The commutative ring for prime near-rings by involving derivation

IOP Conf. Series: Journal of Physics: Conf. Series **1218** (2019) 012029 doi:10.1088/1742-6596/1218/1/012029

The commutative ring for prime near-rings by involving derivation

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Abstract. A near-ring is an extension of the ring without having to fulfill the commutative of the sum and left distributive of the addition and multiplication operations. This paper shows that a prime and zero symmetric near-ring with derivation on near-ring satisfies proposed conditions is a commutative ring.

1. Introduction
A non-empty set N equipped with two binary operation "+" and "." is called near-ring denoted by $(N,+,.)$ near-ring if $(N,+)$ forms group, $(N,.)$ forms semigroup and $(N,+,.)$ satisfies right distributive. A near-ring N is said to be prime if for all $x,y \in N$, $xNy = \{0\}$ implies $x = 0$ or $y = 0$. A group homomorphism on a near-ring N is called derivation on N if for all $x,y \in N$ satisfies $d(xy) = d(x)y + xd(y)$ or $d(xy) = xd(y) + d(x)y$. For any $x,y \in N$, we denote that $[x,y] = xy - yx$ which is called the Lie products and $x \circ y = xy + yx$ which is called the Jordan products. Research related to derivation on near-ring continues to several publications, see e.g. [1,2,3,4,5]

In [5] several theorems have been obtained which related to a prime near-ring which is a commutative ring involving derivation from the Lie products with the Jordan products and viceversa. In this research, we construct new theorems developing the theorem in [5]. We made new condition for the commutative ring for prime near-ring by involving derivation on Lie products and the Jordan products.

2. Derivation in Prime Near-Ring
In this section, several definitions and theorems will be given which support the main result. A near-ring is not required to apply left distributive, but with certain conditions apply partial left distributive as in the following lemma.

Lemma 2.1 [6] *If d is a derivation on a near-ring N then for all $x,y,z \in N$ satisfies:*
1. $z(xd(y) + d(x)y) = zxd(y) + zd(x)y$
2. $z\big(d(x)y + xd(y)\big) = zd(x)y + zxd(y)$

The properties of Lemma 2.1 are called partial left distributive properties. A near-ring N is called zero symmetric if for all $x \in N$ satisfies $x.0 = 0$. The set of integers $\mathbb{Z}$ with the usual addition and multiplication operation then $(\mathbb{Z},+,\cdot)$ is the zero symmetric near-ring while the set of integers Z with the usual addition operations and multiplication operation defined as for all $x,y \in \mathbb{Z}$ applies $x.y = x$ then $(\mathbb{Z},+,\cdot)$ is near-ring but not zero symmetric. The following theorem shows the existence of near-ring derivation.

Theorem 2.2 [2] *A near-ring N contains a derivation if and only if N is a zero symmetric near-ring*

In this research we use near-ring derivation so that the near-ring used is a zero symmetry near-ring. Let N is a near-ring then $Z(N)$ is defined as $Z(N) = \{x \in N | xy = yx, \forall y \in N\}$.

Lemma 2.3 *Let N is a zero symmetry near-ring and d is a derivation on N. If $r \in Z(N)$ then $d(r) \in Z(N)$.*

Lemma 2.4 [7] *Let N is a prime and zero symmetry near-ring. If there exist a non-zero derivation d on N such that $d(N) \subset Z(N)$, then N is a commutative ring.*

The following lemma is the characteristic of the Lie products and the Jordan products on a near-ring N.

Lemma 2.5 *If N is A near-ring then for all $x, y \in N$ and $k \in \mathbb{N}$ satisfies:*

1. $[x, yx^k] = [x, y]x^k$
2. $[xy^k, y] = [x, y]y^k$
3. $(x \diamond yx^k) = (x \diamond y)x^k$
4. $(xy^k \diamond y) = (x \diamond y)y^k$

Using Theorem 2.2 and theorem obtained from paper [5, Theorem 2.1-2.4] then forms the following theorem.

Theorem 2.6 [2,5] *Let N is a prime and zero symmetric near-ring. If there exist non-negative integers i.e., $p \geq 0, q \geq 0$ and there exist a non-zero derivation d on N such that d satisfying one of the following conditions*

(i). $d([x, y]) = x^p(x \circ y)x^q$ *for all* $x, y \in N$
(ii). $d([x, y]) = -x^p(x \circ y)x^q$ *for all* $x, y \in N$
(iii). $d([x, y]) = y^p(x \circ y)y^q$ *for all* $x, y \in N$
(iv). $d([x, y]) = -y^p(x \circ y)y^q$ *for all* $x, y \in N$

Then N is a commutative ring.

Theorem 2.7 [2,5] *Let N is a prime and zero symmetric near-ring. If there exist non-negative integers i.e., $p \geq 0, q \geq 0$ and there exist a non-zero derivation d on N such that d satisfying one of the following conditions*

(i). $d((x \circ y)) = x^p[x, y]x^q$ *for all* $x, y \in N$
(ii). $d((x \circ y)) = -x^p[x, y]x^q$ *for all* $x, y \in N$
(iii). $d((x \circ y)) = y^p[x, y]y^q$ *for all* $x, y \in N$
(iv). $d((x \circ y)) = -y^p[x, y]y^q$ *for all* $x, y \in N$

Then N is a commutative ring.

It can be seen that the derivation requirement in Theorem 2.6 and Theorem 2.7 relate the derivation of the Lie products with the jordan products and viceversa.

3. Main Results

Theorems for the main result are the results of the development of paper [5] by including derivations on the Lie products and the Jordan products. The following theorem is the development of Theorem 2.6.

Teorema 3.1 *Let N is a prime and zero symmetric near-ring. If there exist non-negative integers i.e., $p \geq 0, q \geq 0$ and there exist a non-zero derivation d on N such that d satisfying one of the following conditions*

(i). $[x, d(y)] = x^p(x \circ y)x^q$ *for all* $x, y \in N$
(ii). $[x, d(y)] = -x^p(x \circ y)x^q$ *for all* $x, y \in N$
(iii). $[d(x), y] = y^p(x \circ y)y^q$ *for all* $x, y \in N$
(iv). $[d(x), y] = -y^p(x \circ y)y^q$ *for all* $x, y \in N$

Then N is a commutative ring.

Proof.

(i) Let there exist a nonzero derivation d and non-negative integers i.e., $p \geq 0, q \geq 0$ such that (i) holds. We will prove that N is a commutative ring by using Lemma 2.3 or showing that $d(N) \subset Z(N)$. Since $x \circ (yx) = (x \circ y)x$ by Lemma 2.5, replacing y by yx in (i) then

$$[x, d(yx)] = x^p(x \circ (yx))x^q = x^p(x \circ y)x^{q+1} == [x, d(y)]x \tag{1}$$

Using Lie product and definition of derivation we obtain

$$xd(yx) - d(yx)x = (xd(y) - d(y)x)x = xd(y)x - d(y)x^2$$
$$x\big(d(y)x + yd(x)\big) - \big(d(y)x + yd(x)\big)x = xd(y)x - d(y)x^2$$

Using Lemma 2.1 and $-(x+y) = -y - x$ we obtain

$$xd(y)x + xyd(x) - yd(x)x - d(y)x^2 = xd(y)x - d(y)x^2$$
$$xyd(x) - yd(x)x = 0$$
$$xyd(x) = yd(x)x \qquad (2)$$

For every $z \in N$ we have

$$zxyd(x) = zyd(x)x$$

Replacing y by zy in equation (2) we get

$$xzyd(x) = zyd(x)x$$

Hence we obtain

$$zxyd(x) = xzyd(x)$$
$$zxyd(x) - xzyd(x) = 0$$
$$(zx - xz)yd(x) = 0$$
$$[z,x]yd(x) = 0$$

Since it holds for all $y \in N$ then

$$[z,x]Nd(x) = 0 \qquad (3)$$

Since N is prime then for each $x \in N$ we have $d(x) = 0$ or $[x,z] = 0$. But d is nonzero derivation hence we have $[x,z] = 0$ or $x \in Z(N)$ for every $x \in N$. By using Lemma 2.4 we have $d(x) \in Z(N)$ for evey $x \in N$ which implies $d(N) \subset Z(N)$ so that N is commutative ring ■

(ii) Let there exist a nonzero derivation d and non-negative integers i.e., $p \geq 0, q \geq 0$ such that (ii) holds. Similarly since $x \circ (yx) = (x \circ y)x$ by Lemma 2.5, replacing y by yx in (ii) then

$$[x, d(yx)] = -x^p\big(x \circ (yx)\big)x^q = -x^p(x \circ y)x^{q+1} = [x, d(y)]x$$

The rest of the proof follows from equation (1) ■

(iii) Let there exist a nonzero derivation d and non-negative integers i.e., $p \geq 0, q \geq 0$ such that (iii) holds. Since $(xy) \circ y = (x \circ y)y$ by Lemma 2.5, replacing x by xy in (iii) then

$$[d(xy), y] = y^p\big((xy) \circ y\big)y^q = y^p(x \circ y)y^{q+1} = [d(x), y]y \qquad (4)$$

Using Lie product and definition of derivation we obtain

$$d(xy)y - yd(xy) = \big(d(x)y - yd(x)\big)y = d(x)y^2 - yd(x)y$$

$$(d(x)y + xd(y))y - y\big(d(x)y + xd(y)\big) = d(x)y^2 - yd(x)y$$

Using Lemma 2.1 and $-(x+y) = -y - x$ we obtain

$$d(x)y^2 + xd(y)y - yxd(y) - yd(x)y = d(x)y^2 - yd(x)y$$
$$xd(y)y - yxd(y) = 0$$
$$xd(y)y = yxd(y) \qquad (5)$$

For every $z \in N$ we have

$$zxd(y)y = zyxd(y)$$

Replacing x by zx in equation (5) we get

$$zxd(y)y = yzxd(y)$$

Hence we obtain

$$zyxd(y) = yzxd(y)$$
$$zyxd(y) - yzxd(y) = 0$$
$$(zy - yz)xd(y) = 0$$
$$[z,y]xd(y) = 0$$

Since it holds for all $x \in N$ then

$$[z,y]Nd(y) = 0$$

The rest of the proof follows from equation (3) ■

(iv) Let there exist a nonzero derivation d and non-negative integers i.e., $p \geq 0, q \geq 0$ such that (iv) holds. Similarly since $(xy) \circ y = (x \circ y)y$ by Lemma 2.5, replacing x by xy in (iv) then

$$[d(xy), y] = -y^p\big((xy) \circ y\big)y^q = -y^p(x \circ y)y^{q+1} = [d(x), y]y$$

The rest of the proof follows from equation (4) ■

The following theorem is the development of Theorem 2.7

Theorem 3.2 *Let N is a prime and zero symmetric near-ring. If there exist non-negative integers i.e., $p \geq 0, q \geq 0$ and there exist a non-zero derivation d on N such that d satisfying one of the following conditions*

$(i). x \circ d(y) = x^p[x, y]x^q$ for all $x, y \in N$

$(ii). x \circ d(y) = -x^p[x, y]x^q$ for all $x, y \in N$

$(iii). d(x) \circ y = y^p[x, y]y^q$ for all $x, y \in N$

$(iv). d(x) \circ y = -y^p[x, y]y^q$ for all $x, y \in N$

Then N is a commutative ring.

Proof.

(i) Let there exist a nonzero derivation d and non-negative integers i.e., $p \geq 0, q \geq 0$ such that (i). Since $[x, yx] = [x, y]x$ by Lemma 5, replacing y by yx in (i) then

$$x \circ d(yx) = x^p[x, yx]x^q = x^p[x, y]x^{q+1} = \big(x \circ d(y)\big)x \tag{6}$$

Using Jordan product and definition of derivation we obtain

$$xd(yx) + d(yx)x = (xd(y) + d(y)x)x = xd(y)x + d(y)x^2$$
$$x\big(d(y)x + yd(x)\big) + (yd(x) + d(y)x)x = xd(y)x + d(y)x^2$$

Using Lemma 2.1 we obtain

$$xd(y)x + xyd(x) + yd(x)x + d(y)x^2 = xd(y)x + d(y)x^2$$
$$xyd(x) + yd(x)x = 0$$
$$xyd(x) = -yd(x)x \tag{7}$$

Replacing y by zy in equation (7) and using $-xy = (-x)y$ we get

$$xzyd(x) = -zyd(x)x = (-z)yd(x)x = (-z)\big(-xyd(x)\big) = (-z)(-x)yd(x)$$
$$xzyd(x) - (-z)(-x)yd(x) = 0 \tag{8}$$

Replacing x by $-x$ in equation (8) we get

$$-xzyd(-x) - (-z)xyd(-x) = 0$$
$$-xzyd(-x) + zxyd(-x) = 0$$
$$zxyd(-x) = xzyd(-x)$$
$$zxyd(-x) - xzyd(-x) = 0$$
$$(zx - xz)yd(-x) = 0$$
$$[z, x]yd(-x) = 0$$

Since it holds for all $y \in N$ then

$$[z, x]Nd(-x) = 0$$

The rest of the proof follows from Theorem 3.1 equation (3) ■

(ii) Let there exist a nonzero derivation d and non-negative integers i.e., $p \geq 0, q \geq 0$ such that (ii). Similarly since $[x, yx] = [x, y]x$ by Lemma 2.5, replacing y by yx in (ii) then

$$x \circ d(yx) = -x^p[x, yx]x^q = -x^p[x, y]x^{q+1} = \big(x \circ d(y)\big)x$$

The rest of the proof follows from equation (6) ■

(iii) Let there exist a nonzero derivation d and non-negative integers i.e., $p \geq 0, q \geq 0$ such that (iii). Since $[xy, y] = [x, y]y$ by Lemma 2.5, replacing x by xy in (iii) then

$$d(xy) \circ y = y^p[xy, y]y^q = y^p[x, y]y^{q+1} = (d(x) \circ y)y \tag{9}$$

Using Jordan product and definition of derivation we obtain

$$d(xy)y + yd(xy) = (d(x)y + yd(x))y = d(x)y^2 + yd(x)y$$
$$(d(x)y + xd(y))y + y(xd(y) + d(x)y) = d(x)y^2 + yd(x)y$$

Using Lemma 2.1 we obtain

$$d(x)y^2 + xd(y)y + yxd(y) + yd(x)y = d(x)y^2 + yd(x)y$$
$$xd(y)y + yxd(y) = 0$$
$$yxd(y) = -xd(y)y \tag{10}$$

Replacing x by zx in equation (10) and using $-xy = (-x)y$ we get

$$yzxd(y) = -zxd(y)y = (-z)xd(y)y = (-z)\big(-yxd(y)\big) = (-z)(-y)xd(y)$$
$$yzxd(y) - (-z)(-y)xd(y) = 0 \tag{11}$$

replacing y by $-y$ in equation (10), we get

$$-yzxd(-y) - (-z)yxd(-y) = 0$$
$$-yzxd(-y) + zyx(-y) = 0$$
$$zyxd(-y) = yzxd(-y)$$
$$zyxd(-y) - yzxd(-y) = 0$$
$$(zy - yz)xd(-y) = 0$$
$$[z,y]xd(-y) = 0$$

Since it holds for all $x \in N$ then

$$[z,x]Nd(-x) = 0$$

The rest of the proof follows from Theorem 3.1 equation (3) ■

(iv) Let there exist a nonzero derivation d and $p, q \in \mathbb{N} \cup \{0\}$ such that (iv) holds. Similarly since $[xy, y] = [x, y]y$ by Lemma 2.5, replacing x by xy in (iv) then

$$d(xy) \circ y = -y^p[xy, y]y^q = -y^p[x, y]y^{q+1} = (d(x) \circ y)y$$

The rest of the proof follows from equation (9) ■

A near-ring N is called 2-torsion-free if for all $x \in N$ satisfies $2x = 0$ then $x = 0$. Set $\mathbb{Z}_n$, the integers modulo n, is defined addition and multiplication operation as : for all $[a]_n, [b]_n \in \mathbb{Z}_n$ applies $[a]_n + [b]_n$=$[a + b]_n$ and $[a]_n[b]_n$=$[ab]_n$ so that $\mathbb{Z}_n$, n odd integers, is 2-torsion-free but $\mathbb{Z}_n$, n even integers, is not 2-torsion-free

Theorem 3.3 *Let N is a prime and zero symmetry near-ring. If $(N, +)$ is 2-torsion-free then there is no non-negative integers i.e., $p \geq 0, q \geq 0$ and a nonzero derivation d such that*
$(i).\ x \circ d(y) = x^p[x, y]x^q$ for all $x, y \in N$
$(ii).\ x \circ d(y) = -x^p[x, y]x^q$ for all $x, y \in N$
$(iii).\ d(x) \circ y = y^p[x, y]y^q$ for all $x, y \in N$
$(iv).\ d(x) \circ y = -y^p[x, y]y^q$ for all $x, y \in N$

Proof.

(i) Let there exist a nonzero derivation d and non-negative integers i.e., $p \geq 0, q \geq 0$ such that (i). Then N is commutative ring by Theorem 3.2 (i). Moreover from Theorem 3.2 equation (7) we have for all $x, y \in N$

$$2xyd(x) = 0 \tag{12}$$

Since N is 2-torsion free, it implies

$$xyd(x) = 0$$

for all $x, y \in N$ so that

$$xNd(x) = 0$$

Since N is prime then for each $x \in N$ we have $d(x) = 0$ or $x = 0$. But d is nonzero derivation hence we have $x = 0$ for every $x \in N$. Contradiction, therefore there is no such derivation. ■

(ii) The proof is similar with (i) and we omit the details. ■

(iii) Let there exist a nonzero derivation d and non-negative integers i.e., $p \geq 0, q \geq 0$ such that (iii). Then N is commutative ring by Theorem 3.2 (iii). Moreover from Theorem 3.2 equation (10) we have for all $x, y \in N$

$$2xyd(x) = 0$$

The rest of the proof follows from equation (12) ■

(iv) The proof is similar with (iii) and we omit the details. ■

Example 3.4 :
The set of all polynomials over $\mathbb{Z}_2$ denoted by $\mathbb{Z}_2[x]$ with usual polynomial addition and multiplication operations, ($\mathbb{Z}_2[x]$, +, ·) is a prime and zero symmetry near-ring. Mapping d on $\mathbb{Z}_2[x]$ which is defined $d\big(p(x)\big) = p'(x)$, for all $p(x) \in \mathbb{Z}_2[x]$ is a derivation mapping and satisfies $[d(p(x)), q(x)] = p(x) \circ q(x)$ so that according to Theorem 3.1 then ($\mathbb{Z}_2[x]$, +, ·) is commutative ring

4. Conclusions

In the main result, Theorem 3.1 and Theorem 3.2 show that a prime and zero symmetry near-ring with derivation on near-ring satisfies proposed conditions is a commutative ring. Theorem 3.3 shows that a prime, zero symmetry and *2*-torsion free near-ring is not possible to fulfill theorem 3.2 so that we must use the other theorem to check whether it is commutative ring.

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