Journal of Physics: Conference Series

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Journal of Physics: Conference Series **128** (2008) 012007 doi:10.1088/1742-6596/128/1/012007

Noncanonical embedded rational map soliton in quantum SU(3) Skyrme model

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Abstract. The model is considered in non canonical bases SU(3) $\supset$ SO(3) for the state vectors. The rational map ansatz are used to describe the soliton with topological number bigger than one. The canonical quantization generates five different momenta of inertia in Hamiltonian and quantum mass corrections, which can stabilize the quantum soliton solution. The explicit expressions of the Lagrangian and Hamiltonian are derived for this model.

1. Introduction

The Skyrme model was introduced as an effective theory of baryons [1]. Recently the topological soliton solutions – skyrmions find the application in quantum Hall effect, Bose-Einstein condensate and black hole physics. The general exact solutions of the model are unknown even in a classical case and the approximate solitonic solutions are under intensive consideration.

The original model was defined for a unitary field $U(\mathbf{x},t)$ that belongs to fundamental representation of the SU(2) group. Semiclassical quantization suggests that the skyrmion rotates as a "rigid body" [2]. Collective coordinates approach separate the variables which depend on the time and spatial coordinates. The structure of the ansatz which depend on spatial coordinates determine the solitonic solutions. Constructive realization of canonical quantization provides in Hamiltonian a term which may be interpreted as an effective pion mass term [3]. The extension of the model to SU(N) group [4] represents the common structure of the Skyrme Lagrangian.

The aim of this work is to discuss the group-theoretical aspects of the canonical quantization of the SU(3) Skyrme model in rational map antsatz approximation with baryon number $B \geq 2$. The ansatz is defined in noncanonical SU(3) $\supset$ SO(3) bases as SO(3) solitonic solution. The canonical quantization generates five different momenta of inertia. The proposed ansatz can be used to determine light nuclei as special skyrmions.

2. Noncanonical embedding of rational map soliton

The Skyrme model is a Lagrangian density for unitary field $U(\mathbf{x},t)$ that can be defined to the general representation of SU(3) group [5]. Here we consider the unitary field in fundamental representation (1,0) of SU(3) group. The chiraly symmetric Lagrangian density has the form:

$$\mathcal{L} = -\frac{f_\pi^2}{4}\mathrm{Tr}\{\mathbf{R}_\mu\mathbf{R}^\mu\} + \frac{1}{32\mathrm{e}^2}\mathrm{Tr}\{[\mathbf{R}_\mu, \mathbf{R}_\nu][\mathbf{R}^\mu, \mathbf{R}^\nu]\}, \tag{1}$$

where the "right" and "left" chiral currents are defined as

$$R_\mu = (\partial_\mu U)\overset{+}{U} = \partial_\mu \alpha^i C_i^{(B)}(\alpha)\left\langle \left| J_{(B)}^{(1,1)} \right| \right\rangle, \tag{2}$$
$$L_\mu = \overset{+}{U}(\partial_\mu U) = \partial_\mu \alpha^i C_i^{\prime(B)}(\alpha)\left\langle \left| J_{(B)}^{(1,1)} \right| \right\rangle \tag{3}$$

and have the values on the SU(3) algebra. The f_π and e in (1) are the phenomenological parameters of the model. The functions $C_i^{(B)}(\alpha)$ and $C_i^{\prime(B)}(\alpha)$ depend on parameters α^i of the group. $J_{(B)}^{(1,1)}$ are the generators of the group.

The system of noncanonical SU(3) generator in terms of canonical generators $J_{(Z,I,M)}^{(1,1)}$ which are defined in [6] can be expressed as follows:

$$\begin{aligned}
J_{(1,1)} &= \sqrt{2}\left(J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} - J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}\right), & J_{(1,0)} &= 2J^{(1,1)}_{(0,1,0)},\\
J_{(1,-1)} &= \sqrt{2}\left(J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})} + J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}\right), & J_{(2,2)} &= -2J^{(1,1)}_{(0,1,1)},\\
J_{(2,1)} &= -\sqrt{2}\left(J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} + J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}\right), & J_{(2,0)} &= -2J^{(1,1)}_{(0,0,0)},\\
J_{(2,-1)} &= -\sqrt{2}\left(J^{(1,1)}_{(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})} - J^{(1,1)}_{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}\right), & J_{(2,-2)} &= 2J^{(1,1)}_{(0,1,-1)}.
\end{aligned} \tag{4}$$

They satisfy the commutation relations

$$\left[J_{(L',M')}, J_{(L'',M'')}\right] = -2\sqrt{3}\begin{bmatrix} (1,1) & (1,1) & (1,1)_a \\ L' & L'' & L \end{bmatrix}\begin{bmatrix} L' & L'' & L \\ M' & M'' & M \end{bmatrix} J_{(L,M)}. \tag{5}$$

The coefficient on the rhs of (5) is SU(3) noncanonical isofactor and SO(3) Clebsch-Gordan coefficient. The state vectors for canonical bases SU(3) $\supset$ SU(2) and non-canonical bases SU(3) $\supset$ SO(3) in fundamental representation are equivalent. The rational map antsatz in SO(3) case we suggest as the matrix

$$\begin{aligned}
(U_R)_{a,a'} &= \left(\exp(2i\hat{n}_m J_{(m)}F(r))\right)_{a,a'} = 2\sin^2 F(-1)^a \hat{n}_{-a}\hat{n}_{a'}\\
&\quad + i\sqrt{2}\sin 2F\begin{bmatrix} 1 & 1 & 1 \\ a & u & a' \end{bmatrix}\hat{n}_u + \cos 2F\delta_{a,a'}.
\end{aligned} \tag{6}$$

The unit vector $\hat{\mathbf{n}}$ are defined in [7] in terms of a rational complex function $R(z)$ as

$$\hat{\mathbf{n}}_R = \frac{1}{1+|R|^2}\{2\mathrm{Re}(R), 2\mathrm{Im}(R), 1-|R|^2\}, \tag{7}$$

for every baryon number

$$B = \frac{1}{24\pi^2}\int \mathrm{d}^3 r \mathcal{I}(\theta,\varphi)\frac{F'(r)\sin^2 F}{r^2}. \tag{8}$$

The symbol $\mathcal{I}(\theta,\varphi)$ denotes the function

$$\mathcal{I}(\theta,\varphi) = \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^2. \tag{9}$$

which depends on angles θ, φ only.

3. Canonical quantization

The quantization of the model can be carried out by means of collective coordinates that separate the variables, which depend on the time and spatial coordinates

$$U(\vec{r}, q(t)) = A(q(t))U_R(\vec{r})A^{\dagger}(q(t)). \tag{10}$$

Here eight SU(3) group parameters $q^i(t)$ are quantum variables. The Skyrme Lagrangian is considered quantum mechanically *ab initio* in contrast to the conventional semiclassical quantization of the soliton as a rigid body. Generalized coordinates $q^i(t)$ and the corresponding velocities $\dot{q}^i\,(t)$ satisfy the following commutation relations

$$\left[\dot{q}^{\alpha}, q^{\beta}\right] = -if^{\alpha\beta}(q), \tag{11}$$

where $f^{\alpha\beta}(q)$ are functions of q only. Weyl ordering of the operators has been employed for arbitrary function $G(q)$

$$\partial_0 G(q) = \frac{1}{2}\left\{\dot{q}^{\alpha}, \frac{\partial}{\partial q^{\alpha}}G(q)\right\}. \tag{12}$$

The curly brackets denote an anticommutator. With thus choice of operator ordering (12) no further ordering ambiguity appears in Lagrangian or Hamiltonian.

After substitution of the ansatz (10) into the model Lagrangian density (1) and integration over spatial coordinates the Lagrangian has this form:

$$L = \frac{1}{2}\dot{q}^{\alpha}g_{\alpha\beta}(q,F)\dot{q}^{\beta} + a^0\frac{1}{2}\left\{\dot{q}^{\alpha}, C'^{(2,0)}_{\alpha}(q)\right\} + \left[(\dot{q})^0 \text{ order terms}\right], \tag{13}$$

where the metric tensor is

$$g_{\alpha\beta}(q,F) = C'^{(L,M)}_{\alpha}(q)E_{(L,M)(L',M')}(F)C'^{(L',M')}_{\beta}(q), \tag{14}$$

and

$$E_{(L,M)(L',M')}(F) = -(-1)^M a_{L,M}(F)\delta_{L,L'}\delta_{M,-M'}. \tag{15}$$

Note that the exact expression of the coefficient a^0 is not important for the calculation of $g_{\alpha\beta}$. In this case we have five different quantum momenta of inertia – integrals beside the terms of the "physical" SU(3) Casimir operator:

$$a_{(1,0)}(F) = \frac{1}{e^3 f_\pi} \int d\theta d\varphi d\tilde{r}\tilde{r}^2 \sin^2 F \left(n_0^2 - 1\right) \left[1 + F'^2 + \frac{\mathcal{I}}{r^2}\sin^2 F\right]; \tag{16a}$$

$$a_{(1,1)}(F) = \frac{1}{2e^3 f_\pi} \int d\theta d\varphi d\tilde{r}\tilde{r}^2 \sin^2 F \left(n_0^2 + 1\right) \left[1 + F'^2 + \frac{\mathcal{I}}{r^2}\sin^2 F\right]; \tag{16b}$$

$$\begin{aligned} a_{(2,0)}(F) = \frac{1}{e^3 f_\pi} \int d\theta d\varphi d\tilde{r}\tilde{r}^2 \sin^2 F \left(n_0^2 - 1\right) \Big(&\cos^2 F + n_0^2 \sin^2 F \\ &- (n_0^2 - 4\cos^2 F + 2n_0^2 \cos 2F)F'^2 \\ &+ \frac{\mathcal{I}}{r^2}\sin^2 F(2\cos^2 F + n_0^2(4 - \cos 2F))\Big); \end{aligned} \tag{16c}$$

$$\begin{aligned} a_{(2,1)}(F) = \frac{1}{2e^3 f_\pi} \int d\theta d\varphi d\tilde{r}\tilde{r}^2 \sin^2 F \Big(&3 + 2\cos 2F - 3n_0^2 + 4n_0^4 \sin^2 F \\ &+ (9 + 8\cos 2F - 3n_0^2 - 4n_0^4(1 + 2\cos 2F))F'^2 \\ &+ \frac{\mathcal{I}}{r^2}\sin^2 F(9 + 4\cos 2F - 15n_0^2 + 4n_0^4(4 - \cos 2F))\Big); \end{aligned} \tag{16d}$$

$$\begin{aligned} a_{(2,2)}(F) = \frac{1}{4e^3 f_\pi} \int d\theta d\varphi d\tilde{r}\tilde{r}^2 \sin^2 F \Big(&-3 - \cos 2F - 12n_0^2 \cos^2 F + 2n_0^4 \sin^2 F \\ &- 2(3 + 2\cos 2F - 24n_0^2 \cos^2 F + n_0^4(1 + 2\cos 2F))F'^2 \\ &- \frac{2\mathcal{I}}{r^2}\sin^2 F(6 + \cos 2F - 12n_0^2 \cos^2 F - n_0^4(4 - \cos 2F))\Big); \end{aligned} \tag{16e}$$

which depend on chiral angle function $F(r)$, one component n_0 of rational map vector and function $\mathcal{I}(\theta, \varphi)$.

The canonical momenta are defined as

$$p_\beta = \frac{\partial L}{\partial \dot{q}^\beta} = \frac{1}{2}\{\dot{q}^\alpha, g_{\alpha\beta}\} + a^0 C_\beta^{\prime(2,0)}(q). \tag{17}$$

Is notable what the momenta have a term which do not contains velocity. The canonical momenta and conjugate coordinates satisfy the commutation relations $[p_\beta, q^\alpha] = -i\delta_{\alpha\beta}$. The commutation relations fix the explicit expressions of the functions

$$f^{\alpha\beta}(q) = (g_{\alpha\beta}(q))^{-1}. \tag{18}$$

It is possible to choose the parametrization on the SU(3) group manifold so that the eight operators

$$\hat{R}_{(LM)} = \frac{i}{2}\left\{p_\beta, C'^\beta_{(L,M)}(q)\right\} \tag{19}$$

are defined as the group generators satisfying the commutation relations (5). Here the functions $C'^\alpha_{(L,M)}(q)$ are dual to the functions defined in (3)

$$\sum_\alpha C'^\alpha_{(L,M)}(q) C_\alpha^{\prime(L',M')}(q) = \delta_{(L,M)(L',M')}, \tag{20a}$$

$$\sum_{L,M} C'^\alpha_{(L,M)}(q) C_{\alpha'}^{\prime(L,M)}(q) = \delta_{\alpha\alpha'}. \tag{20b}$$

The generators (19) act on the Wigner matrix of the SU(3) irreducible representation as right transformation generators:

$$\left[\hat{R}_{(LM)}, D^{(\lambda,\mu)}_{(\alpha_1 L_1 M_1)(\alpha_2 L_2 M_2)}(q)\right] = D^{(\lambda,\mu)}_{(\alpha_1 L_1 M_1)(\alpha_2' L_2' M_2')}(q) \left\langle \begin{matrix} (\lambda,\mu) \\ (\alpha_2' L_2' M_2') \end{matrix} \middle| J_{(LM)} \middle| \begin{matrix} (\lambda,\mu) \\ (\alpha_2 L_2 M_2) \end{matrix} \right\rangle. \tag{21}$$

Journal of Physics: Conference Series **128** (2008) 012007 doi:10.1088/1742-6596/128/1/012007

The indices α_1 and α_2 label the multiplets of (L, M).

R. Sugano and collaborators [8] developed the q-number variational method to formulate a theory that has the consistency between the Lagrangian and the Hamiltonian formalisms on the curved space of generalized coordinates. In terms of (19) generators we construct the Hamiltonian of the model:

$$\begin{aligned} H &= \frac{1}{8}\Big\{\dot{q}^{\alpha}, C_{\alpha}^{\prime(L_1,M_1)}(q)\Big\}E_{(L_1M_1)(L_2M_2)}\Big\{\dot{q}^{\beta}, C_{\beta}^{\prime(L_2,M_2')}(q)\Big\} - V_3 + M_{\mathrm{cl}} \\ &= -\frac{1}{2}\hat{R}_{(L_1M_1)}E^{(L_1M_1)(L_2M_2)}\hat{R}_{(L_2M_2)} - \frac{2V_2}{a_{(2,0)}}\hat{R}_{(2,0)} - 2\left(\frac{V_2}{a_{(2,0)}}\right)^2 - V_3 + M_{\mathrm{cl}}. \end{aligned} \tag{22}$$

where $E^{(L_1M_1)(L_2M_2)}$ is inverse matrix of eq. (15), M_{cl} is classical mass. The functions V_2 and V_3 results from the trace containing three and four group generators respectively.

$$\begin{aligned} V_2 &= \int \mathrm{d}^3x \frac{f_\pi^2}{4}\Bigg(\frac{(-1)^{M_2+M_1}}{a_{(L,M_2)}}\frac{\sqrt{2}}{\sqrt{3}}\sqrt{L^2+L+1}\left[\begin{array}{ccc} L & L & 2 \\ M_2 & -M_2' & M_1 \end{array}\right]\left(D^L_{M_2',M_2}(\varkappa) - \overset{+}{D}{}^L_{M_2',M_2}(\varkappa)\right) \\ &- \frac{1}{a_2}\left(D^2_{0,M_1}(\varkappa) - \overset{+}{D}{}^2_{0,M_1}(\varkappa)\right)\Bigg) \\ &+ \frac{1}{4\mathrm{e}^2}(-1)^{M_1'+M_2}\frac{\sqrt{2\cdot 3}}{a_{(L,M_2)}}\frac{\sqrt{L^2+L+1}}{\sqrt{5-2L}}B_{m,m'}(\varkappa)\left[\begin{array}{ccc} 2 & 1 & 2 \\ M_1'' & m & M_1' \end{array}\right]\left[\begin{array}{ccc} L & 1 & L \\ M_2'' & m' & M_2' \end{array}\right] \\ &\times \left[\begin{array}{ccc} L & L & 2 \\ M_2' & M_2''' & -M_1' \end{array}\right]\Big(\delta_{M_1M_1''}\delta_{M_2M_2''}\overset{+}{D}{}^L_{M_2''',-M_2'}(\varkappa) - \delta_{M_1M_1''}\delta_{M_2M_2''}\delta_{-M_2M_2'''} \\ &- \delta_{M_1M_1''}\delta_{-M_2M_2'''}D^L_{M_2'',M_2}(\varkappa) + \delta_{M_2M_2''}\delta_{-M_2M_2'''}\overset{+}{D}{}^2_{M_1'',M_1}(\varkappa)\Big); \end{aligned} \tag{23}$$

$$\begin{aligned} V_3 &= \int \mathrm{d}^3x \frac{f_\pi^2}{4}\Big(\frac{4(2L_1+1)(2L_2+1)}{a_{(L_1,M_1)}a_{(L_2,M_2)}}\left\{\begin{array}{ccc} L_1 & L_2 & k \\ 1 & 1 & 1 \end{array}\right\}^2\left[\begin{array}{ccc} L_1 & L_2 & k \\ M_1 & M_2 & u \end{array}\right]^2 D^k_{u,u}(\varkappa) + \frac{3}{a_0^2} \\ &+ \frac{1}{a_2^2}\left(1 + \overset{+}{D}{}^2_{0,0}(\varkappa)\right) - \frac{4}{a_{(L,M)}}\Bigg(\frac{1}{a_0}D^L_{M,M}(\varkappa) - (-1)^M\frac{1}{a_2}\frac{\sqrt{L^2+L+1}}{\sqrt{2\cdot 3}} \\ &\times \left[\begin{array}{ccc} L & L & 2 \\ M & -M & 0 \end{array}\right]D^L_{M,M}(\varkappa)\Bigg) \\ &- \frac{3}{2\mathrm{e}^2}\frac{(2L_1+1)(2L_2+1)}{\sqrt{(5-2L_1)\,(5-2L_2)}}(-1)^{M_1+M_2+u}B_{m,m'}(\varkappa)\left\{\begin{array}{ccc} L_1 & L_1 & k \\ 1 & 1 & 1 \end{array}\right\}\left\{\begin{array}{ccc} L_2 & L_2 & k \\ 1 & 1 & 1 \end{array}\right\} \\ &\times \left[\begin{array}{ccc} L_1 & 1 & L_1 \\ M_1' & m & M_1'' \end{array}\right]\left[\begin{array}{ccc} L_2 & 1 & L_2 \\ M_2' & m' & M_2'' \end{array}\right]\left[\begin{array}{ccc} L_1 & L_1 & k \\ M_1 & -M_1'' & u \end{array}\right]\left[\begin{array}{ccc} L_2 & L_2 & k \\ -M_2 & M_2'' & u \end{array}\right] \\ &\times \Bigg(\frac{1}{a_{(L_1,M_1)}a_{(L_2,M_2)}}\Big(\delta_{M_2M_2'}D^{L_1}_{M_1',M_1}(\varkappa)\left(1-(-1)^k\right) - \delta_{M_1M_1'}\delta_{M_2M_2'} \\ &- D^{L_1}_{M_1',M_1}(\varkappa)D^{L_2}_{M_2',M_2}(\varkappa)\Big) + \frac{1}{a_{(L_1,M_1')}a_{(L_2,M_2)}}\delta_{M_2M_2'}D^{L_1}_{M_1',M_1}(\varkappa)\left(1+(-1)^k\right)\Bigg). \end{aligned} \tag{24}$$

where $B_{m,m'}(\varkappa)$ are:

$$B_{m,m'}(\varkappa) = 8(-1)^{m+m'}\hat{n}_{-m}\hat{n}_{-m'}\left(\frac{1}{r^2}\mathcal{I}\sin^2 F - F'^2\right) - (-1)^m\delta_{m,-m'}\frac{8}{r^2}\mathcal{I}\sin^2 F. \tag{25}$$

Journal of Physics: Conference Series **128** (2008) 012007 doi:10.1088/1742-6596/128/1/012007

We define the state vectors as the complex conjugate Wigner matrix elements of the (Λ, Θ) representation depending on eight quantum variables q^α:

$$\left| \begin{array}{c} (\Lambda, \Theta) \\ \alpha, S, N; \beta, S', N' \end{array} \right\rangle = \sqrt{\dim(\Lambda, \Theta)} D^{*(\Lambda,\Theta)}_{(\alpha,S,N)(\beta,S',N')}(q) \left|0\right\rangle , \tag{26}$$

The indices α and β label the multiplets of the SO(3) group. $|0\rangle$ denotes the vacuum state. Because of five different moments of inertia the vectors (26) are not the eigenstates of the Hamiltonian (22). The action of the Hamiltonian on vectors (26) following (21) can be expressed in terms of the moments of inertia $a_{(L,M)}$ and the SU(3) group Clebsch-Gordan coefficients.

4. Conclusions

We considered a new rational map approximation ansatz for Skyrme model which is noncanonical embedded SU(3) $\supset$ SO(3) soliton with baryon number $B \geq 2$. The canonical quantization leads to five different quantum momenta of inertia in Hamiltonian and quantum corrections V_2 and V_3. The state vectors defined as SU(3) group representation (Λ, Θ) matrix which depend on quantum variables q. Because the momenta $a_{(2,1)}$ and $a_{(1,1)}$ are not equivalent the state vectors (26), they are not eigenstate vectors of the Hamiltonian for the higher representations. The mixing is small but to find eigenstates we must diagonolize the Hamiltonian matrix in every (Λ, Θ) representation. For baryon number $B = 1$ case $\hat{n} = \hat{x}$ and we get soliton with two different momenta of inertia which was considered in [6]. The SU(2) rational map ansatzes are not spherically symmetric and "rotation" in SU(3) manifold leads to five momenta of inertia in Hamiltonian.

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