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On radicals and coordinate partial orders

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Abstract. The solution of equations and systems of equations over real, complex, rational and integer numbers is a classic topic of research in various areas of mathematics for several thousand years. In the last 20 years, the so-called universal algebraic geometry has been actively developed, in which systems of equations over arbitrary algebraic systems are researched. Many practically important problems on finite graphs, finite fields, and finite orders can be formulated as problems related to solving systems of equations over these systems, which leads to the need to develop algebraic geometry. Many modern models of informational defence represents by graphs and partial orders (posets). This article presents polynomial algorithms for constructing radical and coordinate partial order of systems of equations over finite partial orders in a language without constants.

1. Introduction

The solution of equations and systems of equations over real, complex, rational and integer numbers is a classic topic of research in various areas of mathematics for several thousand years. Classical algebraic geometry researches the solution sets of algebraic equations over the fields of real and complex numbers. In the area of diophantine geometry and diophantine analysis, solutions of algebraic equations over integer and rational numbers are studied. In the last 20 years, the so-called universal algebraic geometry has been actively developed [1, 2, 3, 4, 5], in which systems of equations over arbitrary algebraic systems are researched. The equations are understood as atomic formulas of the language of an algebraic system. Many concepts and algorithms of classical algebraic geometry are mapped to arbitrary algebraic systems. The main task of algebraic geometry is the search for a set of solutions of a system of equations. This set can be specified through a general solution of the system of equations. In linear algebra, the general solution is given by a linear combination of free variables. A particular solution of the system is obtained by substituting any real or rational numbers into free variables. The structure of the general solution of a system of equations over an arbitrary algebraic structure is called the coordinate algebra for system of equations over algebraic structure. This is a rather important object of algebraic geometry, through which you can get any particular solution of the system of equations over the algebraic structure for which the system is defined. The coordinate algebra is determined through a congruence, which is defined by the radical of the system of equations. In essence, the construction of the radical of a system of equations is the key to the search for coordinate algebra.

In the 20th century, in connection with the rapid development of computer technology and applied mathematics, the study of various finite combinatorial and algebraic objects came to the fore. First of all, these are finite graphs, finite fields, finite orders (partially ordered sets).



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Classical approaches to the study of finite algebraic systems are algebraic and combinatorial. A new approach to the study of these objects – logical and model-theoretic – was born within the area of the universal algebraic geometry [10]. Many practically important problems on finite graphs, finite fields, and finite orders can be formulated as problems related to solving systems of equations over these systems, which leads to the need to develop algebraic geometry. In universal algebraic geometry, there is an algorithmic aspect – pointing out an algorithm for constructing an important object or a property of this object, whether it is solution set of system of equations (an algebraic set of a system), radical, coordinate algebra, or equationally Noetherian property of algebraic structure. From the practice point of view, the problems of solvability and computational complexity of algorithms for constructing objects or checking properties are extremely important. Despite the fact that many algorithmic problems of constructing radical or checking the solvability of system of equations are computationally complex, there are effective algorithms, such as the Gauss algorithm in classical algebraic geometry. This article presents polynomial algorithms for constructing radical and coordinate partial order of systems of equations over finite partial orders in a language without constants.

2. Preliminaries

In this section, we recall the basic definitions from partial orders theory[6], graph theory[8] and universal algebraic geometry[10].

Partially ordered set (partial order) is algebraic system $\mathcal{P} = \langle P | \leq^{(2)}, A \rangle$, there $\leq$ is predicate symbol of the order relation and A is the set of constant symbols on which 3 axioms are fulfilled:

- (i) $\forall p \in P \ p \leq p$ (*reflexivity*);
- (ii) $\forall p_1, p_2 \in P \ p_1 \leq p_2 \wedge p_2 \leq p_1 \rightarrow p_1 = p_2$ (*antisymmetry*);
- (iii) $\forall p_1, p_2, p_3 \in P \ p_1 \leq p_2 \wedge p_2 \leq p_3 \rightarrow p_1 \leq p_3$ (*transitivity*).

Further, we will consider finite partial orders in the language L without constants. Also, the set of variables will be denoted as $X_n = \{x_1, \dots, x_n\}$.

Elements x and y of partial order $\mathcal{P}$ are called *comparable* if either $x \leq y$ or $y \leq x$. Otherwise, elements are called *incomparable*.

Term in L language of variables X_n is any constant in language and any variable in X_n . We say that ϕ is an *Atomic formula* in language L of variables X_n if ϕ is either

- (i) $t_1 = t_2$, where t_1 and t_2 are terms;
- (ii) $t_1 \leq t_2$, where t_1 and t_2 are terms;

An *equation* in the language L of variables X_n we will call any atomic formula in the language L . In the language of partial order L without constants there will be only two types of equations:

- (i) $x_i = x_j$, where $x_i, x_j \in X_n$,
- (ii) $x_i \leq x_j$, where $x_i, x_j \in X_n$.

The equations of each of the types will be denoted respectively by $S_{x=x}$ and $S_{x \leq x}$. Also, the full set of equations in the language L of variables X_n is denoted as $At_L(X_n)$.

The *system of equations* $S(X_n)$ of variables X_n in the language L is an arbitrary set of equations in the language L of variables X_n .

We introduce the types of closed systems of equations in accordance with the axioms of partially ordered sets. System $S(X_n)$ is *transitively closed* if with every equations $x_i \leq x_j$ and $x_j \leq x_k$ it contains equation $x_i \leq x_k$. System of equations is called *antisymmetrically closed* if with every equations $x_i \leq x_j$ and $x_j \leq x_i$ it contains equations $x_i = x_j$ and $x_j = x_i$. System of equations is called *reflexively closed* if with every system's variable $\forall x_i \in X_n$ it contains equation $x_i \leq x_i$.

Set

$$P^n = \{(p_1, \dots, p_n) \mid p_i \in P\}$$

is called *affine n-dimensional space* over poset $\mathcal{P}$ and it's elements are *points*.

Let $S(X_n)$ be a system of equations of the language L of poset $\mathcal{P}$ in variables X_n . Set of all solutions of the system S in affine n-dimensional space P^n is denoted $V(S)$,

$$V(S) = \{(p_1, \dots, p_n) \in P^n \mid \mathcal{P} \models \varphi(p_1, \dots, p_n) \forall \varphi \in S\},$$

and is called *algebraic set* over poset $\mathcal{P}$ defined by system S .

The system S is called *consistent* over $\mathcal{P}$ if $V(S) \neq \emptyset$. Otherwise it is called *inconsistent*.

The *radical* of the system of equations $S(X_n)$ over poset $\mathcal{P}$ is the following subset of the set $At_L(X_n)$:

$$Rad_{\mathcal{P}}(S) = \{\varphi \in At_L(X_n) \mid \mathcal{P} \models \varphi(p_1, \dots, p_n) \forall (p_1, \dots, p_n) \in V(S)\}.$$

The radical of system $S(X_n)$ defines *coordinate partially ordered set*. This object represents the general solution of the system of equations. The coordinate partial order of system $S(X_n)$ of the language L is the partial order $\Gamma_{\mathcal{P}}(S)$, which is the factor system $\langle X_n \mid \leq \rangle / Rad(S(X_n))$, where ratios on variables are given by $Rad(S(X_n))$. More general definition of coordinate algebra can be found in [10].

Two systems of equations S_1 and S_2 is called *equivalent* if $V(S_1) = V(S_2)$ (denotes as $S_1 \sim S_2$).

Partial orders $\mathcal{P}_1$ and $\mathcal{P}_2$ of language L is called *geometrically equivalent* if for every set of variables X every system of equations $S \subseteq At_L(X)$ there is equality

$$Rad_{\mathcal{P}_1}(S) = Rad_{\mathcal{P}_2}(S).$$

We single out the class of partial orders that is important for this article. A *degenerate* partial order is a partial order in which each element is comparable only to itself and to no other element.

We also need to introduce some types of partial orders. *V-shaped partial order* is a three-element partial order on elements a, b, c , in which either $a \geq c, b \geq c$ and elements a, b are incomparable, or $a \leq b, c \leq b$ and elements a, c are incomparable. Hasse diagrams of such partial orders are shown in fig. 1.



Figure 1. V-shaped partial orders

A *partial diamond type order* is a partial order on 4 elements a, b, c, d , where $b \leq a \leq d$ and $b \leq c \leq d$, and elements a and c are incomparable. The Hasse diagram of such a partial order is shown in fig. 2.

Partial orders are closely connected with graphs, namely: each partial order can be assigned a one-to-one graph. This connection is well-known [8], but, for understanding, we describe the algorithm for transition from partial order $\mathcal{P} = \langle P \mid \leq \rangle$ to graph $\Gamma = \langle V \mid E \rangle$. Elements of $\mathcal{P}$ will correspond to the vertices of the graph Γ . In the graph Γ will be an arc (p_1, p_2) if $p_1 \geq p_2$

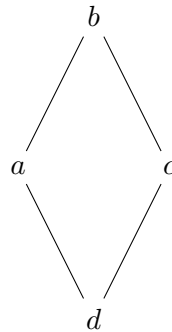


Figure 2. A partial diamond type order.

is true in $\mathcal{P}$. The resulting graph Γ will be transitively closed with loops at each vertex. Also, if we remove all loops from Γ , then we get an acyclic oriented graph. We call such graphs corresponding to partial orders *p-graphs*.

Also, we need a definition of an undirected simple chain for a directed graph. Let $G = \langle V|E \rangle$ be an undirected graph. *edge sequence* in G is such a finite or infinite sequence of edges

$$S = (\dots, E_0, E_1, \dots, E_n, \dots),$$

that every two adjacent edges E_{i-1} and E_i have a common endpoint. A edge sequence is called a *chain* if each of its edges occurs no more than once in it. A non-cyclic chain is called *simple chain*. For directed graphs, you can enter an analog of a simple chain – *path*. In the classical understanding of the path, the end of one edge is the beginning of the next edge in the edge sequence. In our article, the path will be an oriented simple chain, but the end of one edge can be both the beginning and the end for the next edge in the edge sequence.

Let two graphs H and G be given. *Homomorphism* of the graph H to the graph G is the mapping $\varphi : V(H) \rightarrow V(G)$ in which for each two vertices $u, v \in V(H)$ if $(u, v) \in E(H)$ then $(\varphi(u), \varphi(v)) \in E(G)$. The set of homomorphisms from the graph H to the graph G is denoted $Hom(H, G)$.

3. Radical and coordinate partial ordered set

For every system of equations $S(X_n)$ in the language L of variables X_n , the following is true: (1) system S is always consistent and (2) system S consists of two types of equations: $S_{x=x}$ and $S_{x \leq x}$. The system is always consistent, because any point of type $p = (p_i, \dots, p_i), p_i \in P$ will be a solution to the system of equations in this language.

Let us describe an algorithm for constructing the radical of an arbitrary system of equations over a partial order of n variables in a language without constants.

Input: finite system of equations $S(X_n)$ of variables X_n over finite non-degenerated poset $\mathcal{P}$ in language without constants;

Output: $Rad(S)$.

Step 1. $S(X_n) = S_{x=x} \cup S_{x \leq x}$. All equalities are replaced by equivalent pairs of inequalities. That is, the equality $x_i = x_j$ is equivalent to a pair of inequalities $x_i \leq x_j$ and $x_j \leq x_i$. Get a new system of equations $S'(X_n) = S'_{x \leq x}$ that equivalent to origin system.

Step 2. A transition is made from the system of equations $S'(X_n)$ to the oriented graph Γ in such a way that the variables X_n correspond to the vertices of the graph Γ and the equation $x_i \leq x_j \in S'(X_n)$ will correspond to the arc of the graph Γ from x_j to x_i .

Step 3. Doing procedure of the transitive closure of the graph Γ . The result graph is $Tr(\Gamma)$.

Step 4. Doing back procedure of transition graph $Tr(\Gamma)$ to the system of equations $S''(X_n)$.

Step 5. Doing procedure of the antisymmetric closure of the system $S''(X_n)$ and then reflexive closure of the result system. The finally result system is W . The system of equations W will be the radical of the system of equations $S(X_n)$.

To substantiate the algorithm, we need to formulate auxiliary lemmas.

Let a system of equations $S(X_n)$ in the language L of partial order $\mathcal{P}$ be given. A $\mathcal{P}$ -graph of partial order $\mathcal{P}$ is denoted by H , and a graph constructed using the system of equations $S(X_n)$ is denoted by G . The graph G is constructed in the manner described in the algorithm in step 2. It was proved in [7] that the set of isomorphisms between the graph G and all possible subgraphs H are in one-to-one correspondence with the set of solutions of the system of equations $S(X_n)$ over $\mathcal{P}$ provided That $x_i \neq x_j, i, j \in \{1, \dots, n\}, i \neq j$. From this statement the following lemma is easily derived.

Lemma 1 *The solutions of the system of equations $S(X_n)$ over $\mathcal{P}$ correspond one-to-one to the set of homomorphisms $Hom(H, G)$.*

We formulate the second auxiliary lemma.

Lemma 2 *The set of all paths between two incomparable elements in a partial order $\mathcal{P}$ are homomorphically mapped either on a v-shaped partial order, or on a partial diamond type order.*

Proof. We describe the procedure for mapping a single path to a v-shaped partial order and prove that a homomorphism can be defined in this way.

Let the path $(p_1, p_2, \dots, p_n)$ between the incomparable elements p_1 and p_n be given. The first step of the procedure is that each element of the path p_i , comparable to p_1 , is combined with this element and, as a result, all elements comparable to p_i become comparable to p_1 . If p_1 has become comparable with p_n , then such an element p_i is not merged with p_n . Got the path $(p_1, p_{\alpha_1}, \dots, p_n)$.

The second step of the procedure consists in a symmetric action on the part of p_n . Get the path of the vertices (p_1, p_{α}, p_n) . Such a point p_{α} , comparable to p_1 and p_n , will be unique. If there were several such points, then this would mean that there are several different paths in the original path $(p_1, p_2, \dots, p_n)$. Further, it is clear that such a path of three elements, where p_1 is incomparable with p_n , is homomorphically mapped onto a v-shaped partial order.

Such a mapping will be a homomorphism, since all points of p_{α_i} that are combined with p_1 preserve the order relation. If the point p_{α_i} was comparable to p_1 , then when p_{α_i} is mapped to p_1 , $p_1 \leq p_1$ is true, which means that the order relation is preserved. For points that are combined with p_n , the reasoning is similar.

Further, let 2 different paths $(p_1, \dots, p_n)$ and $(q_1, \dots, q_m)$ be given between two incomparable elements, where $p_1 = q_1$ and $p_n = q_m$. Let these 2 paths represent a partial order of $\mathcal{M}$. We prove that $\mathcal{M}$ is homomorphically mapped either onto a v-shaped partial order or onto a partial diamond type order.

First we find the elements p_i, q_j with the maximal index, which are $p_i = q_j$, but $p_{i-1} \neq q_{j-1}$. Now, all elements of p_k, q_l are equated to p_n , for all $k \geq i, l \geq j$. We repeat the procedure: we find the elements p_{i_1}, q_{j_1} with the maximum index that are equal to each other, but not equal to p_n . Further, we equate all p_k and q_l to p_1 for all $k \leq i_1$ and $l \leq j_1$. It turns out 2 new paths: $(p_1, p_{\alpha_1}, \dots, p_{\alpha_k}, p_n)$ and $(p_1, q_{\beta_1}, \dots, q_{\beta_l}, p_n)$ where the interior of the paths has no common elements. Internal paths are not empty in fact that the original paths are different. Now these paths can, independently of each other, be mapped onto v-shaped partial orders. Depending on which v-shaped orders are displayed, these paths can turn out to be either a v-shaped partial order or a partial diamond type order.

It is easy to see that this procedure extends to any number of paths between incomparable elements in a partial order. ■

Now everything is ready to prove the correctness of the described algorithm.

Theorem 1 *The described algorithm correctly constructs the radical of the system of equations $S(X_n)$ over a non-degenerate partial order $\mathcal{P}$ in a language without constants.*

Proof.

The algorithm describes the steps for obtaining new equations through the closures of a system of equations. It is necessary to make sure that for non-degenerate posets the radical is determined correctly, that is, the set W from the algorithm is a radical. Let us reformulate the statement of the theorem as follows: W is the radical of the system of equations $S(X_n)$ over the partial order $\mathcal{P}$ if and only if $\mathcal{P}$ is non-degenerate.

First we prove the sufficiency of this statement. Let the partial order $\mathcal{P}$ be degenerate. Then, for the equation $x_i \leq x_j$ from $S(X_n)$, inequality $x_j \leq x_i$ appears in the radical $Rad(S)$ and, as a result, the equality $x_i = x_j$. This will happen because $\forall p_k \in P$ with a value of $x_i = p_k$ only $x_j = p_k$ can get into the solution, which does not violate the $x_i \leq x_j$ equation. In $\mathcal{P}$, all elements are incomparable with each other, so any two comparable variables will “stick together” at one point. Therefore, if $\mathcal{P}$ is a degenerate partial order, then the algorithm determines the radical incorrect.

We prove the opposite, the set of equations W constructed by the algorithm is the radical of the system of equations $S(X_n)$. In poset $\mathcal{P}$, by definition, there must be such two elements p_k, p_l , that $p_k < p_l$. We also note that $W \sim Rad_{\mathcal{P}}(S)$, $W \subseteq Rad_{\mathcal{P}}(S)$ and W are transitively, reflexively and antisymmetrically closed.

We show that the set of inequalities in the radical coincides with the set of inequalities in W . Consider any 2 variables $x_i, x_j \in X_n$. Suppose they are incomparable in W . If the variables x_i and x_j are incomparable, then they are either connected by a path in a graph constructed from W or not. If the variables are not connected by a path, then for the variable x_i and all variables connected to it by the path in W , we assign the value p_k , and for the variable x_j and all variables connected to it by the path, we assign the value p_l . Similarly, you can assign x_i the value p_l , and the variable x_j the value p_k . This means that in the solution set of system of equations there will be 2 points, the i and j coordinates of which are incomparable. This means that x_i and x_j are incomparable in $Rad(S)$. In other words, it was shown that if there is no path from one variable to another in W , then this path will not appear in the radical.

Now, let the incomparable variables x_i and x_j in W be connected by path. Then, by the lemma 2, there is a homomorphism of this path either into a v-shaped partial order, or into a partial diamond type order. If such partial orders can be homomorphically mapped to the original partial order, then by the lemma 1 this composition of homomorphisms determines the solution of the system of equations W . The partial diamond type order can be mapped to (p_k, p_l) in such a way that x_i will be assigned the value p_k , the variable x_j assigns to p_l and vice versa. This means that in the set of solutions of the system, the i and j coordinates are incomparable. And this means that in $Rad(S)$ these variables will be incomparable. A V-shaped partial order is a special case of a partial diamond type order. This proves that if two variables are incomparable in the system of equations W , then they are incomparable in the radical.

Let the variables $x_i \neq x_j$, but $x_i \leq x_j$ in W . That is, they are comparable, but not equal. Let us prove that in the radical these variables will also be unequal. If $x_i \leq x_j$, then map these variables to p_k and p_l in such a way that x_i is assigned the value p_k , and x_j is the value p_l . All variables that are larger than x_j relative to W are assigned the value p_l , and the remaining variables are assigned the value p_k . It is seen that such a value of variables is a solution to the system. It is also clear that the i -th coordinate is strictly less than j -th, therefore, in the radical, the variables x_i and x_j will not be equal.

It was shown that if the variables x_i and x_j have a certain relationship to each other, then other relations to these variables are not added in the radical. Further, we note that W and $Rad(S)$ are transitive and reflexively closed. This means that the set of inequalities between variables in these two systems coincide. In view of the antisymmetric closure of the system W , the set of equalities between the variables in these two systems also coincide. Therefore, $W = Rad(S)$. ■

From the algorithm it can be seen that the construction of the radical does not depend on the structure of a non-degenerate partial order. Consequently, it is also possible to formulate the following result.

Consequence 1 *Any non-degenerate partial orders in a language L without constants are geometrically equivalent.*

We now show an algorithm of constructing a radical of degenerate partial orders.

Input: finite system of equations $S(X_n)$ of variables X_n over a finite degenerate partial order $\mathcal{P}$ in a language without constants;

Output: $Rad(S)$.

Step 1. $S(X_n) = S_{x=x} \cup S_{x \leq x}$. All equalities are replaced by equivalent pairs of inequalities. Get a new system of equations $S'(X_n) = S'_{x \leq x}$ that equivalent to origin system.

Step 2. A transition is made from the system of equations $S'(X_n)$ to the oriented graph Γ .

Step 3. Search for connected components of the graph Γ . Denote these connected components by $\mathcal{C} = \{C_i\}_{i \in I}$.

Step 4. A new system of equations W is constructed as follows. For each connected component $C_i \in \mathcal{C}$, all equalities and inequalities between all variables from C_i are added to W . The resulting system of equations W will be the radical of the system $S(X_n)$.

From the proof of the theorem 1 it is clear that this algorithm works correctly. Finally, we estimate the complexity of the described algorithms.

The radical construction algorithm for non-degenerate partial orders consists of five steps. At the input, the finite system of equations $S(X_n)$ of m equations in n variables. At the first step, the equalities are replaced by pairs of inequalities. There will be no more such replacements than m . At the second step, a transition occurs from the system of equations to the graph Γ . This transition means the construction of the adjacency matrix $n \times n$ with no more than $2m$ arcs. In the third step, the transitive closure of the graph Γ is sought. This construction can be carried out using the classical Floyd-Warshall algorithm [9], whose complexity is estimated as $O(n^3)$. At the fourth step, a transition occurs from the adjacency matrix of the graph $Tr(\Gamma)$ to the system of equations. And at the fifth step, the antisymmetric and reflexive closure of the system of equations are consistently constructed. The antisymmetric closure of a system of n variables will be evaluated as $O(n^2 \log(n))$. Total inequalities in the system can be n^2 . For each inequality, a matching inequality is sought for this over $\log(n^2)$ time. This is where the estimate comes from. The reflexive closure of a system is obviously estimated to be $O(n)$. It turns out that the algorithm for constructing a radical over a non-degenerate partial order works in polynomial time on the number of variables and equations in the system of equations $S(X_n)$.

The radical construction algorithm for degenerate partial orders also works in polynomial time. The first two steps in it are similar to the first algorithm. In the third step, a search is made for the connected components of the graph. This search can be done through a depth traversal of the graph, which is estimated at $O(n+m)$ [9]. And in the fourth step, the equations are constructed on n vertices, that is, the enumeration of C_n^2 possible pairs. Now it is clear that both algorithms are polynomial in the number of variables and equations in the $S(X_n)$ system.

Finally, it remains to show how the coordinate partial order of the radical is constructed.

Input: the system of equations $S(X_n)$ in the language L , which coincides with its radical $Rad(S)$;

Output: coordinate poset $\Gamma_{\mathcal{P}}(S)$.

We will work with the system $\bar{S}$, which at the beginning of the algorithm coincides with $S(X_n)$.

Step 1. Discrimination of equivalence classes. For all equalities $x_i = x_j$ in the system $\bar{S}$, replace the occurrence of all x_j with x_i . After that, all equalities are removed from $\bar{S}$.

Step 2. Building of p-graph. We construct the graph G in such a way that the vertices of the graph are the variables of the system $\bar{S}$. In the graph G there will be an arc from x_j to x_i if the system $\bar{S}$ contains the equation $x_i \leq x_j$. The resulting graph is a p-graph, so it corresponds to the partial order $\Gamma_{\mathcal{P}}(S)$, which is the coordinate for the system of equations $S(X_n)$.

From the properties of the radical, all equalities in $Rad(S)$ form an equivalence relation on the set of variables X_n . The first step of the algorithm selects representatives of equivalence classes. The change of variables does not change the radical property of the system of equations for partial orders. $\bar{S}$ is transitive and reflexively closed. Therefore, the graph constructed by the system is a p-graph. Therefore, this p-graph corresponds to a partial order, which is coordinate for the system $S(X_n)$.

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