w_1(-1/2, u_1) \end{pmatrix} \quad (30)$$

and construct stochastic matrix

$$M_{q1} = \begin{pmatrix} w_1(+1/2, u_a) & w_1(+1/2, u_d) \\ w_1(-1/2, u_a) & w_1(-1/2, u_d) \end{pmatrix}, \quad (31)$$

taking two values of unitary 2×2 -matrix u_1 denoted u_a and u_d corresponding to two directions \vec{n}_a and \vec{n}_d , respectively.

Then we consider the second qubit with density matrix ρ_2 , which is determined by the tomographic-probability vector

$$\vec{w}_2(u_2) = \begin{pmatrix} w_2(+1/2, u_2) \\ w_2(-1/2, u_2) \end{pmatrix}, \quad (32)$$

and construct stochastic matrix

$$M_{q2} = \begin{pmatrix} w_2(+1/2, u_b) & w_2(+1/2, u_c) \\ w_2(-1/2, u_b) & w_2(-1/2, u_c) \end{pmatrix}, \quad (33)$$

taking two values of unitary 2×2 -matrix u_2 denoted u_b and u_c corresponding to two directions \vec{n}_b and \vec{n}_c , respectively.

The tensor product of two stochastic matrices

$$M = M_{q1} \otimes M_{q2} \quad (34)$$

is 4×4 -matrix with the column being tomographic-probability distributions of a two-qubit quantum state with density matrix $\rho_{12} = \rho_1 \otimes \rho_2$ given by (13), in view of the corresponding choice of matrices u_1 and u_2 .

The stochastic matrix of the form M has a specific property. Let us consider 4×4 -matrix [10, 12, 13]

$$I = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \quad (35)$$

and calculate the trace

$$B = \text{Tr } IM. \quad (36)$$

One can show that for any stochastic 4×4 -matrix M of the form (34) the following inequality holds: $|B| \leq 2$. The Shannon entropy for matrices (29), (33), and (34) read

$$H_{q1} = H(u_a) + H(u_d), \quad H_{q2} = H(u_b) + H(u_c), \quad (37)$$

where the Shannon entropies depending on unitary 2×2 -matrices are given by (22).

One can consider Rényi entropy [24] of the stochastic matrix. For matrix M_q (25), the Rényi entropy is

$$\begin{aligned} R(u_1, u_2) &= \frac{1}{1-q} \ln \left\{ \left[w^q(+1/2, u_1) + w^q(-1/2, u_1) \right] \left[w^q(+1/2, u_2) + w^q(-1/2, u_2) \right] \right\} \\ &= R_1(u_1) + R_2(u_2), \end{aligned} \quad (38)$$

where the number q provides parametric dependence of the Rényi entropy and in the limit $q \rightarrow 1$ the Rényi entropy becomes the Shannon entropy. In (38)

$$R_j(u_j) = \frac{1}{1-q} \ln \left[w^q(+1/2, u_j) + w^q(-1/2, u_j) \right], \quad j = 1, 2 \quad (39)$$

is the Rényi entropy associated to the probability vector in j th column of matrix (25).

Some new inequalities for the Rényi entropy of symplectic tomogram describing the quantum state of a system with continuous variables were obtained in [28–33].

4. Bell's inequalities

The inequality properties for stochastic matrices constructed by means of tomographic probability vectors correspond to Bell's inequalities [6–8]. The inequality given in [7] is written for correlations of spin projections onto different directions. The inequality [8] generalizing the inequality [7] is written for the following probabilities.

Given two qudits with equal j . We produce a shift labelling the spin projections as follows:

$$-j \rightarrow 0, -j + 1 \rightarrow 1, \dots, j \rightarrow d, \quad d = 2j + 1.$$

This means that in the tomographic probability distribution all spin projections are denoted by integer nonnegative numbers. Let us have for two-qudit state the joint tomographic probability distribution $w(m_1, m_2, u)$, where now $m_1, m_2 = 0, 1, \dots, 2j$. Now we consider the following probability of the event that the results of measuring m_1 and m_2 differ by integer k , i.e.,

$$P(m_1 = m_2 + k, u) = \sum_{m=0}^{2j} w(m, m - k, u), \quad (40)$$

where the difference $m - k$ means the nonnegative number equal to $m - k$ modulo d . For example, if $d = 2$, one has $0 - 1 = 1$, and if $d = 3$, one has $2 - 3 = 2$, $0 - 2 = 1$.

If one uses the matrix $u = u_1 \otimes u_2$ the probability can be rewritten in the form

$$P(m_1 = m_2 + k, u_1, u_2) = \sum_{m=0}^{2j} w(m, m - k, u_1, u_2), \quad (41)$$

where now u_1 and u_2 are local unitary transforms.

Other notation can be used for these probabilities and for the tomogram [8, 11, 34]

$$P(A_a = B_b + k) = \sum_{m=0}^{d-1} P(A_a = m, B_b = m - k), \quad (42)$$

where $d = 2j + 1$ and subscripts a and b correspond to u_1 and u_2 . There exists the inequality formulated in this notation as follows:

$$\begin{aligned} I_d = & \sum_{k=0}^{\left[\frac{d}{2}\right]-1} \left(1 - \frac{2k}{d-1}\right) \\ & \times \left\{ \left[P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) + P(A_2 = B_2 + k) + P(B_2 = A_1 + k) \right] \right. \\ & \left. - \left[P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) + P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1) \right] \right\} \leq 2. \end{aligned} \quad (43)$$

Using notation (41) in terms of qudit-state tomogram for the probabilities, one can check that for $d = 2$ inequality (43) is identical to the inequality for Bell's number (38).

The trace of product of real $N \times N$ -matrices can be constructed as scalar product of N^2 -vectors in N^2 -dimensional Hilbert space. Thus the inequality $|\text{Tr } IM| \leq 2$ can be considered as the inequality $|\vec{I} \vec{M}| \leq 2$, where vectors \vec{I} and \vec{M} have matrix elements of the matrices I and M as their components.

The inequality [7] has the form

$$\left| \sum_{m_1 m_2} \sum_k w(m_1, m_2, u_k) c(m_1, m_2, k) \right| \leq 2, \quad (44)$$

where the coefficients of the linear form $c(m_1, m_2, k)$ depend on the index of unitary matrix u_k and the spin projections m_1 and m_2 .

For Bell's inequality [7], $c(m_1, m_2, k)$ are just matrix elements of the matrix I (37). For two qutrits, the coefficients can be obtained too [11]. For entangled states, Bell's inequalities can be violated. Thus, for two qubits, the number $|B|$ (38) can reach value $2\sqrt{2}$ which is the Cirelson bound [35]. One can construct the stochastic matrix M for two qudit states with the probability vector $\vec{w}(u)$ using these vectors as columns $\vec{w}(u)$ of the matrix. The index $k = 1, 2, \dots, d^2$ can be considered as a pair of indices $k = (\alpha, \beta)$ where $\alpha, \beta = 1, 2, \dots, d$. For the matrices $u_{(\alpha, \beta)} = u_\alpha \otimes u_\beta$, where α and β are unitary $d \otimes d$ -matrices. the simply separable two-qudit state has the stochastic matrix M of the tensor-product form $M = M_1(u_\alpha) \otimes M_2(u_\beta)$, where the stochastic matrices M_1 and M_2 are constructed from the tomographic-probability vectors of qudits $\vec{w}(u_\alpha)$ and $\vec{w}(u_\beta)$ used as columns of these matrices. The separable state of two qudits can be associated with convex sum of the stochastic matrices of the given tensor-product form. Both the tensor product of such matrices and convex sum of the tensor products satisfy the same inequality of the form $|\sum_{i,j} c_{ij} M_{ij}| \leq L$. The bound L is different for the matrix M corresponding to entangled state. The coefficients c_{ij} have some geometrical meaning which one needs to clarify.

5. Conclusions

To conclude, we formulate our main results.

We reviewed the probability representation of quantum mechanics in which quantum states are described by standard probability distributions. The probability distribution (or tomographic probability distribution) can be considered as the probability vector. For qudits, the probability vector depends on random spin projections and unitary group. Using the tomographic probability vectors we constructed the stochastic matrices. The columns of the stochastic matrices were taken as tomographic probability vectors at different values of the unitary-group elements. The properties of the constructed stochastic matrices associated with quantum states and forming semigroups provide the possibility to understand mathematical mechanism of Bell's inequality violations for entangled quantum states of multipartite systems.

We pointed out that the joint probability distributions used to study Bell's inequality violations in [8] are just spin tomograms introduced to describe quantum states of bipartite systems in [3] (see, also [9]).

The example of two qubit states considered in this paper demonstrates that Bell's inequality violations is a specific property of joint probability distributions which can be used either in classical statistics or in quantum mechanics. This observation is coherent with considerations presented in recent works [19, 20] where Bell's inequalities were discussed for classical probability distributions.

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References

- [1] Dodonov V V and Man'ko V I 1997 *Phys. Lett. A* **229** 335
- [2] Man'ko V I and Man'ko O V 1997 *Zh. Éksp. Teor. Fiz.* **112** 796 [1997 JETP **85** 430]

- [3] Man'ko V I and Safonov S S 1998 *Yad. Fiz.* **4** 658
- [4] Stayer D F et al. 2002 *Amer. J. Phys.* **70** 288
- [5] Schrödinger E 1935 *Proc. Cambridge Philos. Soc.* **31** 555
- [6] Bell J S 1964 *Physics* (Long Island City, N.Y.) **1** 195
- [7] Clauser J F, Horne M A, Shimony A and Holt R A 1969 *Phys. Rev. Lett.* **23** 880
- [8] Collins D, Gisin N, Linden N, Massar S and Popescu S 2002 *Phys. Rev. Lett.* **88** 040404
- [9] Man'ko V I, Marmo G, Sudarshan E C G and Zaccaria F 2004 *Phys. Lett. A* **327** 353
- [10] Lupo C, Man'ko V I and Marmo G and Sudarshan E C G 2005 *J. Phys. A: Math. Gen.* **38** 10377
- [11] Lupo C, Man'ko V I and Marmo G 2007 *J. Phys. A: Math. Gen.* **40** 13091
- [12] Andreev V A, Man'ko V I, Man'ko O V and Shchukin E V 2006 *Teor. Mat. Fiz.* **146** 140 [*Theor. Math. Phys.* 146 140]
- [13] Chernega V N and Man'ko V I 2007 *J. Russ. Laser Res.* **28** 103
- [14] Man'ko M A, Man'ko V I and Mendes R V 2006 *J. Russ. Laser Res.* **27** 507 [quant-ph/0602129]
- [15] Man'ko V I 2007 *Probability instead of wave function and Bell inequalities as entanglement criterion* 2007 Talk at IV Intern. Conf. "Quantum Theory: Reconsideration of Foundations-4" (Vaxjo, Sweden, June 2007), in: G. Adenier, A. Yu. Khrennikov, P. Lahti, V. I. Man'ko and T. Nieuwenhuizen (eds.), AIP Conference Proceedings Series, Vol. 962, p. 140
- [16] Brunner N, Branciard C and Gisin N 2008 *Can one see entanglement?* quant-ph/0802.0472v1
- [17] Gisin N 2008 quant-ph/0702.021v2
- [18] Acin A, Chen J L, Gisin N et al. 2004 *Phys. Rev. Lett.* **92** 250404
- [19] Matzkin A 2007 *Classical statistical distribution can violate Bell's inequalities* quant-ph/0703251
- [20] Matzkin A 2007 *Bell's theorem as signature of nonlocality: a classical counterexample* quant-ph/07019.2114
- [21] Aliski R and Lendi K 1987 *Dynamical Semigroups and Applications* (Springer: Berlin)
- [22] Bohm A, Kaldass H and Komy S 2007 *Time-asymmetry in quantum physics* quant-ph/0703162
- [23] Shannon C E 1948 *Bell Systems Tech. J.* **27** 379
- [24] Rényi A 1970 *Probability Theory* (North-Holland: Amsterdam)
- [25] Man'ko M A, Man'ko V and Mendes R V 2001 *J. Phys. A: Math. Gen.* **34** 8321
- [26] Klimov A B, Man'ko O V, Man'ko V I, Smirnov Y F and Tolstoy V N 2002 *J. Phys. A: Math. Gen.* **35** 6101
- [27] Man'ko O V and Man'ko V I 2004 *J. Russ. Laser Res.* **25** 115 [quant-ph/041131]
- [28] Man'ko M A, Man'ko O V and Man'ko V I 2006 *Intern. J. Mod. Phys. B* **20** 1399
- [29] De Nicola S, Fedele R, Man'ko M A and Man'ko V I 2006 *Eur. Phys. J. B* **52** 191 [quant-ph/0607200v1]
- [30] Man'ko M A *New inequalities for tomograms in the probability representation of quantum states* 2006 Talk at the 3d Int Workshop "Nonlinear Physics. Theory and Experiment. III (Gallipoli, Lecce, Italy, June 2006) De Nicola S, Fedele R, Man'ko M A and Man'ko V I 2007 *Teor. Mat. Fiz.* **152** 241 [*Theor. Math. Phys.* **152** 1081] [quant-ph/0611114v1]
- [31] Man'ko M A 2006 *J. Russ. Laser Res.* **27** 405
- [32] Man'ko M A 2006 *Symplectic entropy* Talk at the 3d Feynman Festival (25–29 August 2006, University of Maryland, USA) De Nicola S, Fedele R, Man'ko M A and Man'ko V I 2007 *J. Phys.: Conf. Ser.* **70** 012007
- [33] Man'ko M A 2007 *Tomographic entropy and new entropic uncertainty relations* 2007 Talk at IV Intern. Conf. "Quantum Theory: Reconsideration of Foundations-4" (Vaxjo, Sweden, June 2007), in: G. Adenier, A. Yu. Khrennikov, P. Lahti, V. I. Man'ko and T. Nieuwenhuizen (eds.), AIP Conference Proceedings Series, Vol. 962, p. 132
- [34] Chen J L, Deng D L and Hu M G 2008 *Analytic proof of Gisin's theorem for two d-dimensional systems* quant-ph/0802.0125v1
- [35] Cirel'son B S 1980 *Lett. Math. Phys.* **4** 93