Hamiltonian $U(2)$-actions and Szegö kernel asymptotics

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Abstract. In this paper we shall review some recent results on the asymptotic expansion of the equivariant components of an algebro geometric Szegö kernel determined by the linearization of a Hamiltonian action of $U(2)$ (with certain assumptions). We shall build on the techniques developed in [13], [1], and [11], and therefore ultimately on the microlocal description of the Szegö kernel as a Fourier integral operator in [3].

1. Introduction
Suppose that $M$ is a connected projective manifold of complex dimension $d$, and let $A$ be an ample line bundle on it. Let $h$ be an Hermitian metric on $A$, such that the unique connection on $A$ compatible with both the complex structure and the Hermitian metric has curvature $\Theta = -2i\omega$, where $\omega$ a Hodge form. In particular, $\omega^d$ is a volume form on $M$; we shall denote by $dV_M$ the associated density.

Let $A^\vee$ be the dual line bundle to $A$, endowed with the induced metric, and consider the unit circle bundle $X \subseteq A^\vee$; thus $X$ is a principal $S^1$-bundle on $M$, with projection $\pi : X \to M$. The connection on $A$ corresponds to a (normalized) connection 1-form $\alpha$ on $X$, such that $d\alpha = 2\pi^*(\omega)$. Then $(\alpha/2\pi) \wedge \pi^*(\omega)^d$ is a volume form on $X$; $dV_X$ will denote the associated density.

Since $\omega$ is Kähler, $(X, \alpha)$ is a contact CR manifold. Let us denote by $H(X) \subseteq L^2(X)$ its Hardy space, by $\Pi : L^2(X) \to H(X)$ the orthogonal projector (the Szegö projector), and by $\Pi \in \mathcal{D}'(X \times X)$ its distributional kernel (the Szegö kernel). It is a well-known foundational result, due to Boutet de Monvel and Sjöstrand, that $\Pi$ is a Fourier integral operator with complex phase ([3], [2], [11]).

Since the CR structure of $X$ is $S^1$-invariant, there is a naturally induced unitary representation of $S^1$ on $H(X)$; therefore, $H(X)$ splits unitarily and equivariantly as a direct sum of isotypical components:

$$H(X) = \bigoplus_{k \in \mathbb{Z}} H(X)_k,$$

where $H_k(X) \subset H(X)$ is the subspace of CR functions that transform like the character $e^{ik\theta}$. We have $H(X)_k = 0$ if $k < 0$, and for $k \geq 0$ there is a natural unitary isomorphism between $H(X)_k$ and $H^0 \left( M, A^{\otimes k} \right)$. In particular, every $H(X)_k$ is finite-dimensional.
Thus we have
\[ \Pi = \sum_{k \geq 0} \Pi_k, \]
where \( \Pi_k : L^2(X) \to H(X)_k \) is the orthogonal projector. The distributional kernel of \( \Pi_k \) (or reproducing kernel of \( H(X)_k \)) is a function \( \Pi_k \in C^\infty(X \times X) \), and its asymptotic behavior has been extensively investigated, starting with the on-diagonal asymptotics of \([12],[4],[13]\). The most relevant approach for our discussion is the one developed in \([13],[1],[11]\) (we refer to the introductions of \([6]\) and \([7]\) for a somewhat wider discussion and reference list).

Specifically, as \( k \to +\infty \) we have

(i) if \( x, y \in X \), and \( x \not\in S^1 \cdot y \), then
\[ \Pi_k(x,y) = O(k^{-\infty}); \]

(ii) if \( x, y \in X \) and \( g, h \in S^1 \), then (denoting \( r_g : X \to X \) the action of \( g \) on \( X \))
\[ \Pi_k(r_g(x), r_h(y)) = g^k h^{-k} \Pi_k(x,y); \]

(iii) uniformly on \( x \in X \), there is an asymptotic expansion of the form
\[ \Pi_k(x,x) \sim \left( \frac{k}{\pi} \right)^d \cdot \left[ 1 + \sum_{j \geq 1} k^{-j} a_j(m_x) \right], \]

where \( a_j : M \to \mathbb{R} \) is \( C^\infty \), and we have set \( m_x := \pi(x) \). Each \( a_j \) is describable in terms of the curvature tensor of \( M \) (for instance, as first remarked by \([8]\), we define \( \psi_2 : C^d \times C^d \to C \) by
\[ \psi_2(v_1,v_2) := -i \omega_0(v_1,v_2) - \frac{1}{2} ||v_1 - v_2||^2. \]

In so-called Heisenberg local coordinates (henceforth, HLC’s) centered at \( x \in X \), we have
\[ \Pi_k \left( x + \left( \frac{\theta_1}{\sqrt{k}}, \frac{\theta_2}{\sqrt{k}} \right), x + \left( \frac{\theta_1}{\sqrt{k}}, \frac{\theta_2}{\sqrt{k}} \right) \right) \sim \left( \frac{k}{\pi} \right)^d \cdot e^{i k (\theta_1 - \theta_2) + \psi_2(v_1,v_2)} \cdot \left[ 1 + \sum_{j \geq 1} k^{-j/2} R_j(m_x; v_1, v_2) \right]. \]

Here \( R_j(m_x; \cdot, \cdot) \) is a polynomial of degree \( \leq 3 j \). To appreciate the significance of (2), recall that in the expression \( x + (\theta, v) \) the coordinate \( \theta \) expresses the action of \( e^{i \theta} \in S^1 \), i.e. \( x + (\theta, v) = r_{e^{i \theta}}(x + (0, v)) \), while \( v \) represents an ‘horizontal’ displacement from \( x \) (see \([11]\) for a detailed description of HLC’s). Hence, \( v_1 - v_2 \) represents a displacement from \( (\pi \times \pi)^{-1}(\Delta_M) \).

For fixed \( C > 0 \) and \( \epsilon \in (0, 1/6) \), the asymptotic expansion (2) holds uniformly for \( ||v_j|| \leq C k^{\epsilon} \).

The general issue we are involved with is the following: how does the previous analysis carry over when the standard \( S^1 \)-action is replaced by a more general action of a connected compact Lie group \( G \) on \( X \), yielding a unitary representation on \( H(X) \)?

As a motivation for the following arguments, let us remark that the standard circle action \( r : S^1 \times X \to X \) descends to the trivial action on \( M \), and the latter may be viewed as a
Hamiltonian action with constant moment map equal to 1. Thus we may interpret \( r \) as the contact lift of the trivial action on \( M \) associated to the moment map \( \Phi_{S^1} \equiv 1 \).

The latter interpretation yields a much more general setting for the previous result. Namely, given any Hamiltonian action of a connected compact Lie group \( G \), with moment map \( \Phi_G : M \to \mathfrak{g}^\ast \), we have a built-in infinitesimal contact action of the Lie algebra \( \mathfrak{g} \) on \( X \). Explicitly, for any \( \xi \in \mathfrak{g} \) let \( \xi_M \in X(M) \) be the associated Hamiltonian vector field on \( M \), and let \( \xi_M \in X(X) \) be its horizontal lift to \( X \). Then

\[
\xi_X := \xi_M^\sharp - (\Phi_G, \xi) \partial_\theta
\]

is a contact vector field on \( X \); here \( \partial_\theta \) is the generator of the \( S^1 \)-action.

Let us assume, as is the case in many natural and interesting situations, that this infinitesimal action is the differential of a genuine contact action \( \tilde{\mu} \) of \( G \) on \( X \). For instance, when \( G = S^1 \), \( \mu \) is trivial and \( \Phi_G = 1 \), as we have remarked we recover the (reverse) standard action on \( X \).

Let us also assume that \( \mu \) is holomorphic. Then \( \tilde{\mu} \) preserves the contact and CR structures of \( X \), and induces a unitary representation on \( H(X) \).

Hence, by the Theorem of Peter and Weyl, \( H(X) \) splits equivariantly and unitarily over the irreducible representations of \( G \), which are all finite dimensional:

\[
H(X) = \bigoplus_{\mathcal{D} \in G} H(X)_{\mathcal{D}}.
\]  

(3)

If \( 0 \not\in \Phi_G(M) \), then every isotypical component \( H(X)_{\mathcal{D}} \) is finite-dimensional. Therefore, the corresponding projection operator \( \Pi_{\mathcal{D}} : L^2(X) \to H(X)_{\mathcal{D}} \) is smoothing, i.e., its distributional integral kernel is in fact a \( C^\infty \) function \( \Pi_{\mathcal{D}} \in C^\infty(X \times X) \). We are interested in the pointwise asymptotics of \( \Pi_{\mathcal{D}} \) as \( \mathcal{D} \to \infty \) along a ‘ray’ in weight space, in a sense to be specified.

When \( G \) is a torus, this problem has been investigated in [9], [10], [5]. The cases where \( G = U(2) \) and \( G = SU(2) \) have been studied in [6] and [7], respectively. We shall give an overview of some of the results in [6].

As is well-known, the irreducible representations of \( U(2) \) are indexed by the pairs \( \mathcal{D} = (\nu_1, \nu_2) \in \mathbb{Z}^2 \) with \( \nu_1 > \nu_2 \), the irreducible representation associated to \( \mathcal{D} \) being \( V_{\mathcal{D}} := \text{Sym}^{\nu_1-\nu_2+1}(\mathbb{C}^2) \). We shall fix a weight \( \mathcal{D} \), and consider the asymptotics of \( \Pi_{k, \mathcal{D}}(x,y) \) when \( k \to +\infty \).

Let us set from now on \( G = U(2) \) and \( \mathfrak{g} = \mathfrak{su}(2) \), and let \( \mathcal{O} \subseteq \mathfrak{g} \) be the coadjoint orbit through \( \mathcal{D} \) (with some abuse of language, we view \( \mathcal{D} \) as an element of \( \mathfrak{g} \), and identify \( g \equiv g^\vee \) equivariantly)). Furthermore, let \( T \leq G \) be the standard maximal torus, with Lie algebra \( \mathfrak{t} \subseteq \mathfrak{g} \).

The restricted action of \( T \) is also Hamiltonian, and its moment map \( \Phi_T : M \to \mathfrak{t} \) is the composition of \( \Phi_G \) with the orthogonal projection \( \mathfrak{g} \to \mathfrak{t} \).

We shall make the following assumptions:

(i) \( 0 \not\in \Phi_T(M) \), and therefore also \( 0 \not\in \Phi_G(M) \);

(ii) \( \Phi_G \) is transverse to the ray \( \mathbb{R}_+ \cdot \mathcal{D} \), or equivalently to the cone over the coadjoint orbit, \( \mathbb{R}_+ \mathcal{O} \);

(iii) \( \Phi_T \) is transverse to \( \mathbb{R}_+ \mathcal{D} \).

For example, consider the unitary representation of \( G \) on \( \mathbb{C}^4 \cong \mathbb{C}^2 \times \mathbb{C}^2 \) given by \( A \cdot (Z,W) := (AZ,AW) \). By restriction we obtain a contact action \( \tilde{\mu} \) on \( S^7 \), and by passing to projective space an Hamiltonian action \( \mu \) on \( \mathbb{P}^3 \). The positive line bundle \( A \) is of course the hyperplane line bundle, \( X = S^7 \) and \( \pi : S^7 \to \mathbb{P}^3 \) is the Hopf map. All the previous assumptions are then satisfied for any pair \( \mathcal{D} \) with \( \nu_1 > \nu_2 \). An explicit plethysm computation yields that \( \dim H_{\mathcal{D}}(X) = O(k^2) \) (see [6] for a detailed discussion of explicit examples).

To begin with, \( \Pi_{k, \mathcal{D}}(x,x) \) does not have a uniform asymptotic expansion in this case, but rather it concentrates over a certain hypersurface in \( M \). More precisely, let us set
\(M_G := \pi^{-1}(R_+ \mathcal{O})\). Under the previous assumptions, \(M_G \subseteq M\) is a connected real hypersurface in \(M\), and \(M \setminus M_G\) has two connected components, that we shall euphemistically call the ‘inside’, \(A\), and the ‘outside’, \(B\). Explicitly, if \(m \in M\) the image in \(t \cong R^2\) of the orbit \(G \cdot m\) is a closed segment \(J_m\) on the line \(x + y = \text{trace}(\Phi_G(m))\). Then \(m \in B\) if and only if \(R_+ \mathcal{V}\) does not intersect \(J_m\), \(m \in M_G\) if and only if \(R_+ \mathcal{V}\) intersects \(J_m\) in an endpoint, and \(m \in A\) if and only if \(R_+ \mathcal{V}\) intersects \(J_m\) in an interior point.

Then \(\Pi_k \mathcal{V}(x, y) = O(k^{-\infty})\), unless \((m_x, m_y) \in M_G \times M_G^c\), and \(x \in G \cdot y\). Rapid decrease holds uniformly for pairs \((x, y)\) converging from the ‘outside’ \(B\) at a sufficiently slow pace to the locus where the previous conditions are satisfied.

We are thus led to consider the asymptotics of \(\Pi_k \mathcal{V}(x, x)\) when \(m_x \in M_G^c\). Under the previous assumptions, the action of \(G\) on the inverse image \(X_G^c := \pi^{-1}(M_G^c) \subseteq X\) is locally free. We shall make the stronger simplifying assumption that \(G\) acts freely on \(X_G^c\) (this is the case in the examples discussed in [6]). Then \(\Pi_k \mathcal{V}(x, x)\) admits an asymptotic expansion for \(k \to +\infty\) of the following form:

\[
\Pi_k \mathcal{V}(x, x) \sim \frac{C}{\|\Phi_G(m)\|^{d+1/2}} \cdot D_\mathcal{V}(m) \cdot \left( \frac{\|\mathcal{V}\| \cdot k}{\pi} \right)^{d-1/2} \cdot \left[ 1 + \sum_{j \geq 1} k^{-j/4} a_j(m_x) \right],
\]

for appropriate \(C^\infty\) functions \(a_j\). Here \(C\) is a universal constant, and \(D_\mathcal{V}(m)\) is a distortion function in the normal direction to \(M_G^c\) (we refer to [6] for precise definitions).

We can give a rescaled version of the previous result, at least for ‘horizontal displacements’ along direction perpendicular to the orbits of the complexified action. Namely, if \(x \in X_G^c\) and \(v_j \in T_{m_x} M, j = 1, 2\), satisfy \(h_{m_x}(v_j, \xi_M(m)) = 0\) for every \(\xi \in g\), then the previous pointwise asymptotics may be refined as follows:

\[
\Pi_k \mathcal{V} \left( x + \frac{v_1}{\sqrt{k}} x + \frac{v_2}{\sqrt{k}} \right) \sim \frac{C \cdot e^{\psi_2(v_1, v_2)/k} \mathcal{V}(m_x)}{\|\Phi_G(m)\|^{d+1/2}} \cdot D_\mathcal{V}(m) \cdot \left( \frac{\|\mathcal{V}\| \cdot k}{\pi} \right)^{d-1/2} \cdot \left[ 1 + \sum_{j \geq 1} k^{-j/4} a_j(\mathcal{V}, m_x; v_1, v_2) \right],
\]

where \(a_j(\mathcal{V}, m_x; \cdot, \cdot)\) is a polynomial of degree \(\leq \lfloor 3j/2 \rfloor\); the asymptotic expansion holds uniformly for \(\|v_j\| \leq C k^\epsilon\), for any \(C > 0\) and \(\epsilon \in (0, 1/6)\).

For normal displacements, there is an asymptotic expansion whose terms are expressed by a less terse integral formula; however, it can be used to obtain an estimate on the dimension of the isotypical component:

\[
\dim H_k \mathcal{V}(X) \geq C \cdot \left( \frac{k \|\mathcal{V}\|}{\pi} \right)^{d-1} \cdot \int_{M_G^c} \frac{D_\mathcal{V}(m)}{\|\Phi_G(m)\|^d} \, dV_{M_G^c}(m) + O \left( k^{d-3/4} \right).
\]

Although our presentation is limited to the Kähler setting for the sake of simplicity, the previous results may be naturally extended to the more general almost complex framework of [2] and [11].
References

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