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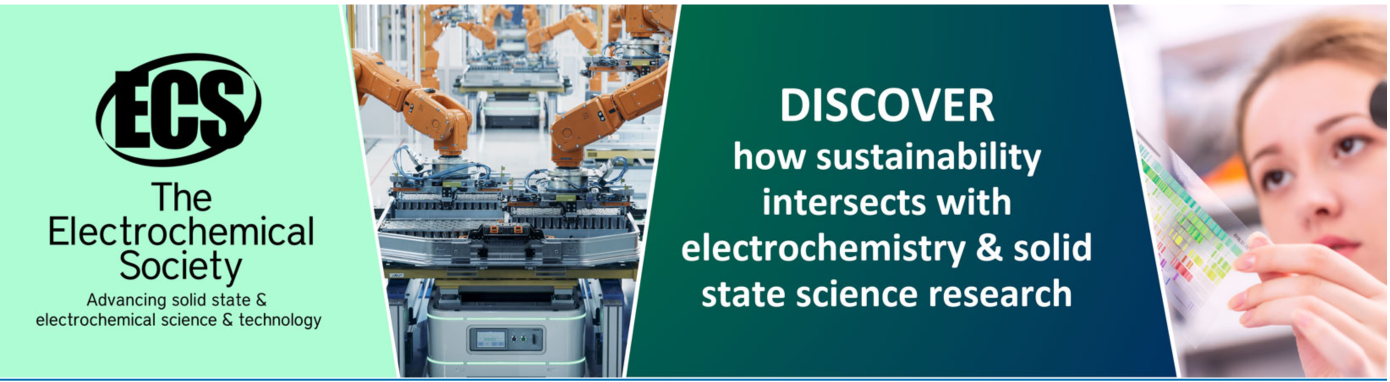

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Mathematical modeling and simulation of the coupled strain space thermoplasticity problems

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Abstract. Using the strain space thermoplasticity theory, proposed by the first author, the coupled dynamic thermomechanical boundary value problems are formulated. The strain space thermoplasticity theory, in contrast to the existing one, allows to formulate the coupled thermoplastic boundary value problems for the displacement and temperature increments. The explicit and implicit finite difference equations for two dimensions case of the boundary value problems are constructed. The numerical solution of the explicit finite difference equations reduced to the application of the recurrent formulas, whereas the implicit scheme reduced to the application of the elimination method. Comparison shows that the numerical results obtained using the explicit and implicit schemes for aforementioned methods are coincides.

1. Introduction

Thermo-mechanical coupling is the most common class of coupled problems, in which the mechanical response of the structure depends on its thermal behavior and vice versa. The investigation of the joint influence of the thermomechanical forces on the deformation process of materials is an actual problem of solid mechanics and is usually referred as the coupled problem of the thermoelasticity or thermoplasticity. Studies in the field of coupled thermoelasticity [1-4] and thermoplasticity [5-13] are widely developed due to their many applications in the advanced structural design problems.

The coupled thermoelasticity problems investigated by Biot[1], Lord and Shulman[2], Youssef[3], introduced a generalized coupled theory with a wave-type heat equation. In [4] develops the theoretical framework appertaining to coupled thermomechanical deformations of solids, subject to large as well as inelastic deformations. The essential feature of the analysis is a consistent natural formulation which encompasses also all thermodynamic aspects.

Simo and Miehe [5&29] present a complete formulation of a model of coupled associative thermoplasticity at finite strains, addresses in detail the numerical analysis aspects involved in its finite element implementation, and assesses the performance of the proposed mechanical and finite element models in a comprehensive set of numerical simulations. The coupled thermoplasticity problems are considered in [6-12]. The coupled visco-plasticity are considered by Stainier and Ortiz [8]. In [13], a variational formulation of the coupled thermo-mechanical boundary-value problem for general dissipative solids is presented.

It is known that depending on the loading surfaces considered in the stress and strain spaces may be formulated two types of constitutive relations of plasticity. The strain space formulation of plasticity theory was proposed by Naghdi and Trapp [14], Casey and Naghdi [15] and was showed that plastic strain rate is normal to the loading surface, whereas Yoder and Iwan [16] considered an alternative associated flow law using so called a stress relaxation tensor normal to the loading surface in strain space. A comprehensive review for thermoplasticity theory at finite strains can be found in [25-27].

In [30] the strain and stress space thermoplasticity theories are considered and compared and is shown that the strain space constitutive relations and loading conditions depend only on strain tensor deviator and temperature and is convenient for formulation and numerical solution of the coupled boundary value problems. In [10] using the strain space constitutive relation the coupled thermoplasticity boundary value problem is formulated. The coupled and uncoupled thermomechanical boundary value problems are numerically solved in following works [10-12, 17-20, 28].

This paper deals with the numerical solution of the 2D coupled thermoplastic boundary value problems formulated using the strain space thermoplasticity theory [30]. Usually in numerical solution of strain space thermoplasticity problems the original problem is partitioned into several smaller sub-problems, which are solved sequentially.

In Section 2 the constitutive relations for strain space and stress space thermoplasticity theories are given. These constitutive relations considered in the case of piecewise linear approximation of the deformation diagram. By comparison, it is shown that the strain space thermoplastic constitutive relation is more convenient for the modeling and numerical solution of the coupled thermoplasticity boundary value problems than the stress space thermoplasticity theory.

In Sections 3 based on strain space thermoplastic theory the coupled boundary value problem consisting of the motion equation, constitutive relations and heat equations with a corresponding initial and boundary conditions are presented.

In Sections 4, using the finite difference method, for coupled thermoplasticity boundary value problem, the explicit and implicit schemes are constructed. The explicit and implicit finite difference equations are solved using the recurrent formulas and elimination method, respectively. Note that in numerical solution of the coupled boundary value problems the external thermomechanical forces are gradually applied with a small increments and the results are found as a sum of the increments of corresponding values.

In Section 5, the numerical examples for coupled boundary value problems are solved. Comparison of the numerical results received using the explicit and implicit finite difference equations shows a good coincidence.

2. Constitutive relations for strain space thermoplasticity theory

There are two types of thermoplasticity theories, depending on the loading surfaces considered in the stress and strain spaces, in the theory of plasticity. Note that the right hand side of the strain space thermoplasticity constitutive relations depend on strain tensor and its deviators and temperature, whereas the stress space theory constitutive relations depend on the stress tensor and strain tensor deviators and temperature. For that the strain space thermoplasticity constitutive relations are convenient for formulation and numerical solution of the coupled boundary value problems.

Let's consider the constitutive relations of thermoplasticity with a loading surface in the strain space [10, 30]

$$d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl} - H\left(\frac{\partial F}{\partial \varepsilon_{kl}} d\varepsilon_{kl} + \frac{\partial F}{\partial T} dT\right)\frac{\partial F}{\partial \varepsilon_{ij}} - C_{ijkl}\alpha\delta_{kl} dT,$$
$$at \;\; F = 0 \;\; and \;\; dF = \frac{\partial F}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial F}{\partial T} dT \geq 0 \tag{2.1}$$

where F is the loading function of the following form

$$F \equiv \frac{1}{2} e_{ij} e_{ij} - R(\omega, T) = 0, \quad H = \left(2 \frac{\partial R}{\partial \omega} \varepsilon_u^2 \right)^{-1}, \quad \varepsilon_u^2 = \frac{1}{2} e_{ij} e_{ij} \tag{2.2}$$

e_{ij}, ω, T are the strain tensor deviator, hardening parameter and absolute temperature, respectively; R – is an experimentally determined function. In the case of piecewise linear approximation, the constitutive relation (2.1) takes the form

$$d\sigma_{ij} = \lambda d\theta \delta_{ij} + 2\mu d\varepsilon_{ij} - (3\lambda + 2\mu)\alpha dT \delta_{ij} - \frac{\mu - \mu'}{\varepsilon_u^2}(e_{kl} de_{kl}) e_{ij} - \frac{\mu - \mu'}{\varepsilon_u^2} e_{ij} \frac{\partial F}{\partial T} dT,$$
$$at \ F = 0 \quad and \quad dF = e_{ij} de_{ij} + \frac{\partial F}{\partial T} dT \geq 0. \tag{2.3}$$

For comparison, the constitutive relation of the stress space thermoplasticity theory is given [10]

$$d\sigma_{ij} = \lambda d\theta \delta_{ij} + 2\mu d\varepsilon_{ij} - (3\lambda + 2\mu)\alpha dT \delta_{ij} - \frac{\mu - \mu'}{\sigma_u^2}(S_{kl} de_{kl}) S_{ij} - \frac{\mu - \mu'}{\sigma_u^2} S_{ij} \frac{\partial f}{\partial T} dT,$$
$$at \ f = 0 \quad and \quad df = S_{ij} dS_{ij} + \frac{\partial f}{\partial T} dT \geq 0. \tag{2.4}$$

The right hand side and the loading condition of (2.4) depend on the stress and strain deviators and temperature, whereas (2.3) depends only on strain tensor deviator and temperature. The dependence of the constitutive relations on strain tensors is convenient for formulation and numerical solution of the coupled thermoplasticity boundary value problems. It can be seen that the third term of the constitutive relations (2.3) and (2.4) are responsible for thermoelastic deformations, whereas the fifth is for thermoplastic deformations. If we neglect the temperature components in (2.1, 2.4), then follows from them the strain space plasticity theory proposed in [23].

3. The coupled thermoplasticity boundary value problem

Note that constitutive relations of the thermoplasticity theories have an incremental form. In formulating the coupled thermoplasticity boundary value problems, according to incremental constitutive relations, all equations, initial and boundary conditions should be written with respect to the increments of the unknowns and other quantities. So, the coupled boundary problem based on the flow theory consists of the motion equation

$$d\sigma_{ij,j} + dX_i = \rho d\ddot{u}_i, \tag{3.1}$$

the constitutive relations of the strain space thermoplasticity theory (2.3)

$$d\sigma_{ij} = \lambda d\theta \delta_{ij} + 2\mu d\varepsilon_{ij} - \alpha\gamma dT \delta_{ij} - \frac{\mu - \mu'}{\varepsilon_u^2}(e_{kl} de_{kl}) e_{ij} - \frac{\mu - \mu'}{\varepsilon_u^2} e_{ij} \frac{\partial F}{\partial T} dT,$$
$$F = 0 \quad and \quad dF = e_{ij} de_{ij} + \frac{\partial F}{\partial T} dT \geq 0; \tag{3.2}$$

the heat conduction equation for isotropic materials [21]

$$\lambda_0 dT_{,ii} - C_\varepsilon d\dot{T} - T_0 \alpha\gamma d\dot{\varepsilon}_{ij} = 0 \tag{3.3}$$

and the Cauchy relations

$$d\varepsilon_{ij} = \frac{1}{2}(du_{i,j} + du_{j,i}) \tag{3.4}$$

with initial

$$\begin{cases} du_i\big|_{t=t_0} = \phi_i, \quad d\dot{u}_i\big|_{t=t_0} = \psi_i, \\ dv_j\big|_{t=t_0} = \tilde{\phi}_j, \quad d\dot{v}_j\big|_{t=t_0} = \tilde{\psi}_j, \\ \qquad dT\big|_{t=t_0} = T_0 \end{cases} \tag{3.5}$$

and boundary conditions

$$\begin{cases} du_i\big|_{\Sigma} = u_i^0, \quad dv_j\big|_{\Sigma} = v_j^0, \\ dT\big|_{\Sigma} = \overline{T}_0, \quad d\sigma_{ij} n_j\big|_{\Sigma} = S_i^0, \end{cases} \tag{3.6}$$

where $C_\varepsilon -$ a heat coefficient at a constant temperature, $\alpha -$ a thermal expansion coefficient, $\lambda_0 -$ the heat flow coefficient and $F -$ a loading function in the strain space [21, 31]. Taking into account the loading function F from Eq. (2.2), the constitutive relation (3.2) can be written in the form

$$d\sigma_{ij} = \lambda d\theta \delta_{ij} + 2\mu d\varepsilon_{ij} - \alpha\gamma dT \delta_{ij} - \frac{\mu - \mu'}{\varepsilon_u^2} e_{ij} (e_{kl} de_{kl} + \beta dT), \tag{3.7}$$

where $\beta = \partial F / \partial T$, $\gamma = 3\lambda + 2\mu$.

The Eq.(3.1-3.6) in two-dimensional case take the following forms, respectively

$$\begin{cases} \dfrac{\partial (d\sigma_{11})}{\partial x} + \dfrac{\partial (d\sigma_{12})}{\partial y} + dX_1 = \rho \dfrac{\partial^2 (du)}{\partial t^2} \\ \dfrac{\partial (d\sigma_{21})}{\partial x} + \dfrac{\partial (d\sigma_{22})}{\partial y} + dX_2 = \rho \dfrac{\partial^2 (dv)}{\partial t^2} \end{cases} \tag{3.8}$$

$$\begin{cases} d\sigma_{11} = \lambda d\theta + 2\mu d\varepsilon_{11} - \alpha\gamma dT - \dfrac{\mu - \mu'}{\varepsilon_u^2} e_{11} (e_{11} de_{11} + e_{22} de_{22} + 2e_{12} de_{12} + \beta dT), \\ d\sigma_{22} = \lambda d\theta + 2\mu d\varepsilon_{22} - \alpha\gamma dT - \dfrac{\mu - \mu'}{\varepsilon_u^2} e_{22} (e_{11} de_{11} + e_{22} de_{22} + 2e_{12} de_{12} + \beta dT), \\ d\sigma_{12} = 2\mu d\varepsilon_{12} - \dfrac{\mu - \mu'}{\varepsilon_u^2} e_{12} (e_{11} de_{11} + e_{22} de_{22} + 2e_{12} de_{12} + \beta dT); \end{cases} \tag{3.9}$$

where

$$d\theta = d\varepsilon_{11} + d\varepsilon_{22}, \quad \varepsilon_u = \sqrt{\frac{1}{2}(e_{11}^2 + e_{22}^2 + 2e_{12}^2)},$$

$$d\varepsilon_{11} = \frac{\partial (du)}{\partial x}, \; d\varepsilon_{22} = \frac{\partial (dv)}{\partial y}, \; d\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial (du)}{\partial y} + \frac{\partial (dv)}{\partial x} \right).$$

Substituting last expressions into Eq. (3.9) gives

$$\begin{cases} d\sigma_{11} = (\lambda + 2\mu)\dfrac{\partial(du)}{\partial x} + \lambda d\,\dfrac{\partial(dv)}{\partial y} - \alpha\gamma dT - \dfrac{\mu - \mu'}{\varepsilon_u^2} e_{11}(e_{11}de_{11} + e_{22}de_{22} + 2e_{12}de_{12} + \beta dT), \\ d\sigma_{22} = \lambda d\,\dfrac{\partial(du)}{\partial x} + (\lambda + 2\mu)\dfrac{\partial(dv)}{\partial y} - \alpha\gamma dT - \dfrac{\mu - \mu'}{\varepsilon_u^2} e_{22}(e_{11}de_{11} + e_{22}de_{22} + 2e_{12}de_{12} + \beta dT), \\ d\sigma_{12} = \mu(\dfrac{\partial(du)}{\partial y} + \dfrac{\partial(dv)}{\partial x}) - \dfrac{\mu - \mu'}{\varepsilon_u^2} e_{12}(e_{11}de_{11} + e_{22}de_{22} + 2e_{12}de_{12} + \beta dT); \end{cases} \tag{3.10}$$

and then inserting into Eq.(3.8) gives the motion and heat equations for displacement and temperature increments

$$\begin{cases} (\lambda + 2\mu)\dfrac{\partial^2(du)}{\partial x^2} + \mu\dfrac{\partial^2(du)}{\partial y^2} + (\lambda + \mu)\dfrac{\partial^2(dv)}{\partial x\partial y} - \alpha\gamma\dfrac{\partial(dT)}{\partial x} - \xi = \rho\dfrac{\partial^2(du)}{\partial t^2}, \\ \mu\dfrac{\partial^2(dv)}{\partial x^2} + (\lambda + 2\mu)\dfrac{\partial^2(dv)}{\partial y^2} + (\lambda + \mu)\dfrac{\partial^2(du)}{\partial x\partial y} - \alpha\gamma\dfrac{\partial(dT)}{\partial y} - \zeta = \rho\dfrac{\partial^2(dv)}{\partial t^2}, \\ \lambda_0\left[\dfrac{\partial^2(dT)}{\partial x^2} + \dfrac{\partial^2(dT)}{\partial y^2}\right] - C_\varepsilon\dfrac{\partial(dT)}{\partial t} - \alpha\gamma T_0\left[\dfrac{\partial^2(du)}{\partial x\partial t} + \dfrac{\partial^2(dv)}{\partial y\partial t}\right] = 0; \end{cases} \tag{3.11}$$

where

$$\begin{cases} \xi = \dfrac{\mu - \mu'}{\varepsilon_u^2}\begin{bmatrix} e_{11}(e_{11}\dfrac{\partial(de_{11})}{\partial x} + e_{22}\dfrac{\partial(de_{22})}{\partial x} + 2e_{12}\dfrac{\partial(de_{12})}{\partial x} + \beta\dfrac{\partial(dT)}{\partial x}) + \\ +e_{12}(e_{11}\dfrac{\partial(de_{11})}{\partial y} + e_{22}\dfrac{\partial(de_{22})}{\partial y} + 2e_{12}\dfrac{\partial(de_{12})}{\partial y} + \beta\dfrac{\partial(dT)}{\partial y}) \end{bmatrix}, \\ \zeta = \dfrac{\mu - \mu'}{\varepsilon_u^2}\begin{bmatrix} e_{22}(e_{11}\dfrac{\partial(de_{11})}{\partial y} + e_{22}\dfrac{\partial(de_{22})}{\partial y} + 2e_{12}\dfrac{\partial(de_{12})}{\partial y} + \beta\dfrac{\partial(dT)}{\partial y}) + \\ +e_{12}(e_{11}\dfrac{\partial(de_{11})}{\partial x} + e_{22}\dfrac{\partial(de_{22})}{\partial x} + 2e_{12}\dfrac{\partial(de_{12})}{\partial x} + \beta\dfrac{\partial(dT)}{\partial x}) \end{bmatrix}; \end{cases}$$

with an appropriate initial and boundary conditions

$$\begin{cases} du(x,y,t)\big|_{t=0} = \phi(x,y), & \dfrac{\partial(du)}{\partial t}\Big|_{t=0} = \psi(x,y), \\ dv(x,y,t)\big|_{t=0} = \tilde{\phi}(x,y), & \dfrac{\partial(dv)}{\partial t}\Big|_{t=0} = \tilde{\psi}(x,y), \\ dT(x,y,t)\big|_{t=0} = T_0 & \end{cases} \tag{3.12}$$

$$\begin{cases} du(x,y,t)\big|_{x=0,\ell_1} = u_0, & du(x,y,t)\big|_{y=0,\ell_2} = \bar{u}_0, \\ dv(x,y,t)\big|_{x=0,\ell_1} = v_0, & dv(x,y,t)\big|_{y=0,\ell_2} = \bar{v}_0, \\ dT(x,y,t)\big|_{x=0,\ell_1} = T_1(t), & dT(x,y,t)\big|_{y=0,\ell_2} = T_2(t) \end{cases} \tag{3.13}$$

where $\lambda,\ \mu,\mu',\alpha,\beta,C_\varepsilon,\ \lambda_0 -$ are the given constants, $\ell_1,\ell_2 -$ the length of the rectangle sides, $\varphi,\ \psi,\tilde{\varphi},\ \tilde{\psi},\ T_0,\ T_1,T_2 -$ are the known values. The Eq.(3.11-3.13) present the 2D coupled strain space thermoplasticity boundary value problem.

4. Finite difference equations for 2D coupled thermoplasticity boundary problem

The Eq. (3.11-3.13) consisting of the two motions and one heat equations present the 2D coupled strain space thermoplasticity boundary value problem. Note that the equations depend on displacement and temperature increments du, dv and dT, respectively. In order to construct the finite difference equations, the derivatives of displacement and temperature increments in Eq.(3.11-3.13) replacing by the corresponding difference quotients, we obtain

$$\begin{cases} (\lambda+2\mu)\dfrac{du_{i+1j}^k-2du_{ij}^k+du_{i-1j}^k}{h_1^2}+\mu\dfrac{du_{ij+1}^k-2du_{ij}^k+du_{ij-1}^k}{h_2^2}-\alpha\gamma\dfrac{dT_{i+1j}^k-dT_{i-1j}^k}{2h_1}-\xi_{ij}^k+ \\ \qquad +(\lambda+\mu)\dfrac{dv_{i+1j+1}^k-dv_{i+1j-1}^k-dv_{i-1j+1}^k+dv_{i-1j-1}^k}{4h_1h_2}=\rho\dfrac{du_{ij}^{k+1}-2du_{ij}^k+du_{ij}^{k-1}}{\tau^2} \\ \mu\dfrac{dv_{i+1j}^k-2dv_{ij}^k+dv_{i-1j}^k}{h_1^2}+(\lambda+2\mu)\dfrac{dv_{ij+1}^k-2dv_{ij}^k+dv_{ij-1}^k}{h_2^2}-\alpha\gamma\dfrac{dT_{ij+1}^k-dT_{ij-1}^k}{2h_2}-\zeta_{ij}^k+ \\ \qquad +(\lambda+\mu)\dfrac{du_{i+1j+1}^k-du_{i+1j-1}^k-du_{i-1j+1}^k+du_{i-1j-1}^k}{4h_1h_2}=\rho\dfrac{dv_{ij}^{k+1}-2dv_{ij}^k+dv_{ij}^{k-1}}{\tau^2} \end{cases} \tag{4.1}$$

$$\begin{aligned} &\lambda_0\left(\frac{dT_{i+1j}^k-2dT_{ij}^k+dT_{i-1j}^k}{h_1^2}+\frac{dT_{ij+1}^k-2dT_{ij}^k+dT_{ij-1}^k}{h_2^2}\right)-C_\varepsilon\frac{dT_{ij}^{k+1}-dT_{ij}^{k-1}}{2\tau}- \\ &-\alpha\gamma T_0\left(\frac{du_{i+1j}^{k+1}-du_{i-1j}^{k+1}-du_{i+1j}^{k-1}+du_{i-1j}^{k-1}}{4h_1\tau}+\frac{dv_{ij+1}^{k+1}-dv_{ij-1}^{k+1}-dv_{ij+1}^{k-1}+dv_{ij-1}^{k-1}}{4h_2\tau}\right)=0. \end{aligned} \tag{4.2}$$

These equations can be resolved by means of $du_{i,j}^{k+1}$, $dv_{i,j}^{k+1}$ and $dT_{i,j}^{k+1}$ respectively, and we get the following recurrent formulas

$$\begin{cases} du_{ij}^{k+1}=\dfrac{\tau^2}{\rho}\left(\begin{array}{l}(\lambda+2\mu)\dfrac{du_{i+1j}^k-2du_{ij}^k+du_{i-1j}^k}{h_1^2}+\mu\dfrac{du_{ij+1}^k-2du_{ij}^k+du_{ij-1}^k}{h_2^2}-\xi_{ij}^k- \\ \alpha\gamma\dfrac{dT_{i+1j}^k-dT_{i-1j}^k}{2h_1}+(\lambda+\mu)\dfrac{dv_{i+1j+1}^k-dv_{i+1j-1}^k-dv_{i-1j+1}^k+dv_{i-1j-1}^k}{4h_1h_2}\end{array}\right)+2du_{ij}^k-du_{ij}^{k-1} \\ dv_{ij}^{k+1}=\dfrac{\tau^2}{\rho}\left(\begin{array}{l}(\lambda+2\mu)\dfrac{dv_{ij+1}^k-2dv_{ij}^k+dv_{ij-1}^k}{h_2^2}+\mu\dfrac{dv_{i+1j}^k-2dv_{ij}^k+dv_{i-1j}^k}{h_1^2}-\zeta_{ij}^k- \\ \alpha\gamma\dfrac{dT_{ij+1}^k-dT_{ij-1}^k}{2h_2}+(\lambda+\mu)\dfrac{du_{i+1j+1}^k-du_{i+1j-1}^k-du_{i-1j+1}^k+du_{i-1j-1}^k}{4h_1h_2}\end{array}\right)+2dv_{ij}^k-dv_{ij}^{k-1} \end{cases} \tag{4.3}$$

$$dT_{ij}^{k+1}=-\frac{2\tau}{C_\varepsilon}\left(\begin{array}{l}\alpha\gamma T_0\left(\dfrac{du_{i+1j}^{k+1}-du_{i-1j}^{k+1}-du_{i+1j}^{k-1}+du_{i-1j}^{k-1}}{4h_1\tau}+\dfrac{dv_{ij+1}^{k+1}-dv_{ij-1}^{k+1}-dv_{ij+1}^{k-1}+dv_{ij-1}^{k-1}}{4h_2\tau}\right)- \\ -\lambda_0\left(\dfrac{dT_{i+1j}^k-2dT_{ij}^k+dT_{i-1j}^k}{h_1^2}+\dfrac{dT_{ij+1}^k-2dT_{ij}^k+dT_{ij-1}^k}{h_2^2}\right)-\dfrac{C_\varepsilon}{2\tau}dT_{ij}^{k-1}\end{array}\right). \tag{4.4}$$

The finite-difference analogues of the initial conditions (3.12) have the form

$$\begin{gathered} du_{ij}^{0}=\phi_{ij}, \qquad dv_{ij}^{0}=\tilde{\phi}_{ij}; \\ du_{ij}^{1}=du_{ij}^{0}+\tau\psi_{ij}, \qquad dv_{ij}^{1}=dv_{ij}^{0}+\tau\tilde{\psi}_{ij}; \\ dT_{ij}^{0}=\bar{T}_{ij}. \end{gathered} \tag{4.5}$$

The boundary conditions (3.13) in the finite-difference case take the form

$$\begin{aligned} &\left.\begin{aligned} du_{0j}^{k}=u_{0}^{y},\ du_{N_1 j}^{k}=\bar{u}_{0}^{y} \\ dv_{0j}^{k}=v_{0}^{y},\ dv_{N_1 j}^{k}=\bar{v}_{0}^{y} \end{aligned}\right\} \qquad j=\overline{0,N_2} \\ &\left.\begin{aligned} dv_{i0}^{k}=v_{0}^{y},\ dv_{iN_2}^{k}=\bar{v}_{0}^{y} \\ du_{i0}^{k}=u_{0}^{x},\ du_{iN_2}^{k}=\bar{u}_{0}^{x} \end{aligned}\right\} \qquad i=\overline{0,N_1} \\ &\left.\begin{aligned} dT_{0j}^{k}=T_{xj}^{k},\ dT_{N_1 j}^{k}=\bar{T}_{xj}^{k} \\ dT_{i0}^{k}=T_{iy}^{k},\ dT_{iN_2}^{k}=\bar{T}_{iy}^{k} \end{aligned}\right\} \qquad k=\overline{0,M}. \end{aligned} \tag{4.6}$$

Taking into account the initial and boundary conditions (4.5 & 4.6), we can see that the finite difference Eq.(4.1 & 4.2) are explicit. Using the Eq.(4.3 & 4.4) and the initial and boundary conditions (4.5 & 4.6) we can find the displacement $du_{i,j}^{k+1}$, $dv_{i,j}^{k+1}$ and temperature $dT_{i,j}^{k+1}$ increments, according to the increments of the external thermomechanical factors. Then the total displacement and temperature are found as a sum of increments of $du(x,y,t)$, $dv(x,y,t)$ and $dT(x,y,t)$ i.e.

$$u=\sum_{p} du, \qquad T=\sum_{p} dT$$

where p-is a number of thermomechanical force increments. Note that the finite difference Eq. (4.1 & 4.2) are implicit and the convergence of these schemes is slow and depends on mesh step lengths of h_1, h_2 and τ. In order to construct the schemes without convergence limitations, we should replace in Eq.(4.1 & 4.2) the index k by $k+1$ for the first terms. Then we receive the implicit finite difference equations, which after some transformations may be reduced to the following forms

$$a_{ij}du_{ij+1}^{k+1}+b_{ij}du_{ij}^{k+1}+c_{ij}du_{ij-1}^{k+1}=f_{ij} \tag{4.7}$$

$$A_{ij}dv_{i+1,j}^{k+1}+B_{ij}dv_{ij}^{k+1}+C_{ij}dv_{i-1j}^{k+1}=F_{ij} \tag{4.8}$$

$$\tilde{a}_{ij}dT_{i+1j}^{k+1}+\tilde{b}_{ij}dT_{ij}^{k+1}+\tilde{c}_{ij}dT_{i-1j}^{k+1}=\varphi_{ij} \tag{4.9}$$

where

$$a_{ij}=\frac{\mu}{h_2^2}, \qquad b_{ij}=-2\frac{\mu}{h_2^2}-\frac{\rho}{\tau^2}, \qquad c_{ij}=\frac{\mu}{h_2^2},$$

$$\begin{aligned} f_{ij}=\alpha\gamma\frac{dT_{i+1j}^{k}-dT_{i-1j}^{k}}{2h_1}-(\lambda+2\mu)\frac{du_{i+1j}^{k}-2du_{ij}^{k}+du_{i-1j}^{k}}{h_1^2}+\xi_{ij}^{k}- \\ -(\lambda+\mu)\frac{dv_{i+1j+1}^{k}-dv_{i+1j-1}^{k}-dv_{i-1j+1}^{k}+dv_{i-1j-1}^{k}}{4h_1h_2}+\rho\frac{u_{ij}^{k-1}-2u_{ij}^{k}}{\tau^2} \end{aligned}$$

$$A_{ij}=\frac{\mu}{h_1^2},\quad B_{ij}=-2\frac{\mu}{h_1^2}-\frac{\rho}{\tau^2},\quad C_{ij}=\frac{\mu}{h_1^2},$$

$$F_{ij}=\alpha\gamma\frac{dT_{ij+1}^k-dT_{ij-1}^k}{2h_2}-(\lambda+2\mu)\frac{dv_{ij+1}^k-2dv_{ij}^k+dv_{ij-1}^k}{h_2^2}+\zeta_{ij}^k-$$
$$-(\lambda+\mu)\frac{du_{i+1j+1}^k-du_{i+1j-1}^k-du_{i-1j+1}^k+du_{i-1j-1}^k}{4h_1h_2}+\rho\frac{v_{ij}^{k-1}-2v_{ij}^k}{\tau^2}$$

$$\tilde{a}_{ij}=\frac{\lambda_0}{h_1^2},\quad \tilde{b}_{ij}=-2\frac{\lambda_0}{h_1^2}-\frac{C_\varepsilon}{2\tau},\quad \tilde{c}_{ij}=\frac{\lambda_0}{h_1^2},$$

$$\varphi_{ij}=\alpha\gamma T_0\left(\frac{du_{i+1j}^{k+1}-du_{i-1j}^{k+1}-du_{i+1j}^{k-1}+du_{i-1j}^{k-1}}{4h_1\tau}+\frac{dv_{ij+1}^{k+1}-dv_{ij-1}^{k+1}-dv_{ij+1}^{k-1}+dv_{ij-1}^{k-1}}{4h_2\tau}\right)-$$
$$-\lambda_0\frac{dT_{ij+1}^k-2dT_{ij}^k+dT_{ij-1}^k}{h_2^2}-\frac{C_\varepsilon}{2\tau}dT_{ij}^{k-1}.$$

For solving the Eqs. (4.7-4.9), the elimination method [24, 22] is used.

5. Numerical tests

Consider the aforementioned coupled boundary value problem Eq. (3.11-3.13) in a rectangular area with sides ℓ_1,ℓ_2. Let's all four sides of the rectangle be fixed and putted on them the temperature in a sinusoidal form as a boundary condition. At the initial moment $t=0$ the inner temperature in the rectangular is T_0. The described process is modeled as a boundary value problem by the Eqs.(3.11–3.13) with a following initial and boundary conditions:

$$u(x,y,t)\big|_{t=0}=0,\qquad \left.\frac{\partial u(x,y,t)}{\partial t}\right|_{t=0}=0,$$
$$v(x,y,t)\big|_{t=0}=0,\qquad \left.\frac{\partial v(x,y,t)}{\partial t}\right|_{t=0}=0,$$
$$T(x,y,t)\big|_{t=0}=T_0,$$

$$u(x,y,t)\big|_{x,y=0}=0,\qquad u(x,y,t)\big|_{x,y=1}=0,$$
$$v(x,y,t)\big|_{x,y=0}=0,\qquad v(x,y,t)\big|_{x,y=1}=0,$$
$$T(x,y,t)\big|_{x=0,x=1}=T_0\cdot\sin(y_j\cdot\pi),$$
$$T(x,y,t)\big|_{y=0,y=1}=T_0\cdot\sin(x_i\cdot\pi).$$

The values of the constants are chosen as follows:

$\lambda=1,\ \lambda_0=1,\alpha=0.05,\mu=0.5,\ \mu'=0.3,\ \rho=1.0,\ C_\varepsilon=2.7,\ n=12,\ \tau=0.01,\ \ell=1,\ T_0=75$

Note that in case of strain space thermoplasticity theory the considered boundary value problem, the applied external thermomechanical forces is partitioned into several smaller partitions. Then the general solution of the boundary problem may be represented as a sum of the solutions of the sub-problems corresponding to the increments of the external load. Note that, Eqs. (3.11-3.13) are written with respect to increments (differentials) of temperature and displacement.

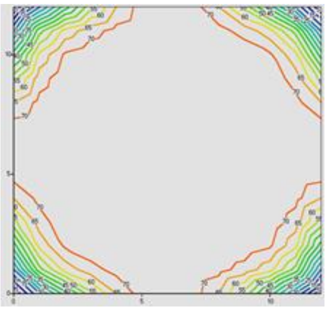
Using the finite difference method the explicit and implicit schemes for the coupled thermoplasticiy boundary value problem were constructed. The explicit schemes are solved using the recurrent formulas for displacement U and temperature T. For solving the implicit schemes, it is applied the elimination method [24, 22]. For the numerical solution of the finite difference equations was developed a software in C# integrated with MathCAD [32]. In Tables 1-2 the values of temperature obtained on explicit (Table 1.) and implicit (Table 2.) schemes are given. Comparison of the tables shows that the values of T are coincide. Note that, taking into account the symmetricity of the boundary problem, the values of temperature are given in one fourth part of the considered rectangle.

Table 1. Distribution of the temperature by implicit scheme method

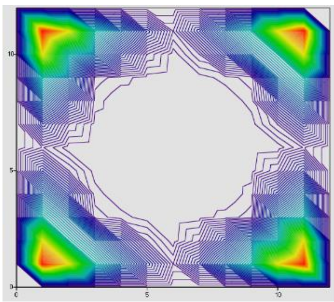
y \ x	*0*	*0,1*	*0,2*	*0,3*	*0,4*	*0,5*	*0,6*
0	0	19,411	37,50	53,033	64,952	72,444	75
0,1	19,411	47,449	61,285	67,765	71,627	73,929	74,708
0,2	37,5	61,285	70,807	73,435	74,338	74,783	74,929
0,3	53,033	67,765	73,435	74,678	74,907	74,972	74,991
0,4	64,952	71,627	74,338	74,907	74,986	74,997	74,999
0,5	72,444	73,929	74,783	74,972	74,997	74,999	75
0,6	75	74,708	74,929	74,991	74,999	75	75

Table 2. Distribution of the temperature according to the method explicit scheme

y \ x	*0*	*0,1*	*0,2*	*0,3*	*0,4*	*0,5*	*0,6*
0	0	19,411	37,50	53,033	64,952	72,444	75
0,1	19,411	48,029	61,758	68,032	71,756	73,978	74,731
0,2	37,5	61,328	70,795	73,408	74,321	74,776	74,925
0,3	53,033	67,704	73,373	74,637	74,886	74,964	74,987
0,4	64,952	71,583	74,31	74,894	74,981	74,995	74,998
0,5	72,444	73,892	74,766	74,968	74,996	74,999	75
0,6	75	74,672	74,914	74,987	74,999	75	75



a).



b).

Figure.1 a). Distribution of the temperature T in the rectangle. b). Distribution of the plasticity zones in the rectangle.

In the Fig. 1 using the values given in the Tables 1-2 the distribution of the temperature and plasticity zones are shown. We can see from the Figure 1, that the highest temperatures are reached at the corners of the rectangle and accordingly, plastic zones also arise near the corners.

6. Conclusions

The nonlinear deformation process of the solids was modelled using the strain space thermoplasticity theory. The modeling equations presented as a coupled thermoplasticity boundary value problem. Note that, in case of coupled boundary value problem, the motion equation and constitutive relations of the

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thermoplasticity should be joint the heat conduction equations with a suitable initial and boundary conditions. Using the finite difference method an explicit and implicit schemes for two dimensions coupled boundary value problem was constructed. The numerical solution of the explicit finite difference equations reduced to the application of the recurrent formulas, whereas the implicit schemes solved using the elimination method. The numerical results obtained using the explicit and implicit schemes shows a good coincidence. For the numerical solution of the finite difference equations a software in C# integrated with MathCAD was developed.

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