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To cite this article: S Giller *et al* 2008 *J. Phys.: Conf. Ser.* **128** 012003

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On some aspects of noncommutative pure Yang-Mills theory

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Abstract. Two types of gauge transformations of noncommutative pure gauge theory are discussed. It is shown that Yang-Mills theory with the so called twisted gauge symmetry is consistent provided it also enjoys the standard noncommutative $\ast$ -gauge symmetry.

1. Introduction

The standard approach to gauge symmetry on noncommutative space-time [1,2,3] basically consists in replacing the product by the star one in both the definitions of gauge transformations as well as their action on gauge potentials. This appears to be quite restrictive in a sense that practically only star counterpart of $U(N)$ group transformations close to form a group [3,4,5]. Such restrictions on gauge groups are absent in the so called twisted gauge transformations approach [6-13]. Here, contrary to the previous case, there is no star product involved in the definitions of gauge transformations and their action on fields. Noncommutativity of transformations is reflected in modified Leibnitz rule. So, by construction, twisted gauge transformations unlike standard $\ast$ -ones impose rather mild restrictions on gauge groups. However, it appears that twisted gauge theory is consistent provided it also possesses the standard noncommutative gauge symmetry. It is just an aim of this short note to show that, in general, the current conservation in the theory with twisted gauge symmetry implies some constraints which are not automatically fulfilled by virtue of field equations (more specifically, these constraints follow by taking the divergence of field equations). If one demands no such constraints exist, the range of gauge potentials must be enlarged so that the action exhibits also standard noncommutative gauge symmetry [14].

The note starts with a flash definitions of noncommutative Yang-Mills theory and both types of gauge transformations; standard noncommutative ones and twisted ones. Then the consistency of theory with twisted gauge symmetry is discussed.

2. Noncommutative pure Yang-Mills theory

Noncommutative pure Yang-Mills theory is defined by the following action

$$S = -\frac{1}{4} \int d^4x \text{Tr}(F_{\mu\nu} \ast F_{\mu\nu}) \quad (1)$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_* , [A_\mu, A_\nu]_* \equiv A_\mu * A_\nu - A_\nu * A_\mu \quad (2)$$

$$\begin{aligned} e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y} A_\mu(x)A_\nu(y) |_{x=y} &= m(F^{-1}A_\mu \otimes A_\nu) \equiv m_*(A_\mu \otimes A_\nu) \\ m(A_\mu \otimes A_\nu) &= A_\mu A_\nu \\ F &= e^{\frac{-i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \end{aligned} \quad (3)$$

and $A_\mu(x)$ is $N \times N$ matrix valued function.

This action can be considered as being obtained from usual commutative pure Yang-Mills one by replacing ordinary product of fields by star product defined by eq.(3). Consequently, ordinary commutators are replaced by corresponding $*$ -ones given by eq. (2).

Varying the action with respect to gauge potentials gives the following field equations

$$\partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}]_* = 0 \quad (4)$$

These equations look as their commutative counterparts however one has to remember that commutator is now $*$ -one. It appears that it is possible to define two types of gauge transformations which leave the action invariant. There are standard noncommutative gauge transformations for gauge potentials belonging to $u(N)$ Lie algebra and twisted gauge ones for potentials belonging a priori to any Lie algebra.

3. Standard $*-U_*(N)$ gauge transformations

Standard noncommutative gauge transformations are given by following equations

$$\begin{aligned} A_\mu &\longmapsto A'_\mu = U_* * A_\mu * U_*^{-1} - i\partial_\mu U_* * U_*^{-1} \\ U_*(x) &= e_*^{i\alpha^a(x)T^a} \equiv I + i\alpha^a T_a + \frac{i^2}{2}\alpha^a * \alpha^b T_a T_b + \dots \end{aligned} \quad (5)$$

where $T_a \in u(N)$. Two points are really important here. The first is the use of $*$ -product in definition of gauge transformations as well as their action on fields. The second point related to the first one is the observation that gauge transformations close to form a group provided matrices T_a span $u(N)$ Lie algebra. So practically, only unitary group $U(N)$ can be $*$ gauged.

Now, invariance of the action S under infinitesimal $*$ -gauge transformations

$$\begin{aligned} \delta A_\mu &\equiv A'_\mu - A_\mu = i[\alpha, A_\mu]_* + \partial_\mu \alpha \\ \delta \partial_\nu A_\mu &= \partial_\nu \delta A_\mu \end{aligned} \quad (6)$$

implies (via noncommutative counterpart of second Noether theorem) that

$$[A_\mu, \frac{\delta L}{\delta A_\mu}]_* + \partial_\nu [A_\mu, F^{\nu\mu}]_* = 0 \quad (7)$$

$$\partial_\mu \frac{\delta L}{\delta A_\mu} = i[A_\mu, \frac{\delta L}{\delta A_\mu}]_* \quad (8)$$

where $\frac{\delta L}{\delta A_\mu} \equiv \partial_\nu F^{\nu\mu} - i[A_\nu, F^{\nu\mu}]_*$ is the Euler - Lagrange expression.

Eq.(7) provides a conserved current $j^\nu \equiv [A_\mu, F^{\nu\mu}]_*$, $\partial_\mu j^\mu = 0$. The conservation of this current (which can also be checked directly using field equations) is crucial for consistency of the theory. In fact, eq.(8) confirms that no constraints on A_μ follow by taking divergence of field equations.

4. Twisted gauge transformations

Twisted gauge transformations and their action on fields are defined exactly as in ordinary commutative situation i.e.

$$A_\mu \longmapsto A'_\mu = U A_\mu U^{-1} - i(\partial_\mu U)U^{-1}$$

$$U(x) = e^{i\alpha^a(x)T^a}. \quad (9)$$

Contrary to the first kind of transformations there is no $*$ -product involved in these definitions. However, an important assumption is added that derivatives entering $*$ -products do not act on gauge parameters. This assumption results in twisted Leibnitz rule (see eq(10) below) which can be expressed in terms of twisted coproduct Δ_F of α which explains the name of symmetry.

$$\delta(\psi * \phi) = i\alpha^a(x)(T_a\psi * \phi + \psi * T_a\phi) = m_*(\Delta_F(i\alpha)\psi \otimes \phi)$$

$$\Delta_F = F\Delta F^{-1}, \quad \Delta(i\alpha) = i\alpha \otimes I + I \otimes i\alpha \quad (10)$$

The above twisted Leibnitz rule is given for functions transforming as matter fields. In the case of gauge fields there would be an extra terms coming from derivative of gauge parameter entering gauge potentials transformation rule.

5. Pure Yang - Mills theory with twisted gauge symmetry

By the very definition twisted gauge transformations impose rather mild restrictions on gauge groups. So, a natural question arises whether it is possible to construct noncommutative gauge theory with some compact Lie group G different from unitary one, playing the role of gauge group and with gauge potentials belonging to a Lie algebra representation of G . To answer this question it is convenient to select some N -dimensional unitary irreducible representation of G (and the corresponding irreducible representation of g , denoted also by g) and to assume that the gauge potential takes its values in g . The Lie algebra g is a subalgebra of $u(N)$. The latter viewed as linear space is equipped with the scalar product $(A, B) \equiv \text{Tr}(AB)$ invariant under the adjoint action of $U(N)$. If B_i , $i = 1, \dots, a$ span an orthonormal bases in the orthogonal complement $g_\perp$ of g in $u(N)$ ($g_\perp = \{B \in u(N); \text{Tr}(AB) = 0, A \in g\}$) then the condition that the potential A_μ belongs to the representation g reads $A \in g \iff \text{Tr}(AB_i) = 0$, $i = 1, \dots, a$.

To define Yang-Mills theory with twisted gauge group G , Lagrange multiplier method can be used. The relevant action reads

$$S = -\frac{1}{4} \int d^4x \text{Tr}(F_{\mu\nu} * F_{\mu\nu}) + \int d^4x \rho_i^\mu \text{Tr}(A_\mu B_i) \quad (11)$$

where $\rho_i^\mu(x)$ are the Lagrange multipliers and where it is assumed that the gauge potential A_μ *a priori* takes its values in $u(N)$.

Eq. (11) implies the following equations of motion

$$\text{Tr}(A_\mu B_i) = 0 \quad (12)$$

$$\partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}]_* + \rho_i^\nu B_i = 0, \quad (13)$$

hence

$$\rho_i^\nu = -\text{Tr}(B_i(\partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}]_*)) \quad (14)$$

Eq.(13) means that in general, $f^\nu \equiv \partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}]_* \in g_\perp$ (in the case of ordinary commutative gauge theory $f^\mu \in g$ by construction and one deals with ordinary field equations $\partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}]_* = 0$, similarly, in standard (i.e. not twisted) approach to gauge theory on noncommutative space-time g must basically be $u(N)$ and $g_\perp = 0$; again one arrives at ordinary form of field equations).

Due to antisymmetry of $F^{\mu\nu}$ tensor the field eqs.(13) imply the followig consistency condition

$$\partial_\nu(i[A_\mu, F^{\mu\nu}]_* - \rho_i^\nu B_i) = 0 \quad (15)$$

ie.

$$\frac{i}{2}[\partial_\nu A_\mu - \partial_\mu A_\nu, F^{\mu\nu}]_* + i[A_\mu, \partial_\nu F^{\mu\nu}]_* - \partial_\nu \rho_i^\nu B_i = 0. \quad (16)$$

Using field equations, Jacobi identities and taking into account that $\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} + i[A_\mu, A_\nu]_*$ eq.(16) can be rewritten in the form

$$[A_\nu, \rho_i^\nu B_i]_* + \partial_\nu \rho_i^\nu B_i = 0 \quad (17)$$

which finally implies

$$Tr(\Gamma[A_\nu, \rho_i^\nu B_i]_*) = 0 \quad (18)$$

for any $\Gamma \in g$.

Now, $*$ -commutator consists of two pieces, one proportional to matrix commutator and the second one involving matrix anticommutator;

$$[A_\nu, \rho_i^\nu B_i]_* = \frac{1}{2}(\{A_\nu^a, \rho_i^\nu\}_* [T_a, B_i] + [A_\nu^a, \rho_i^\nu]_* \{T_a, B_i\}). \quad (19)$$

However, commutators of B 's and T 's belong to orthogonal complement of Lie algebra g

$$[T_a, B_i] \in g_\perp \quad (20)$$

(Eq.(20) results because if $U \in G$ then $U^+ \Gamma U \in g$ and $Tr(\Gamma' U B_i U^+) = Tr(U^+ \Gamma' U B_i) = 0$ so that $U B_i U^+ \in g_\perp$). Consequently, consistency condition (see eq.(18)) reads

$$Tr(\Gamma[A_\nu, \rho_i^\nu B_i]_*) = Tr(\Gamma[A_\nu, \rho_i^\nu]_* \{T_a, B_i\}) = 0 \quad (21)$$

It is clear that eq.(21) does not produce any further constraints on A_μ provided $\{T_a, B_i\} \in g_\perp$ but then eq.(20) would imply

$$T_a B_i \in g_\perp \quad \text{and} \quad B_i T_a \in g_\perp \quad (22)$$

so that

$$\Sigma B_i \in g_\perp \quad \text{and} \quad B_i \Sigma \in g_\perp \quad (23)$$

for each Σ belonging to the enveloping algebra $U(g)$ of g which due to its assumed irreducibility is an algebra M_N of all $N \times N$ matrices (this is Burnside theorem). So, equations (23) imply that orthogonal complement $g_\perp$ of g is a two-sided ideal in M_N . However, it is known (by Wedderburn theorem) that M_N has only two ideals: $I = \phi$ or $I = M_N$. $g_\perp = M_N$ would imply no gauge symmetry, so one is left with $g_\perp = \phi$ ie. g must be $u(n)$ Lie algebra.

In this way one concludes that gauge theory with twisted gauge group G imposes no extra constraints on gauge fields A_μ provided G is unitary group $U(N)$. But then theory also enjoys standard $*$ - $U_*(N)$ symmetry which controls consistency of the theory and provides conserved currents $j^\nu \equiv [A_\mu, F^{\nu\mu}]_*$.

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