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Control problems for the stationary MHD equations under mixed boundary conditions

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Abstract. The optimal control problems for the stationary magnetohydrodynamic equations under inhomogeneous mixed boundary conditions for a magnetic field are considered. The role of control in control problems under study is played by normal component of the magnetic field on the part of the boundary. The existence of optimal solution is proved and optimality system for considered extremum problem is obtained.

1. Introduction. Statement of the boundary value problem

Control problems for models of magnetic hydrodynamics of viscous electric conducting fluids play an important role in a number of applications [1, 2]. In this paper we study control problems for the stationary model of magnetic hydrodynamics, considered under mixed boundary conditions for the electromagnetic field.

Let Ω be a bounded domain of the space $\mathbf{R}^3$ with boundary $\Sigma = \partial\Omega$ consisting of two parts Σ_ν and Σ_τ. We will consider following system of the stationary magnetohydrodynamic equations of viscous incompressible fluid

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - æ\,\mathrm{curl}\,\mathbf{H}\times\mathbf{H} = \mathbf{f}, \quad \mathrm{div}\,\mathbf{u} = 0, \tag{1}$$

$$\nu_1\mathrm{curl}\,\mathbf{H} - \rho_0^{-1}\mathbf{E} + æ\mathbf{H}\times\mathbf{u} = \nu_1\mathbf{j}, \quad \mathrm{div}\,\mathbf{H} = 0, \quad \mathrm{curl}\,\mathbf{E} = \mathbf{0}, \tag{2}$$

together with the following inhomogeneous boundary conditions:

$$\mathbf{u}|_\Sigma = \mathbf{0}, \ \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} = q, \ \mathbf{H}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{0}, \ \ \mathbf{E}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{k}. \tag{3}$$

Here $\mathbf{u}$ is the velocity vector, $\mathbf{H}$ and $\mathbf{E}$ are magnetic and electric fields, respectively, $p = P/\rho_0$, where P is the pressure, $\rho_0 = \mathrm{const}$ is a fluid density, $æ = \mu/\rho_0$, $\nu_1 = 1/\rho_0\sigma = æ\nu_m$, ν and ν_m are constant kinematic and magnetic viscousity coefficients, σ is a constant conductivity, μ is a constant magnetic permeability, $\mathbf{n}$ is the outer normal to Σ, $\mathbf{j}$ is the current density. Below we will refer to problem (1)–(3) for given functions $\mathbf{f}, \mathbf{j}, \mathbf{k}$ and q as Problem 1. We note that all the quantities in (1)–(3) are dimensional and their physical dimensions are defined in terms of SI units. Physically the boundary conditions for the electromagnetic field in (3) correspond to the situation when the part Σ_ν of the boundary Σ is a perfect insulator.

In the particular case when $\Sigma_\nu = \emptyset$ the boundary conditions (3) for the magnetic field take the form $\mathbf{H}\cdot\mathbf{n}|_\Sigma = \mathbf{q}$, $\mathbf{E}\times\mathbf{n}|_\Sigma = \mathbf{k}$. The corresponding boundary value and extremum problems for model (1), (2) were studied by a number of authors (see, for example [3–13]. Beginning from 2014, the authors published papers [14–16] devoted to the analysis of the solvability of boundary value problems for model (1), (2) considered under mixed (homogeneous or inhomogeneous) boundary conditions for the electromagnetic field. Further, authors' papers [17, 18] were published, in which the study of control problems for model (1), (2) in some particular cases was started. Continuing the studies begun in [17, 18], we formulate below the control problem for model (1), (2) in which the boundary function $q \in H^s(\Sigma_\tau)$ where $s \in [0, 1/2]$ plays the role of control. We study the solvability of the optimal control problem and derive an optimality system describing the necessary conditions of extremum.

2. Function spaces. The preliminary results

As in [15, 16, 17, 18] we begin with describing the main functional spaces. Below we will use the Sobolev spaces $H^s(D)$, $s \in \mathbf{R}$, $H^0(D) \equiv L^2(D)$, where D denotes domain Ω, its boundary Σ or open nonempty subset $\Sigma_0 \subset \Sigma$. Corresponding spaces of vector–functions are denoted by $H^s(D)^3$ and $L^2(D)^3$. The inner products and norms in the spaces $H^s(D)$ and $H^s(D)^3$ are denoted by $(\cdot,\cdot)_{s,D}$ and $\|\cdot\|_{s,D}$. The inner products and norms in $L^2(\Omega)$ and $L^2(\Omega)^3$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|_\Omega$. By $H^s(\Sigma)$ or $H^s(\Sigma_0)$ at $s \geq 0$ we denote the usual trace spaces of the space $H^{s+1/2}(\Omega)$ over Σ or over proper (non empty) open subset Σ_0 of Σ. By $H^{-s}(\Sigma)$ and $H^{-s}(\Sigma_0)$ we denote the dual spaces of $H^s(\Sigma)$ and $H^s(\Sigma_0)$, respectively, in the case when Σ_0 is a surface without boundary. By $\langle\cdot,\cdot\rangle_{s,\Sigma}$, $\langle\cdot,\cdot\rangle_{s,\Sigma_0}$ we denote the duality pairing between $H^{-s}(\Sigma)$ and $H^{-s}(\Sigma)$, $H^{-s}(\Sigma_0)$ and $H^{-s}(\Sigma_0)$ respectively.

As in [20, 21] we assume that the following conditions to Ω are satisfied:

(i) Ω is a bounded domain in $\mathbf{R}^3$ and the boundary $\partial\Omega$ is the union of a finite number of disjoint closed C^2 surfaces, each surface having finite surface area;

(ii) Σ_τ is nonempty open subset of $\partial\Omega$ with $M+1$ disjoint nonempty open components $\{\sigma_0, \sigma_1, ..., \sigma_M\}$ and there is a positive d_0 such that dist $d(\sigma_i, \sigma_j) \geq d_0 > 0$ when $i \neq j$ and $M \geq 1$. The boundary of each σ_i is either empty or $C^{1,1}$ curve. We set $\Sigma_\nu = \partial\Omega \setminus \overline{\Sigma}_\tau$.

Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable compactly supported functions in Ω, $H_0^1(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, $V = \{\mathbf{v} \in H_0^1(\Omega)^3 : \operatorname{div}\mathbf{v} = 0\}$, $H^{-1}(\Omega)^3 = (H_0^1(\Omega)^3)^*$, $L_0^2(\Omega) = \{p \in L^2(\Omega) : (p, 1) = 0\}$, $H^1(\Omega, \Sigma_\tau) = \{\varphi \in H^1(\Omega) : \varphi|_{\Sigma_\tau} = 0\}$, $C_{\Sigma_\tau 0}(\overline{\Omega})^3 := \{\mathbf{v} \in C^0(\overline{\Omega})^3 : \mathbf{v}\cdot\mathbf{n}|_{\Sigma_\tau} = 0,\ \mathbf{v}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{0}\}$. In addition to the spaces introduced above we will use the spaces $H(\operatorname{div}, \Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \operatorname{div}\mathbf{v} \in L^2(\Omega)\}$, $H(\operatorname{curl}, \Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \operatorname{curl}\mathbf{v} \in L^2(\Omega)^3\}$, $H^0(\operatorname{curl}, \Omega) = \{\mathbf{v} \in H(\operatorname{curl}, \Omega) : \operatorname{curl}\mathbf{v} = \mathbf{0}\}$ and the space $H_{DC}(\Omega) = H(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega)$, equipped with the Hilbert norm $\|\mathbf{u}\|_{DC}^2 := \|\mathbf{u}\|_\Omega^2 + \|\operatorname{div}\mathbf{u}\|_\Omega^2 + \|\operatorname{curl}\mathbf{u}\|_\Omega^2$.

Let $H_{DC\Sigma_\tau}(\Omega)$ be the closure of $C_{\Sigma_\tau 0}(\overline{\Omega})^3 \cap H^1(\Omega)^3$ with respect to the norm $\|\cdot\|_{DC}$. Let $\mathcal{H}_{\Sigma_\tau}(\Omega) = \{\mathbf{h} \in L^2(\Omega)^3 : \operatorname{div}\mathbf{h} = 0,\ \operatorname{curl}\mathbf{h} = \mathbf{0} \text{ in } \Omega,\ \mathbf{h}\cdot\mathbf{n}|_{\Sigma_\tau} = 0,\ \mathbf{h}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{0}\}$, $\mathcal{H}_{\Sigma_\nu}(\Omega) = \{\mathbf{h} \in L^2(\Omega)^3 : \operatorname{div}\mathbf{h} = 0,\ \operatorname{curl}\mathbf{h} = \mathbf{0} \text{ in } \Omega,\ \mathbf{h}\cdot\mathbf{n}|_{\Sigma_\nu} = 0,\ \mathbf{h}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{0}\}$, $V_{\Sigma_\tau}(\Omega) = \{\mathbf{v} \in H_{DC\Sigma_\tau}(\Omega) : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\} \cap \mathcal{H}_{\Sigma_\tau}(\Omega)^\perp$, $H_s = H_0^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)$.

A number of properties of the function spaces introduced above has been proved in [20, 21]. We formulate these properties as the following theorem.

Theorem 2.1. *We assume that conditions (i), (ii) hold. Then:*

1) the spaces $\mathcal{H}_{\Sigma_\tau}(\Omega)$ and $\mathcal{H}_{\Sigma_\nu}(\Omega)$ are finite dimensional;

2) $H_{DC\Sigma_\tau}(\Omega) \subset H^1(\Omega)^3$ and the norm $\|\cdot\|_{DC}$ is equivalent to the norm $\|\cdot\|_{1,\Omega}$;

3) for all $\mathbf{h} \in V_{\Sigma_\tau}(\Omega)$ the coercitivity inequality $\|\operatorname{curl}\mathbf{h}\|^2 \geq \delta_1\|\mathbf{h}\|_{1,\Omega}^2$ holds where constant δ_1 depends on Ω and Σ_τ;

4) the orthogonal decomposition $L^2(\Omega)^3 = \nabla H^1(\Omega, \Sigma_\tau) \oplus \operatorname{curl} H_{DC\Sigma_\tau}(\Omega) \oplus \mathcal{H}_{\Sigma_\nu}(\Omega)$ holds and $\operatorname{curl} H_{DC\Sigma_\tau}(\Omega) \equiv \operatorname{curl} V_{\Sigma_\tau}(\Omega)$.

Along with spaces $H_{DC}(\Omega)$ and $H^0(\operatorname{curl}, \Omega)$ we will use their subspaces

$$\mathcal{H}_{\operatorname{div}}^{s+1/2}(\Omega) := \{\mathbf{h} \in H^{s+1/2}(\Omega) : \operatorname{curl}\mathbf{h} \in L^2(\Omega)^3, \operatorname{div}\mathbf{h} = 0\} \cap \mathcal{H}_{\Sigma_\tau}(\Omega)^\perp,$$

$$\mathcal{H}^{s+1/2}_{\rm div}(\Omega) := \{h \in \mathcal{H}^{s+1/2}_{\rm div}(\Omega) : \mathbf{h}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{0}\},$$
$$H^0_{\Sigma_\tau}({\rm curl},\Omega) := \{\mathbf{e}\in H^0({\rm curl},\Omega) : \mathbf{e}\times\mathbf{n}|_{\Sigma_\tau} \in L^2_T(\Sigma_\tau)\}$$

equipped with natural norms

$$\|\mathbf{h}\|_{\mathcal{H}^{s+1/2}_{\rm div}(\Omega)} = \|\mathbf{h}\|_{s+1/2,\Omega} + \|{\rm curl}\,\mathbf{h}\|_\Omega,\ \|\mathbf{e}\|_{H^0_{\Sigma_\tau}({\rm curl},\Omega)} := \|\mathbf{e}\|_\Omega + \|\mathbf{e}\times\mathbf{n}\|_{\Sigma_\tau}.$$

The spaces $\mathcal{H}^{s+1/2}_{\rm div}(\Omega,\Sigma_\nu)$ and $H^0_{\Sigma_\tau}({\rm curl},\Omega)$ will be used below for describing properties of the magnetic and electric fields, respectively.

The following technical lemma holds (see for details [6, 20, 21]).

Lemma 2.1. *Under condition (i) there exist constants $\delta_i = \delta_i(\Omega) > 0$ and $\gamma_i = \gamma_i(\Omega) > 0$, $i = 0,1$, $\beta > 0$, depending on Ω such that*

$$(\nabla\mathbf{v},\nabla\mathbf{v}) \geq \delta_0\|\mathbf{v}\|^2_{1,\Omega}\ \ \forall\mathbf{v}\in H^1_0(\Omega)^3,\ ({\rm curl}\,\mathbf{\Psi},{\rm curl}\,\mathbf{\Psi}) \geq \delta_1\|\mathbf{\Psi}\|^2_{1,\Omega}\ \forall\mathbf{\Psi}\in V_{\Sigma_\tau}(\Omega),$$
$$|((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})| \leq \gamma_0\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega},$$
$$|({\rm curl}\mathbf{u}\times\mathbf{v},\mathbf{w})| \leq \gamma_1\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{s+1/2,\Omega}\|\mathbf{w}\|_{1,\Omega}\ \forall\mathbf{u},\mathbf{w}\in H^1(\Omega)^3, \mathbf{v}\in H^{s+1/2}(\Omega)^3,$$
$$\sup_{\mathbf{v}\in H^1_0(\Omega)^3,\mathbf{v}\neq 0} -({\rm div}\mathbf{v},p)/\|\mathbf{v}\|_{1,\Omega} \geq \beta\|p\|_\Omega\ \ \forall p\in L^2_0(\Omega). \tag{4}$$

Furthermore, the following identities hold true:

$$((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) = -((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})\ \forall\mathbf{u}\in H^1(\Omega)^3 : {\rm div}\,\mathbf{u} = 0,\ (\mathbf{v},\mathbf{w})\in H^1_0(\Omega)^3\times H^1(\Omega)^3,$$

$$({\rm rot}\,\mathbf{\Psi}\times\mathbf{H},\mathbf{u}) = (\mathbf{H}\times\mathbf{u},{\rm rot}\,\mathbf{\Psi}) = -({\rm rot}\,\mathbf{\Psi}\times\mathbf{u},\mathbf{H})\ \forall\mathbf{\Psi},\mathbf{u}\in H^1(\Omega)^3, \mathbf{H}\in H^{s+1/2}(\Omega)^3.$$

Let the following conditions hold in addition to (i), (ii):

(iii) $\mathbf{f}\in H^{-1}(\Omega)^3$, $\mathbf{j}\in L^2(\Omega)^3$, $\mathbf{k}\in(\gamma_\tau|_{\Sigma_\tau})H^0_{\Sigma_\tau}({\rm curl},\Omega)$.

In what follows we will deal with a weak form of Problem 1. To this end we multiply the first equation in (1) by $\mathbf{v}\in H^1_0(\Omega)^3$, the first equation in (2) by curl $\mathbf{\Psi}$ where $\mathbf{\Psi}\in V_{\Sigma_\tau}(\Omega)$, integrate over Ω, apply Green's formulas, add the obtained results and make use of the identity [14]

$$(\mathbf{E},{\rm curl}\,\mathbf{\Psi}) = \int_{\Sigma_\tau}(\mathbf{E}\times\mathbf{n}|_{\Sigma_\tau})\cdot\mathbf{\Psi}_T d\sigma = (\mathbf{k},\mathbf{\Psi}_T)_{\Sigma_\tau} = (\mathbf{E}_0,{\rm curl}\,\mathbf{\Psi})\ \forall\mathbf{\Psi}\in V_{\Sigma_\tau(\Omega)}. \tag{5}$$

As a result we arrive at the weak form of Problem 1:

$$\nu(\nabla\mathbf{u},\nabla\mathbf{v}) + \nu_1({\rm curl}\mathbf{H},{\rm curl}\mathbf{\Psi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}) + æ[({\rm curl}\mathbf{\Psi}\times\mathbf{H},\mathbf{u}) - ({\rm curl}\mathbf{H}\times\mathbf{H},\mathbf{v})]-$$

$$-({\rm div}\mathbf{v},p) = \langle\mathbf{f},\mathbf{v}\rangle + (\nu_1\mathbf{j},{\rm curl}\,\mathbf{\Psi}) + \rho_0^{-1}(\mathbf{k},\mathbf{\Psi})_{\Sigma_\tau}\ \forall(\mathbf{v},\mathbf{\Psi})\in H_s, \tag{6}$$
$${\rm div}\mathbf{u} = 0 \text{ in } \Omega,\ \ \mathbf{H}\cdot\mathbf{n} = q \text{ on } \Sigma_\tau. \tag{7}$$

We will refer below to any triple $(\mathbf{u},\mathbf{H},p)\in H^1_0(\Omega)^3\times\mathcal{H}^{s+1/2}_{\rm div}(\Omega,\Sigma_\nu)\times L^2_0(\Omega)$ satisfying (6), (7) as a weak solution to Problem 1.

The identity (6) does not contain electric field $\mathbf{E}\in H^0_{\Sigma_\tau}({\rm curl},\Omega)$ which was eliminated with the help of (5). However, using a condition on a boundary vector $\mathbf{k}$ in (iii) vector $\mathbf{E}$ can be reconstructed uniquely from triple $(\mathbf{u},\mathbf{H},p)$ satisfying (6) so that the first equation in (2) holds a.e. in Ω (see details in [15]).

Let in addition to (i)–(iii) the following condition holds:

(iv) $\overline{\Sigma}_\tau\cap\overline{\Sigma}_\nu = \emptyset$.

The following theorem from [15] provides the existence of a weak solution to Problem 1:

Theorem 2.2. *Under assumptions (i)–(iv) for any* $q \in H^s(\Sigma_\tau)$, $s \in [0, 1/2]$, *there exists a weak solution* $(\mathbf{u}, \mathbf{H}, p)$ *to Problem 1 and the following estimates hold:*

$$\|\mathbf{u}\|_{1,\Omega} \leq M_{\mathbf{u}},\ \|\mathbf{H}\|_{\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)} \leq M_{\mathbf{H}},\ \|p\|_\Omega \leq M_p, \tag{8}$$

wheree $M_{\mathbf{u}}, M_{\mathbf{H}}, M_p$ *are continuous nondecreasing functions of* $\|\mathbf{f}\|_{-1,\Omega}$, $\|\mathbf{j}\|_\Omega, \|\mathbf{k}\|_{\Sigma_\tau}, \|q\|_{s,\Sigma_\tau}$.

If, besides, functions $\mathbf{f}, \mathbf{j}, \mathbf{k}, q$ *are small (or "viscosities"* ν, ν_m *are large) in the sense*

$$\gamma_0 M_{\mathbf{u}} + \gamma_1(\sqrt{æ}/2)M_{\mathbf{H}} < \delta_0\nu,\ \ \gamma_1 M_{\mathbf{u}} + \gamma_1(\sqrt{æ}/2)M_{\mathbf{H}} < \delta_1\nu_m, \tag{9}$$

where constants $\delta_0, \delta_1, \gamma_0, \gamma_1$ *are introduced in Lemma 2.1, then the weak solution is unique.*

The proof of Problem 1' solvability essentially uses the following Lemma [22]:

Lemma 2.2. *Let under assumptions (i), (ii) and (iv)* $q \in H^s(\Sigma_\tau)$ *at* $s \in [0, 1/2]$. *Then there exists a unique function* $\mathbf{H}_0 \in \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega, \Sigma_\nu)$ *such that* $\mathrm{curl}\,\mathbf{H}_0 = \mathbf{0}$, $\mathrm{div}\,\mathbf{H}_0 = 0$ *in* Ω, $\mathbf{H}_0 \cdot \mathbf{n} = q$ *on* Σ_τ, *and* $\|\mathbf{H}_0\|_{s+1/2,\Omega} \leq C_\Sigma \|q\|_{s,\Sigma_\tau}$, *where constant* C_Σ *does not depend on* q.

3. Boundary control problem

Let us formulate control problem for Problem 1. For this purpose we divide the set of all data of Problem 1 into two groups: the first contains the functions $\mathbf{f}, \mathbf{j}$ and $\mathbf{k}$ while the second one contains the function q. We assume that q can be changed in some subset K_s where

(j) $K_s \subset H^s(\Sigma_\tau)$, $0 \leq s \leq 1/2$, is nonempty convex closed set.

Setting $X_s = H^1_0(\Omega)^3 \times \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega, \Sigma_\nu) \times L^2_0(\Omega)$ and $Y_s = H^{-1}(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)^* \times L^2_0(\Omega) \times H^s(\Sigma_\tau)$, $\mathbf{x} = (\mathbf{u}, \mathbf{H}, p) \in X_s$ we introduce an operator $F \equiv (F_1, F_2, F_3) : X_s \times K_s \to Y_s$ by

$$\langle F_1(\mathbf{x}), (\mathbf{v}, \boldsymbol{\Psi})\rangle = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \nu_1(\mathrm{curl}\mathbf{H}, \mathrm{curl}\boldsymbol{\Psi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) - (\mathrm{div}\mathbf{v}, p) +$$

$$+æ[(\mathrm{curl}\boldsymbol{\Psi} \times \mathbf{H}, \mathbf{u}) - (\mathrm{curl}\mathbf{H} \times \mathbf{H}, \mathbf{v})] - \langle \mathbf{f}, \mathbf{v}\rangle - (\nu_1 \mathbf{j}, \mathrm{curl}\,\boldsymbol{\Psi}) - \rho_0^{-1}(\mathbf{k}, \boldsymbol{\Psi}_T)_{\Sigma_\tau}\ \forall(\mathbf{v}, \boldsymbol{\Psi}) \in H_s,$$

$$\langle F_2(\mathbf{x}, r)\rangle = -(\mathrm{div}\mathbf{u}, r)\ \forall r \in L^2_0(\Omega),\ F_3(\mathbf{x}, q) = \mathbf{H}\cdot\mathbf{n} - q \in H^s(\Sigma_\tau)$$

and rewrite the weak form (6), (7) of Problem 1 in the form of the operator equation

$$F(\mathbf{x}, q) = F(\mathbf{u}, \mathbf{H}, p, q) = 0. \tag{10}$$

Let $I : X_s \to \mathbf{R}$ be a weakly lower semicontinuous cost functional. Consider the following optimal control problem:

$$J(\mathbf{x}, q) = (\mu_0/2)I(\mathbf{x}) + (\mu_1/2)\|q\|^2_{s,\Sigma_\tau} \to \inf,\ F(\mathbf{x}, q) = 0,\ (\mathbf{x}, q) \in X_s \times K_s. \tag{11}$$

Here $\mu_0 > 0$ and $\mu_1 \geq 0$ are nonnegative parameters.

As the cost functional we choose one of the following:

$$I_1(\mathbf{v}) = \|\mathbf{v} - \mathbf{v}_d\|^2_Q,\ I_2(\mathbf{H}) = \|\mathbf{H} - \mathbf{H}_d\|^2_Q,\ I_3(p) = \|p - p_d\|^2_Q. \tag{12}$$

Here the function $\mathbf{v}_d \in L^2(Q)^3$ denotes some desired velocity field given in a subdomain $Q \subset \Omega$. Functions $\mathbf{H}_d \in L^2(Q)^3$ and $p_d \in L^2(Q)$ have similar sense for the magnetic field or pressure. The set of possible pairs for problem (11) is denoted by $Z_{ad} = \{(\mathbf{x}, q) \in X_s \times K_s : F(\mathbf{x}, q) = 0, J(\mathbf{x}, q) < \infty\}$.

Let in addition to (i) the following conditions hold:

(jj) $\mu_0 > 0$, $\mu_1 \geq 0$ and K_s is bounded set, or $\mu_0 > 0$, $\mu_1 > 0$ and I is bounded below.

By analogy with [18], the following theorem concerning with solvability of (11) is proved.

Theorem 3.1. *Let under assumptions (i)–(iv) and (j), (jj),* $I : X_s \to \mathbf{R}$ *be a weakly lower semicontinuous functional, where* $s \geq 0$, *and set* Z_{ad} *be nonempty. Then problem (11) has at least one solution* $(\mathbf{x}, q) \in X_s \times K_s$.

4. Derivation of the optimality system

The following stage of study of problem (11) is a derivation of an optimality system describing necessary conditions of extremum.

Let $X_s^* = H^{-1}(\Omega)^3 \times \mathcal{H}_{\rm div}^{s+1/2}(\Omega, \Sigma_\nu)^* \times L_0^2(\Omega)$ and $Y_s^* = H_0^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega) \times L_0^2(\Omega) \times H^s(\Sigma_\tau)^*$ be the duals of spaces X_s and Y_s where $s > 0$. It is easy to show that the Fréchet partial derivative with respect to $\mathbf{x}$ from operator $F : X_s \to Y_s$ at any point $(\hat{\mathbf{x}}, \hat{q}) \equiv (\hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{p}, \hat{q}) \in X_s \times K_s$ is a linear continuous operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{q}) : X_s \to Y_s$ that maps each element $(\mathbf{w}, \mathbf{h}, r) \in X_s$ to an element $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{q})(\mathbf{w}, \mathbf{h}, r) = (\hat{y}_1, \hat{y}_2, \hat{y}_3) \in Y_s$ where the elements $\hat{y}_1 \in H_s^*$, $\hat{y}_2 \in L_0^2(\Omega)$, $\hat{y}_3 \in H^s(\Sigma_\tau)$ are defined by triples $(\hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{p})$ and $(\mathbf{w}, \mathbf{h}, r)$ from relations

$$\langle \hat{y}_1, (\mathbf{v}, \mathbf{\Psi})\rangle = \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + \nu_1(\mathrm{curl}\,\mathbf{h}, \mathrm{curl}\mathbf{\Psi}) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v}) + ((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \mathbf{v}) - (\mathrm{div}\,\mathbf{v}, r) +$$

$$+æ[(\mathrm{curl}\mathbf{\Psi} \times \mathbf{h}, \hat{\mathbf{u}}) + (\mathrm{curl}\mathbf{\Psi} \times \hat{\mathbf{H}}, \mathbf{w}) - (\mathrm{curl}\,\mathbf{h} \times \hat{\mathbf{H}}, \mathbf{v}) - (\mathrm{curl}\,\hat{\mathbf{H}} \times \mathbf{h}, \mathbf{v})] \ \forall (\mathbf{v}, \mathbf{\Psi}) \in H_s,$$

$$\langle \hat{y}_2, r\rangle = -(\mathrm{div}\,\mathbf{w}, r) \ \forall r \in L_0^2(\Omega), \ \hat{y}_3 = \mathbf{h} \cdot \mathbf{n}|_{\Sigma_\tau}. \tag{13}$$

By $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{q})^* : Y_s^* \to X_s^*$ we denote operator adjoint to $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{q})$. Following to the general theory of smooth-convex extremum problems [23] we introduce an element $\mathbf{y}^* = ((\xi, \eta), \sigma, \zeta) \in Y_s^*$ to which we will refer as adjoint state and define the Lagrangian $L : X_s \times K_s \times \mathbf{R} \times Y_s^* \to \mathbf{R}$ by formula $L(\mathbf{x}, q, \lambda_0, \mathbf{y}^*) = \lambda_0 J(\mathbf{x}, q) + \langle \mathbf{y}^*, F(\mathbf{x}, q)\rangle_{Y_s^* \times Y_s} \equiv \lambda_0 J(\mathbf{x}, q) + \langle F_1(\mathbf{x}, q), (\xi, \eta)\rangle_{H_s^* \times H_s} + (F_2(\mathbf{x}, q), \sigma) + \langle \zeta, F_3(\mathbf{x}, q)\rangle_{s, \Sigma_\tau}$.

The following theorem holds.

Theorem 4.1. *Let under assumptions (i)–(iv) and (j), (jj) at* $s \in [0, 1/2]$*, the element* $(\hat{\mathbf{x}}, \hat{q}) \equiv (\hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{p}, \hat{q}) \in X_s \times K_s$ *be a local minimizer in problem (11) and let the cost functional* I *be continuously Fréchet differentiable with respect to state* $\mathbf{x}$ *in point* $\hat{\mathbf{x}}$*. Then there exists a nonzero Lagrange multiplier* $(\lambda_0, \mathbf{y}^*) = (\lambda_0, \xi, \eta, \sigma, \zeta) \in \mathbf{R}^+ \times Y_s^*$ *such that the Euler–Lagrange equation takes place* $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{q})^* \mathbf{y}^* = -\lambda_0 J'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{q})$ *in* X_s^**, which is equivalent to relations*

$$\nu(\nabla \mathbf{w}, \nabla \xi) + \nu_1(\mathrm{curl}\mathbf{h}, \mathrm{curl}\eta) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \xi) + ((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \xi) - (\mathrm{div}\mathbf{w}, \sigma) +$$

$$+æ[(\mathrm{curl}\eta \times \hat{\mathbf{H}}, \mathbf{w}) + (\mathrm{curl}\eta \times \mathbf{h}, \hat{\mathbf{u}})] - æ[(\mathrm{curl}\hat{\mathbf{H}} \times \mathbf{h}, \xi) + (\mathrm{curl}\mathbf{h} \times \hat{\mathbf{H}}, \xi)] + \langle \zeta, \mathbf{h} \cdot \mathbf{n}\rangle_{s, \Sigma_\tau} =$$

$$= -\lambda_0(\mu_0/2)(\langle I'_{\mathbf{u}}(\hat{\mathbf{x}}), \mathbf{w}\rangle + \langle I'_{\mathbf{H}}(\hat{\mathbf{x}}), \mathbf{h}\rangle) \ \forall (\mathbf{w}, \mathbf{h}) \in H_0^1(\Omega)^3 \times \mathcal{H}_{\rm div}^{s+1/2}(\Omega, \Sigma_\nu), \tag{14}$$

$$(\mathrm{div}\xi, r) = \lambda_0(\mu_0/2)(I'_p(\hat{\mathbf{x}}), r) \ \ \forall r \in L_0^2(\Omega), \tag{15}$$

and minimum principle $\mathcal{L}(\hat{\mathbf{x}}, \hat{q}, \lambda_0, \mathbf{y}^*) \le \mathcal{L}(\hat{\mathbf{x}}, q, \lambda_0, \mathbf{y}^*) \ \forall q \in K_s$ *holds, which is equivalent to*

$$\lambda_0 \mu_1(\hat{q}, q - \hat{q})_{s, \Sigma_\tau} - \langle \zeta, q - \hat{q}\rangle_{s, \Sigma_\tau} \ge 0 \ \forall q \in K_s. \tag{16}$$

If, besides, (9) holds for all $q \in K_s$*, then any nontrivial Lagrange multiplier* $(\lambda_0, \mathbf{y}^*)$*, satisfying (14)–(16) is regular, i.e. it has the form* $(1, \mathbf{y}^*)$ *and is determined uniquely for given pair* $(\hat{\mathbf{x}}, \hat{q})$*.*

Proof. According to [23, p. 79], to prove Theorem 4.1, it suffices to show that $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{q}) : X_s \to Y_s$ is a Fredholm operator. By virtue of (13), the operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{q}) : X_s \to Y_s$ can be presented as $F'_{\mathbf{x}} = \Phi + \hat{\Phi} \equiv (\Phi_1, \Phi_2, \Phi_3) + (\hat{\Phi}_1, 0, 0)$ where $\Phi_2(\mathbf{x}) = \mathrm{div}\,\mathbf{w}$, $\Phi_3(\mathbf{x}) = \mathbf{h} \cdot \mathbf{n}|_{\Sigma_\tau}$ and operators $\Phi_1, \hat{\Phi}_1 : X_s \to H_s^* := H^{-1}(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)^*$ act by formulae

$$\langle \Phi_1(\mathbf{w}, \mathbf{h}, r), (\mathbf{v}, \mathbf{\Psi})\rangle = \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + \nu_1(\mathrm{curl}\,\mathbf{h}, \mathrm{curl}\mathbf{\Psi}) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v}) +$$

$$+æ[(\mathrm{curl}\mathbf{\Psi} \times \hat{\mathbf{H}}, \mathbf{w}) - (\mathrm{curl}\,\mathbf{h} \times \hat{\mathbf{H}}, \mathbf{v})],$$

$$\langle \hat{\Phi}_1(\mathbf{w}, \mathbf{h}, r), (\mathbf{v}, \mathbf{\Psi})\rangle = ((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \mathbf{v}) + æ[(\mathrm{curl}\,\mathbf{\Psi} \times \mathbf{h}, \hat{\mathbf{u}}) - (\mathrm{curl}\,\hat{\mathbf{H}} \times \mathbf{h}, \mathbf{v})].$$

Let us show that the operator $\Phi : X_s \to Y_s$ is an isomorphism. For this purpose it is enough to show that for any triple $(\mathbf{F}, \theta, q) \in H_s^* \times \times L_0^2(\Omega) \times H^s(\Sigma_\tau)$ there exists a unique solution $(\mathbf{w}, \mathbf{h}, p) \in X_s$ of problem

$$\nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + \nu_1(\operatorname{curl} \mathbf{h}, \operatorname{curl} \boldsymbol{\Psi}) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p)+$$

$$+æ[(\operatorname{curl} \boldsymbol{\Psi} \times \hat{\mathbf{H}}, \mathbf{w}) - (\operatorname{curl} \mathbf{h} \times \hat{\mathbf{H}}, \mathbf{v})] = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi}) \rangle \ \ \forall (\mathbf{v}, \boldsymbol{\Psi}) \in H_s, \tag{17}$$

$$\operatorname{div} \mathbf{w} = \theta \text{ in } \Omega, \ \ \mathbf{h} \cdot \mathbf{n} = q \text{ on } \Sigma_\tau. \tag{18}$$

that continuously depends on $(\mathbf{F}, q, \theta)$. The existence of the solution of (17), (18) and it's continuous dependence on $(\mathbf{F}, q, \theta)$ is proved using the scheme proposed in [6]. In order to prove the uniqueness let us assume that there exists two solutions $(\mathbf{w}_i, \mathbf{h}_i, p_i) \in X_s$, $i = 1, 2$, of problem (17), (18). Then the difference $\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2$, $\mathbf{h} = \mathbf{h}_1 - \mathbf{h}_2$ and $p = p_1 - p_2$ belongs to $V \times V_{\Sigma_\tau}(\Omega) \times L_0^2(\Omega)$ and satisfies

$$\nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + \nu_1(\operatorname{curl} \mathbf{h}, \operatorname{curl} \boldsymbol{\Psi}) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v})+$$

$$+æ[(\operatorname{curl} \boldsymbol{\Psi} \times \hat{\mathbf{H}}, \mathbf{w}) - (\operatorname{curl} \mathbf{h} \times \hat{\mathbf{H}}, \mathbf{v})] - (\operatorname{div} \mathbf{v}, p) = 0 \ \ \forall (\mathbf{v}, \boldsymbol{\Psi}) \in H_s, \tag{19}$$

$$\operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \ \ \mathbf{h} \cdot \mathbf{n} = 0 \text{ on } \Sigma_\tau. \tag{20}$$

Setting $\mathbf{v} = \mathbf{w}$, $\boldsymbol{\Psi} = \mathbf{h}$ in (19), we arrive at the relation $\nu(\nabla \mathbf{w}, \nabla \mathbf{w}) + \nu_1(\operatorname{curl} \mathbf{h}, \operatorname{curl} \mathbf{h}) = 0$. By virtue of Lemma 2.1 it implies that $\mathbf{w} = \mathbf{0}$ and $\mathbf{h} = \mathbf{0}$ or $\mathbf{w}_1 = \mathbf{w}_2$ and $\mathbf{h}_1 = \mathbf{h}_2$ in Ω. Then, from (19), taking into account (4), we derive that $p_1 = p_2$ in Ω. Thus, we proved that the operator $\Phi : X_s \to Y_s$ is surjective and invertible. Then, by Banach theorem, operator Φ is an isomorphizm. Finally, from compactness of embeddings $\mathcal{H}_{\operatorname{div}}^{1/2+s}(\Omega) \subset L^3(\Omega)^3$ at $s > 0$, and $H^1(\Omega)^3 \subset L^4(\Omega)^3$ and estimates of Lemma 2.1 follows the continuity and compactness of the operator $\hat{\Phi}$.

Let us prove the regularity of the multiplier $(\lambda_0, \mathbf{y}^*)$, i.e. that $\lambda_0 \neq 0$. To this end we denote by $\mathbf{y}^* \equiv (\xi, \eta, \sigma, \zeta)$ an arbitrary solution of system (14)–(16) at $\lambda_0 = 0$. Setting $\mathbf{w} = \xi$, $\mathbf{h} = \eta$ and $r = \sigma$ in this system we come to relation

$$\nu(\nabla \xi, \nabla \xi) + \nu_1(\operatorname{curl} \eta, \operatorname{curl} \eta) + ((\xi \cdot \nabla)\hat{\mathbf{u}}, \xi) + æ[(\operatorname{curl} \eta \times \eta, \hat{\mathbf{u}}) - (\operatorname{curl} \hat{\mathbf{H}} \times \eta, \xi)] = 0. \tag{21}$$

Arguing as in [7] one can easily prove using (21) that $\xi = \mathbf{0}$ and $\eta = \mathbf{0}$ in Ω under condition (9). In this case, from (14), (15) follows that

$$-(\operatorname{div} \mathbf{w}, \sigma) + \langle \zeta, \mathbf{h} \cdot \mathbf{n} \rangle_{s, \Sigma_\tau} = 0 \ \ \forall \mathbf{w} \in H_0^1(\Omega)^3, \ \mathbf{h} \in \mathcal{H}_{\operatorname{div}}^{1/2+s}(\Omega, \Sigma_\nu). \tag{22}$$

Choosing as $\mathbf{h}$ an arbitrary function from $V_{\Sigma_\tau}(\Omega) \subset \mathcal{H}_{\operatorname{div}}^{1/2+s}(\Omega, \Sigma_\nu)$ we derive from (22) that $(\operatorname{div} \mathbf{w}, \sigma) = 0$ for all $\mathbf{w} \in H_0^1(\Omega)^3$. By (4) this identity means that $\sigma = 0$ a.e. in Ω. Then (22) transforms to $\langle \zeta, \mathbf{h} \cdot \mathbf{n} \rangle_{s, \Sigma_\tau} = 0$ for all $\mathbf{h} \in \mathcal{H}_{\operatorname{div}}^{1/2+s}(\Omega, \Sigma_\nu)$. This means that $\zeta = 0$ in $H^s(\Sigma_\tau)^*$ and therefore $\mathbf{y}^* = \mathbf{0}$. Uniqueness of the regular Lagrange multiplier $(1, \mathbf{y}^*)$ under conditions (9) follows from Fredholm property of the operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{q})$. ■

Relations (6), (7) for the main state $(\mathbf{u}, \mathbf{H}, p)$ together with identities (14), (15) for the adjoint state $(\xi, \eta, \sigma, \zeta)$ and variational inequality (16) for control $\hat{q}$ form an optimality system for problem (11). We emphasize that the optimality system plays an important role in the study of the optimal solutions' properties. In particular, based on the analysis of the optimality system one can derive stability estimates of the optimal solutions (see, e.g., [8]) The authors propose to do this in future paper for all three cost functionals defined in (12).

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