Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

Investigation of the nonlinear problem of deforming sandwich plate with transversally soft core

To cite this article: I B Badriev *et al* 2019 *J. Phys.: Conf. Ser.* **1158** 022016

Investigation of the nonlinear problem of deforming sandwich plate with transversally soft core

I B Badriev[1], V V Banderov[1], M V Makarov[1, 2], O V Pankratova[1] and E V Smirnova[1]

[1]Kazan Federal University, 18 Kremlyovskaya Street, 420008, Kazan, Russia
[2]Kazan National Research Technical University, 10 K.Marks Street, 420111, Kazan, Russia

E-mail: ildar.badriev1@mail.ru

Abstract. The study of the problem of finding the stress-strain state of a sandwich plate with a transversally soft core under the action of a transverse load in a one-dimensional geometrically nonlinear formulation was carried out. A generalized formulation of the problem in the form of an operator equation is proposed. The properties of the operator of the equation are established, which make it possible to use the general results of the theory of monotone operators in the study of correctness. To find the stress-strain state of the plate, a two-layer iterative method is proposed with lowering the nonlinearity on the lower layer. A finite-dimensional approximation of the problem and the iterative method was carried out. The convergence of finite-dimensional approximations and the iterative method has been studied. For the numerical implementation of the proposed approximate methods, a software package has been developed in the Matlab environment. Based on it, numerical experiments were carried out.

1. Introduction

Multilayer structures, in particular plates, are widely used in various areas of modern technology: aerospace, aviation, shipbuilding; industrial, civil and transport construction, chemical and power engineering [1–6]. Interest in laminated plates is primarily due to the fact that they have a set of properties and features that qualitatively distinguish them from traditional structures. Multilayer structures usually consist of different materials with significantly different physical and mechanical properties. For bearing layers, as a rule, materials with high elastic moduli are used, which perceive the main part of external force effects. The core serves for the monolithic structure and provides the redistribution of forces between the carrier layers, as well as performs a number of other functions, for example, protection from radiation, heat and sound protection, etc. [7–10]

In this paper, we consider a sandwich plate consisting of two external carrier layers and a transversally soft core located between them and connected to the carrier layers by means of adhesive bonding. A generalized formulation of the problem in the form of an operator equation is proposed. The properties of the operator of the equation are established. This made it possible, in the study of correctness, to use the general results of the theory of monotone operators [11]. To find the stress-strain state of the plate, a two-layer iterative method is proposed with lowering the nonlinearity on the lower layer. A finite-dimensional approximation of the problem and the iterative method was carried out. The convergence of finite-dimensional approximations and the iterative method has been studied.

For the numerical implementation of the proposed approximate methods, a software package has been developed in the Matlab environment. Based on it, numerical experiments were carried out. Note that a physically nonlinear problem of determining the equilibrium position of a soft network shells and methods of solving them have been studied in [12–15]. Geometrically nonlinear problems were considered in [16–20]. The nonlinear problems of the shells theory were studied in [21–26]. The case when both ends of the plate are rigidly fixed was studied in [12, 27].

2. Problem statement

We consider the problem of determining the stress-strain state of an infinitely wide sandwich plate with a transversally soft core. The plate length is equal a, the thickness of the aggregate is *2h*, the thickness of the supporting layers are equal $2h_{(k)}$, where *k* is the layer number. The study of the processes of deformation of such elements, first of all, is dictated by the need to determine the degree of their suitability for further use. To describe the stress-strain state in bearing layers, the equations of the Kirchhoff-Love model are used; in the filler, the equations of elasticity theory, simplified within the accepted model of the transversally soft layer and integrated across the thickness with satisfaction of the conjugation conditions of the layers by displacement. In accordance with [8, 9], we introduce the following notation: $H_{(k)} = h + h_{(k)}$ (Hereinafter we assume that *k* = 1, 2), $X^1_{(k)}$, $X^3_{(k)}$ are the components of the surface load, reduced to the middle surface of the *k*-th layer, $w^{(k)}$, $u^{(k)}$ are the deflections and axial displacements of points the middle surface of the *k*-th layer, respectively, $T^{11}_{(k)}$, $M^{11}_{(k)}$ are membrane forces and internal bending moments in the *k*-th layer, respectively. The edges of the plate are assumed to be fixed, so that the conditions $u^{(k)}(x) = 0$, $w^{(k)}(x) = d\,w^{(k)}/dx = 0$ for $x = 0$, $x = a$ are satisfied. We consider the geometrically nonlinear case: $M^{11}_{(k)} = D_{(k)}\,d^2u^{(k)}/dx^2$, $T^{11}_{(k)} = B_{(k)}\left(d\,u^{(k)}/dx + 0.5(d\,w^{(k)}/dx)^2\right)$, where $B_{(k)} = 2h_{(k)}\,E^{(k)}/(1-\nu^{(k)}_{12}\nu^{(k)}_{21})$ is the tension-compression stiffness of the *k*-th layer, $E^{(k)}$ and $\nu^{(k)}_{12}$, $\nu^{(k)}_{21}$ are the first-kind modulus of elasticity and the Poisson coefficients of the material of the *k*-th carrier layer, $D_{(k)} = B_{(k)}\,h^2_{(k)}/3$ is the flexural rigidity of the *k*-th layer. Let $U = (w^{(1)}, w^{(2)}, u^{(1)} w^{(2)})$ be the vector of displacements of the points of the middle surface of the *k*-th layer, be q^1 the tangential stresses in the core. For q^1 we assume that the boundary conditions $q^1(0) = q^1(a) = 0$ are satisfied. In [28, 29], to describe the stress-strain state of a three-layer plate, the potential energy strain functional was constructed: $L(U,q^1) = P(U,q^1) - A(U,q^1) - A_q(U,q^1)$, $P(U,q^1) = \frac{1}{2}\int_0^a \Big\{ \sum_{k=1}^{2} [B_{(k)}(d\,u^{(k)}/dx + \frac{1}{2}(d\,w^{(k)}/dx)^2)]^2 +$ $+D_{(k)}(d^2\,w^{(k)}/dx^2) + c_1(q^1)^2 + c_2(d\,q^1/dx)^2 + c\,(w^{(2)}_3 - w^{(1)})^2 \Big\}dx$ is the potential energy of deformation, G_{13}, E_3 are the modules of transverse shear and compression of the core, $c_1 = 2h/G_{13}$, $c_2 = h^3/3E_3$, $c_3 = E_3/(2h)$, $A(U,q^1) = \int_0^a \sum_{k=1}^{2} [X^1_{(k)}\,u^{(k)} + M^1_{(k)}\,d\,w^{(k)}/dx + X^3_{(k)}\,w^{(k)}]dx$ is the work of given external forces and moments, $M^1_{(k)}$ is the surface moment of external forces reduced to the middle surface of the *k*-th layer, $A_q(U,q^1) = \int_0^a [(u^{(1)} - u^{(2)}) - \sum_{k=1}^{2} H_{(k)}d\,w^{(k)}/dx + c_1 q^1 - c_2 d^2 q^1/dx^2]\,q^1 dx$ is the work of unknown contact tangential stresses at the corresponding displacements . It was established [29] that the solution of the problem of equilibrium of a sandwich plate are stationary points of the functional *L*.

3. Generalized statement of the problem

Let $V_k = \overset{o}{W}{}_2^{(k)}(0,a)$ be the Sobolev spaces [30] with inner products $(u,\eta)_k = \int_0^a d^k u/dx^k \, d^k \eta/dx^k dx$, k = 1, 2, $V = V_2 \times V_2 \times V_1 \times V_1$. We denote the inner product in V by $(\cdot,\cdot)_V$. The equations for the stationary points of the functional L were obtained by calculating the Gâteaux derivatives [31] of this functional. It was found that the stationary points (U,q^1) are the solution of the variational equation

$$b((U,q^1),(Z,y)) = f(Z) \quad \forall (Z,y) \in W = V \times V_1, \tag{1}$$

where the form $b(\cdot,\cdot)$ given on and the functional f given on V are determined by the formulas

$$\begin{aligned} b((U,q^1),(Z,y)) = & \int_0^a \sum_{k=1}^{2} B_{(k)}\Big[\frac{du^{(k)}}{dx} + \frac{1}{2}\Big(\frac{dw^{(k)}}{dx}\Big)^2\Big]\frac{d\eta^{(k)}}{dx}dx + \int_0^a \sum_{k=1}^{2} B_{(k)}\Big[\frac{du^{(k)}}{dx} + \frac{1}{2}\Big(\frac{dw^{(k)}}{dx}\Big)^2\Big]\frac{dw^{(k)}}{dx} \times \\ & \times \frac{dz^{(k)}}{dx}dx + \int_0^a \sum_{k=1}^{2} D_{(k)} \frac{d^2 w^{(k)}}{dx^2}\frac{d^2 z^{(k)}}{dx^2}dx + \int_0^a \sum_{k=1}^{2} c_3 \int_0^a (w^{(2)} - w^{(1)})(z^{(2)} - z^{(1)})\, dx + \\ & + \int_0^a \Big\{ \sum_{k=1}^{2} H_{(k)} \frac{dz^{(k)}}{dx} + (\eta^{(2)} - \eta^{(1)})\Big\} q^1 dx + \int_0^a \Big\{ \Big[\sum_{k=1}^{2} H_{(k)} \frac{dw^{(k)}}{dx} + (u^{(2)} - u^{(1)}) + c_1 q^1 \Big] y + \\ & + c_2 dq^1/dx\, dy/dx \Big\} dx = 0 \qquad \forall Z = (z^{(1)}, z^{(2)}, \eta^{(1)}, \eta^{(2)}) \in V, \quad \forall y \in V_1, \end{aligned} \tag{2}$$

$$f(Z) = \int_0^a \sum_{k=1}^{2} [X^1_{(k)} \eta^{(k)} + M^1_{(k)} \frac{dz^{(k)}}{dx} + X^3_{(k)} z^{(k)}]\, dx \qquad \forall Z \in V. \tag{3}$$

It is easy to verify that the form $b(\cdot,\cdot)$ is linear in the second argument. In addition, it is limited by the second argument. Therefore, by virtue of the Riesz-Fisher theorem, the form $b(\cdot,\cdot)$ generates an operator $A: W \to W$ defined by the formula

$$b((U,q^1),(Z,y)) = (A(U,q^1),(Z,y))_W \quad \forall (Z,y) \in W, \tag{4}$$

where $(\cdot,\cdot)_W$ is the inner product in W. Moreover, the following result is true.

Theorem 1. *The operator $A: W \to W$ defined by formulas* (2), (4) *is bounded.*

The functional f defined by formula (3) generates an element $F \in V$ by the formula $(F,Z)_V = f(Z)$, $Z \in V$. Thus, problem (1) can be written as an operator equation

$$A(U,q^1) = (F,0), \tag{5}$$

4. Investigation of the generalized statement of the problem.

The study of correctness is based on the following results.

The following Sobolev embedding theorem holds [30, p. 68].

Theorem 2. Let $\Omega \subset R^n$ be a bounded domain with a regular boundary Γ, $1 \le p < \infty$. Then $W_p^{(k)}(\Omega) \subset W_r^{(j)}(\Omega)$ for $0 \le j < k$ and all r such that $1/p - (k-j)/n \le 1/r < 1$; besides, for any function $u \in W_p^{(k)}(\Omega)$ the embedding inequality holds

$$\| u \|_{j,r} \le C_{jr}^{kp} \| u \|_{k,p}, \tag{6}$$

where $\| \cdot \|_{k,p}$ is the norm in $W_p^{(k)}(\Omega)$, and the constant C_{jr}^{kp} depends on Ω, j, k, p, r.

Recall that the operator $A:Y\to Y$ is called pseudo-monotone [11, 32] if it is bounded and for any weakly convergent sequence $\{v_k\}_{k=1}^{+\infty}$ in Y to v^* from the inequality $\limsup\limits_{k\to+\infty}(Av_k,v_k-v^*)_Y\le 0$ it follows that $\liminf\limits_{k\to+\infty}(Av_k,v_k-\zeta)_Y\ge(A\,v^*,v^*-\zeta)_Y$ for all $\zeta\in Y$.

Theorem 3. *The operator* $A:W\to W$ *defined by relations* (2), (4) *is pseudo-monotone.*

We say that the operator $A:Y\to Y$ satisfies a property of the of bounded Lipschitz continuity type (see [17, 33], compare with [34]) if

$$\|AU-AZ\|_Y\le\mu(R)\Phi(\|U-Z\|_Y)\qquad\forall U,Z\in Y,\tag{7}$$

where $R=\max\{\|U\|_Y,\|Z\|_Y\}$, μ is a non-decreasing on $[0,+\infty)$ function, $[0,+\infty)$ is a continuous, increasing function that satisfies the conditions $\lim\limits_{\xi\to+\infty}\Phi(\xi)=+\infty$, $\Phi(0)=0$.

Theorem 4. *The operator* $A:W\to W$ *defined by relations* (2), (4) *satisfies the property of the type of bounded Lipschitz continuity* (7) with functions $\mu(\xi)=c^*(1+\xi+\xi^2)$, $\Phi(\xi)=\xi$, where the positive constant c^* depends on a, G_{13}, E_3, t, $E^{(k)}$, $\nu_{12}^{(k)}$, $\nu_{21}^{(k)}$, $h_{(k)}$, k = 1, 2, and constant in the embedding inequalities (6).

We will say that an operator is quasi-potential [17, 33, 35] if

$$\int_0^1[(A(t(U+\hat{U})),U+\hat{U})_Y-(A(t\hat{U}),\hat{U})_Y]dt=\int_0^1(A(\hat{U}+tU),U)_Y\,dt\quad\forall U,\hat{U}\in Y.$$

Theorem 5. *The operator* $A:W\to W$ *defined by relations* (2), (4) *is quasipotential.*

Let's introduce a functional $\Psi:W\to R^1$ by the formula

$$\Psi(U,q^1)=\int_0^1(A(t(U,q^1)),U,q^1)_W\,dt\quad\forall(U,q^1)\in W.\tag{8}$$

We will say that the functional $\Psi:Y\to R^1$ is coercive [11, 34], if $\Psi:Y\to R^1$ for $\Psi:Y\to R^1$.

Theorem 6. *The functional defined by* (8) *is coercive.*

From Theorems 3–6, using the technique proposed in [17, 36–42], we can verify that the following theorem holds.

Theorem 7. *Problem* (5) *has at least one solution.*

5. Iterative method and numerical experiments

For an approximate solving of problem (5), by analogy with [43-48], its finite difference approximation is constructed in the form

$$(A_{1h}+A_{2h})(U_h,q_h^1)=(F_h,0),\tag{9}$$

where, A_{1h} is a linear operator, A_{2h} is a nonlinear operator.

To solve the difference scheme (9), we will use the following two-layer iterative process with lowering the nonlinearity on the lower layer [49-52]

$$A_{1h}\frac{(U^{(n+1)},q^{1,(n+1)})-(U^{(n)},q^{1,(n)})}{\tau}+(A_{1h}+A_{2h})(U^{(n)},q^{1,(n)})=(F_h,0),\tag{10}$$

where $(U^{(0)},q^{1,(0)})$ is the given initial approximation, $\tau>0$ is an iterative parameter. The convergence of finite-dimensional approximations and the iterative method has been studied.

The numerical implementation of the iterative method (10) is being developed. A software package has been developed in Matlab. For the model problem, numerical experiments were performed. The iteration parameter was chosen empirically. The calculations were carried out for the following characteristics: $a=1$ cm, $h_1=h_2=0.005$ cm, $h=0.05$ cm, $G_{13}=15$ MPa, $E_3=25$ MPa,

$X^3_{(1)} = 0.0319$ MPa, $X^3_{(2)} = 0$, $E^{(k)} = 7 \cdot 10^4$ MPa, $\nu^{(k)}_{12} = \nu^{(k)}_{21} = 0.3$, $X^1_{(k)} = 0$, $M^1_{(k)} = 0$, $k = 1, 2$. The number of mesh points is $N = 100$. The initial approximation $(U^{(0)}, q^{1,(0)})$ was set to zero. Calculations according to (10) were carried out as long as the residual norm remained greater than the specified accuracy $\varepsilon = 5 \cdot 10^{-6}$. The optimal (in terms of the number of iterations) value of the iteration parameter was $\tau = 1$, the number of iterations being equal to 17.

The results of numerical experiments are shown in Fig. 1–3. It should be noted that the formulated for q^1 boundary conditions correspond to the absence of diaphragms at the edges $x = 0$, $x = a$, which leads to the formation of maximum transverse tangential stresses in the aggregate cross sections at a distance of the order of its thickness $2h$, which is observed in Fig. 2. The restriction of the free displacement of the end sections $x = 0$, $x = a$, in the direction of the axis Ox, leads to the formation in the bearing layers of significant membrane forces $T^{11}_{(1)}$, $T^{11}_{(2)}$, from which the force $T^{11}_{(1)}$ in the cross-section $x = a/2$ turns out to be compressing. Because of this, in the vicinity of this cross section, we should expect a loss of stability of the carrier layers in a mixed form (see [53]), the study of which requires the formulation of the corresponding problem.

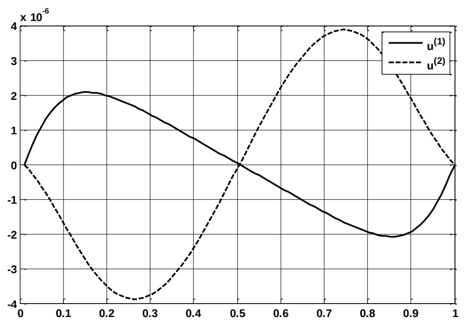

Figure 1. Axial displacements in carrying layers

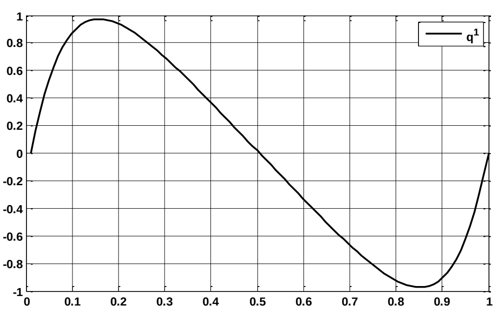

Figure 2. Tangential stresses in core

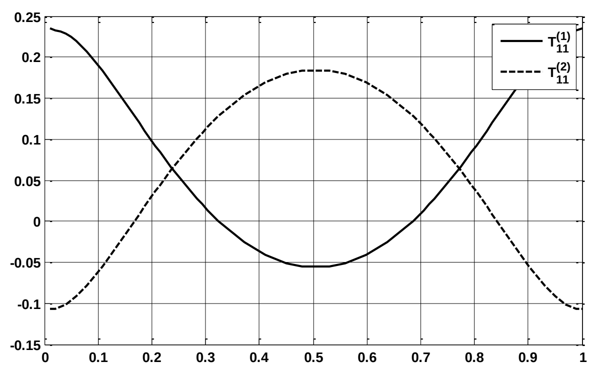

Figure 3. Membrane forces in carrying layers

It is easy to verify that the equality $T^{11}_{(1)} + T^{11}_{(2)} = \text{const}$ holds, which you can be satisfied with on the basis of the results shown in fig. 3.

Acknowledgments
This work was supported by the Russian Science Foundation (project 16-11-10299).

References

[1] Minjing L and Zhanjun W 2016 Application of composite honeycomb sandwich structure in aircraft *J. Science&Technology Review* **34** (8) 21-5
[2] Badriev I B, Makarov M V and Paimushin V N 2016 Mathematical Simulation of Nonlinear Problem of Three-point Composite Sample Bending Test *Procedia Engineering* **150** 1056-62 DOI: 10.1016/j.proeng.2016.07.214
[3] Badriev I B, Makarov M V and Paimushin V N 2016 Numerical Investigation of Physically Nonlinear Problem of Sandwich Plate Bending *Procedia Engineering* **150** 1050-5 DOI: 10.1016/j.proeng.2016.07.213
[4] Badriev I B, Makarov M V and Paimushin V N 2015 On the interaction of composite plate having a vibration-absorbing covering with incident acoustic wave *Russian Mathematics* **59** (3) 66-71 DOI: 10.3103/S1066369X1503007X
[5] Badriev I B, Garipova G Z, Paimushin V N and Makarov M V 2015 Numerical solution of the issue about geometrically nonlinear behavior of sandwich plate with transversal soft filler *Research Journal of Applied Sciences* **10** (8) 428-35 DOI: 10.3923/rjasci.2015.428.435.
[6] Badriev I B, Makarov M V and Paimushin V N 2018 Geometrically Nonlinear Problem of Longitudinal and Transverse Bending of a Sandwich Plate with Transversally Soft Core *Lobachevskii Journal of Mathematics* **39** (3) 448-57 DOI: 10.1134/S1995080218030046
[7] Badriev I B and Paimuhin V N 2018 Mathematical modeling of a dynamic thin plate deformation in acoustoelasticity problems *IOP Conference Series: Earth and Environmental Science* **107** (1) 012095 DOI: 10.1088/1755-1315/107/1/012095
[8] Badriev I B, Makarov M V and Paimushin V N 2017 Contact statement of mechanical problems of reinforced on a contour sandwich plates with transversally-soft core *Russian Mathematics* **61** (1) 69-75 DOI: 10.3103/S1066369X1701008X
[9] Badriev I B and Paimuhin V N 2017 Refined models of contact interaction of a thin plate with positioned on both sides deformable foundations *Lobachevskii Journal of Mathematics* **38**(5) 779-93. DOI: 10.1134/S1995080217050055.
[10] Paimushin V N, Kholmogorov S A and Badriev I B 2017 Theoretical and experimental investigations of the formation mechanisms of residual deformations of fibrous layered structure composites *MATEC Web of Conferences* **129** 02042 DOI: 10.1051/matecconf/201712902042
[11] Lions J L 1969 *Quelque problèmes méthodes de résolution des problèmes aux limites nonlinéaires* (Paris, Dunod).
[12] Badriev I B, Garipova G Z, Makarov M V, Paimushin V N and Khabibullin R F 2015 Solving physically nonlinear equilibrium problems for sandwich plates with a transversally soft core *Lobachevskii Journal of Mathematics* **36** (4) 474-81 DOI: 10.1134/S1995080215040216
[13] Badriev I B, Makarov M V and Paimushin V N 2017 Numerical investigation of a physically nonlinear problem of the longitudinal bending of the sandwich plate with a transversal-soft core *PNRPU Mechanics Bulletin* (1) 39-51 DOI: 10.15593/perm.mech/2017.1.03
[14] Badriev I B, Makarov M V and Paimushin V N 2015 Solvability of physically and geometrically nonlinear problem of the theory of sandwich plates with transversally-soft core *Russian Mathematics* **59** (10) 57-60 DOI: 10.3103/S1066369X15100072
[15] Badriev I B, Banderov V V and Makarov M V 2017 Mathematical Simulation of the Problem of the Pre-Critical Sandwich Plate Bending in Geometrically Nonlinear One Dimensional Formulation *IOP Conference Series: Materials Science and Engineering* **208**(1) 012002 DOI: 10.1088/1757-899X/208/1/012002
[16] Badriev I B and Shagidullin R R 1995 A study of the convergence of a recursive process for solving a stationary problem of the theory of soft shells *Journal of Mathematical Sciences* **73** (5) 519-25 DOI: 10.1007/BF02367668
[17] Badriev I B, Zadvornov O A and Saddek A M 2001 Convergence Analysis of Iterative Methods for Some Variational Inequalities with Pseudomonotone Operators *Differential Equations* **37**

(7) 934-42 DOI: 10.1023/A:1011901503460
[18] Badriev and Shagidullin R R 1992 Study of monomeric equations of static state of soft envelope and algorithm of their solution *Izvestiya vysshikh uchebnykh zavedenii. Matematika* (1) 8-16
[19] Badriev I B, Banderov V V and Zadvornov O A 2013 On the solving of equilibrium problem for the soft network shell with a load concentrated at the point *PNRPU Mechanics Bulletin* (3) 17-35
[20] Badriev I B and Banderov V V 2014 Iterative methods for solving variational inequalities of the theory of soft shells *Lobachevskii Journal of Mathematics* **35** (4) 371-83 DOI: 10.1134/S1995080214040015
[21] Solov'ev S I 2016 Eigenvibrations of a beam with elastically attached load *Lobachevskii Journal of Mathematics* **37** 597-609 DOI: 10.1134/S1995080216050115
[22] Solov'ev S I 2017 Eigenvibrations of a bar with elastically attached load *Differential Equations* **53** (3) 409-23 DOI: 10.1134/S0374064117030116
[23] Abdrakhmanova A I, Gariffulin I R, Davydov R L, Sultanov L U and Fakhrutdinov L R 2015 Investigation of Strain of Solids for Incompressible Materials *Applied Mathematical Sciences* **9** (118) 5907-14 DOI: 10.12988/ams.2015.57507
[24] Davydov R L, Sultanov L U and Kharzhavina V S 2015 Elastoplastic model of deformation of three- dimensional bodies in terms of large strains *Global Journal of Pure and Applied Mathematics* **11** (6) 5099-108
[25] Berezhnoi D V, Balafendieva I S, Sachenkov A A and Sekaeva L R 2016 Modelling of deformation of underground tunnel lining, interacting with water-saturated soil *IOP Conference Series: Materials Science and Engineering* **158**(1) 012018. DOI: 10.1088/1757-899X/158/1/012018
[26] Berezhnoi D V and Sagdatullin M K 2015 Calculation of interaction of deformable designs taking into account friction in the contact zone by finite element method *Contemporary Engineering Sciences* **8**(21-24) 1091-8. DOI: 10.12988/ces.2015.58237.
[27] Badriev I B, Makarov M V and Paimushin V N 2016 Numerical investigation of physically nonlinear problem of sandwich plate bending *Procedia Engineering* **150** 1050-5 DOI: 10.1016/j.proeng.2016.07.213
[28] Paimushin V N and Bobrov S N 2000 Refined geometric nonlinear theory of sandwich shells with a transversely soft core of medium thickness for investigation of mixed buckling forms *Mechanics of Composite Materials* **36** (1) 59-66
[29] Paimushin V N 1987 Generalized Reissner variational principle in nonlinear mechanics of three-dimensional composite solids, with applications to the theory of multilayer shells *Mechanics of solids* **22**(2) 166-74.
[30] Adams R A 1975 *Sobolev Spaces* (New York, San Francisco, London, Academic Press)
[31] Vaĭnberg M M 1974 *Variational method and method of monotone operators in the theory of nonlinear equations* (Wiley)
[32] Brezis H 1968 Équations et inéquations non-linéaires dans les espaces vectoriels en dualité *Annales de l'institut Fourier (Grenoble)* **19** 115-75.
[33] Badriev I B, Zadvornov O A and Lyashko A D 2004 A study of variable step iterative methods for variational inequalities of the second kind *Differential Equations* **40** (7) 971-83 DOI: 10.1023/B:DIEQ.0000047028.07714.df
[34] Gajewskii H, Gröger K and Zacharias K 1974 *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen* (Berlin, Akademie-Verlag)
[35] Vainberg M M and Lavrent'ev I M 1972 Nonlinear quasi-potential operators *Dokl. Akad. Nauk SSSR* **205** (5) 1022-4 (In Russian)
[36] Solov'ev S I 1985 Fast methods for solving mesh schemes of the finite element method of second order accuracy for the Poisson equation in a rectangle *Izvestiya vysshikh uchebnykh zavedenii. Matematika* Mat. (10) 71-4
[37] Badriev I B 1989 Application of duality methods to the analysis of stationary seepage problems

with a discontinuous seepage law *Journal of Soviet Mathematics* **45** (4) 1310-14 DOI: 10.1007/BF01097084

[38] Badriev I B and Fanyuk B Y 2012 Iterative methods for solving seepage problems in multilayer beds in the presence of a point source *Lobachevskii Journal of Mathematics* **33** (4) 386-99 DOI: 10.1134/S1995080212040026

[39] Badriyev I B, Zadvornov O A, Ismagilov L N and Skvortsov E V 2009 Solution of plane seepage problems for a multivalued seepage law when there is a point source *Journal of Applied Mathematics and Mechanics* **73** (4) 434-42 DOI: 10.1016/j.jappmathmech.2009.08.007.

[40] Chebakova V J, Gerasimov A V and Kirpichnikov A P 2016 On the solving of one type of problems of mathematical physics *IOP Conference Series: Materials Science and Engineering* **158** (1) 012023 DOI: 10.1088/1757-899X/158/1/012023

[41] Badriev I B and Zadvornov O A 2003 A decomposition method for variational inequalities of the second kind with strongly inverse-monotone operators *Differential Equations* **39** (7) 936-44 DOI: 10.1023/B:DIEQ.0000009189.91279.93

[42] Badriev I B 2013 On the solving of variational inequalities of stationary problems of two-phase flow in porous media *Applied Mechanics and Materials* **392** 183-7 DOI: 10.4028/www.scientific.net/AMM.392.183

[43] Badriev I B, Banderov V V, Gnedenkova V L, Kalacheva N V, Korablev A I and Tagirov R R 2015 On the finite dimensional approximations of some mixed variational inequalities *Applied Mathematical Science* **9** (113-6) 5697-705 DOI: 10.12988/ams.2015.57480

[44] Badriev I B, Banderov V V, Lavrentyeva E E and Pankratova O V 2016 On the Finite Element Approximations of Mixed Variational Inequalities of Filtration Theory *IOP Conference Series: Materials Science and Engineering* **158** (1) 012012 DOI: 10.1088/1757-899X/158/1/012012

[45] Badriev I B 1983 Difference-schemes for linear-problems of the filtration theory with discontinuous law *Izvestiya Vysshikh Uchebnykh Zavedenii Matematika* **5** 3-12

[46] Solov'ev S I 2016 Approximation of operator eigenvalue problems in a Hilbert space *IOP Conference Series-Materials Science and Engineering* **158** (1) 012087 DOI: 10.1088/1757-899X/158/1/012087

[47] Dautov R Z, Lapin A V and Lyashko A D 1980 Some mesh schemes for quasi-linear elliptic equations *USSR Computational Mathematics and Mathematical Physics* **20**(2) 62-78. DOI: 10.1016/0041-5553(80)90024-5.

[48] Badriev I B and Pankratova O V 1992 Mixed finite-element method for nonlinear stationary problems of seepage theory *Journal of Soviet Mathematics* **61**(6) 2405-16. DOI: 10.1007/BF01100574

[49] Badriev I B, Banderov V V and Zadvornov O A 2013 On the equilibrium problem of a soft network shell in the presence of several point loads *Applied Mechanics and Materials* **392** 188-90 DOI: 10.4028/www.scientific.net/AMM.392.188

[50] Badriev I B and Nechaeva L A 2013 Mathematical simulation of steady filtration with multivalued law *PNRPU Mechanics Bulletin* (3) 37-65

[51] Badriev I B and Karchevskii M M 1994 Convergence of an iterative process in a Banach space *Journal of Mathematical Sciences* **71** (6) 2727-35 DOI: 10.1007/BF02110578

[52] Badriev I B and Karchevskii M M 1989 Convergence of the iterative Uzawa method for the solution of the stationary problem of seepage theory with a limit gradient *Journal of Soviet Mathematics* **45**(4) 1302-9 DOI: 10.1007/BF01097083

[53] Paimushin V N 2001 The theory of stability of three-layer plates and shells (stages of development, current state and areas for further research) *Solid mechanics* (2) 148–62 (in Russian)