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Application of residual power series method to time fractional gas dynamics equations

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Abstract. In this study, we proposed the power series method called Residual Power Series(RPS)method coupled with Fractional Complex Transform(FCT) to solve the time fractional nonlinear gas dynamics equations. FCT is the alternative approach in fractional calculus which transforms the fractional differential equationin to integer order differential equation making the solution in simple manner.The results reveal that RPS method is simple and convenient for similar type of time fractional nonlinear differential equations.

1. Introduction

Consider the gas dynamics equation of fractional order in t-domain:

$$\frac{\partial^{\beta} v}{\partial t^{\beta}} = -\frac{1}{2}(v^2)_x + cv(1-v) + h(x,t) \qquad t > 0, \qquad 0 < \beta \leq 1 \tag{1}$$

with initial condition

$$v(x,0) = g(x) \tag{2}$$

Wherec is a constant and $h(x,t)$ is a known function. Numerical solution of equations was studied by many researchers. Das and Kumar [4] have applied the differential transform method (DTM) to solve time fractional gas dynamics equations. Later, the same types of problems were solved by applying fractional homotopy analysis transform method by Rashidi et al. [10]. In 2013, Jagdevsinghet al. [9] has given the numerical solution of (1) by homotopy perturbation method coupled with Sumudu transform. Aminikanth et al. [8] have solved the same type of problems by new homotopy perturbation method via Laplace transform.

In the present study, we proposed a recently developed method namely Residual Power Series (RPS) Method coupled with FCT to construct an exact solution of time fractional gas dynamics equations. Thismethod was first envisioned by Jordon mathematician Abu Arqub [7] and it has successfully been applied to many situations. Linjunwang et al. [15] used RSP method to obtain semi analytic solutions of time-fractional WBK equations. Later the same technique has been investigated by FeiXu et al. [14] for solving the classical Boussinesq equations. The similar numerical technique was proposed by Alguran [11]forsolving fractional foam drainage problems.

2. Jumaries Riemann-Liouville Fractional Derivatives

Jumaries fractional derivative is a modified Riemann-Liouvilles derivative of order $'\beta'$ defined as

$$D_t^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(-\beta)} \displaystyle\int_0^t (t-\tau)^{-\beta-1} [f(\tau) - f(0)] d\tau & \beta < 0 \\ \dfrac{1}{\Gamma(-\beta)} \dfrac{d}{dt} \displaystyle\int_0^t (t-\tau)^{-\beta-1} [f(\tau) - f(0)] d\tau, & 0 < \beta < 1 \\ \left[f^{(\beta-m)}(t) \right]^{(m)}, & m \le \beta \le m+1, \quad m \ge 1 \end{cases} \tag{3}$$

2.1 Standard properties of modified Riemann-Liouvilles derivatives:

(i) $D_t^{\beta}(k) = 0, \alpha > 0, k$ is a constant

(ii) $D_t^{\beta}[kf(t)] = kD_t^{\beta} f(t), \beta > 0$

(iii) $D_t^{\beta} t^{\alpha} = \dfrac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\beta)} t^{\alpha-\beta}, \alpha > \beta > 0$

(iv) $D_t^{\beta}[g(t)h(t)] = \left[D_t^{\beta} g(t)\right] h(t) + g(t)\left[D_t^{\beta} h(t)\right]$

(v) $D_t^{\beta}\left[g\big(h(t)\big)\right] = g'_h\big(h(t)\big) D_t^{\beta} h(t)$

3. The basic concepts of FCT method

The fractional complex transform[2,3] is defined as

$$T = \frac{kt^{\alpha_1}}{\Gamma 1 + \alpha_1} \tag{4}$$

$$X = \frac{lx^{\alpha_2}}{\Gamma 1 + \alpha_2} \tag{5}$$

$$Y = \frac{my^{\alpha_3}}{\Gamma 1 + \alpha_3} \tag{6}$$

$$Z = \frac{nz^{\alpha_4}}{\Gamma 1 + \alpha_4} \tag{7}$$

where k, l, m, n are constants and $0 < \alpha_1 \le 1, 0 < \alpha_2 \le 1, 0 < \alpha_3 \le 1, 0 < \alpha_4 \le 1$.

4. Outline of RPS Method

Consider the time fractional gas dynamics equation

$$D_t^{\beta} v = -vv_x + cv(1-v) + h(x,t) \qquad t > 0, \qquad 0 < \beta \le 1, \quad c > 0 \tag{8}$$

with initial condition

$$v(x,0) = g(x) \tag{9}$$

Applying the complex transformation [1]

$$D_T v(x,T) = -vv_x + cv(1-v) + h(x,T) \tag{10}$$

According to RPS Method, we expand $v(x,T)$ as a power series about T = 0 in the following manner:

$$v(x,T) = \sum_{n=0}^{\infty} g_n T^n \tag{11}$$

Let $v_k(x,T)$ denotes the k^{th} truncated series of $v(x,T)$.

$$v_k(x,T) = g_0 + \sum_{n=1}^{k} g_n T^n, \quad k = 1,2,3 \ldots \tag{12}$$

From Eqn. (9), it is easy to verify initial residual power series solution $v_0(x,T)$ is given by

$$v_0(x,T) = g_0 = v(x,0) = g(x) \tag{13}$$

From Eqn. (12), the k[th] residual power series approximation v_k will be obtained by computing the components $g_1, g_2, \dots g_k$.
Before computing these components, we define the residual function

$$\mathrm{Resi}(x,T) = D_T v + v v_x - c v(1-v) - h(x,T) \tag{14}$$

and k[th] residual function $\mathrm{Resi}_k(x,T)$ as

$$\mathrm{Resi}_k(x,T) = D_T v_k + v_k v_{k_x} - c v_k(1-v_k) - h(x,T) \tag{15}$$

Here, we mention some useful results described in [5, 6, 12] which are essential in RPS method

(i) $\mathrm{Resi}\,(x,T) = 0$

(ii) $\lim_{k\to\infty} \mathrm{Resi}_k(x,T) = \mathrm{Res}\,(x,T)$ for all $x \in I$ and $t \geq 0$ (16)

(iii) $D_T^m \mathrm{Resi}_k(x,0) = 0,\ \ m = 0,1,2,3,..k$

Now, we have to find the coefficients $g_1(x), g_2(x), \dots$ of the RPS solution (12) as follows:
Substitute the k[th] truncated series in to the Eqn. (15) and calculate the derivative D_t^{k-1} of $\mathrm{Resi}_k(x,T)$, k =1, 2, 3 …together with Eqn. (16), we obtain the following algebraic system:

$$D_T^{k-1}\mathrm{Resi}_k(x,0) = 0, \quad k = 1,2,3,.. \tag{17}$$

From (15) and (17), we have

$$\left[D_T v_1 + v_1 v_{1_x} - c v_1(1-v_1) - h(x,T)\right]_{T=0} = 0 \tag{18}$$

$$D_T^1\left[D_T v_2 + v_2 v_{2_x} - c v_2(1-v_2) - h(x,T)\right]_{T=0} = 0 \tag{19}$$

$$D_T^2\left[D_T v_3 + v_3 v_{3_x} - c v_3(1-v_3) - h(x,T)\right]_{T=0} = 0 \tag{20}$$

$$D_T^3\left[D_T v_4 + v_4 v_{4_x} - c v_4(1-v_4) - h(x,T)\right]_{T=0} = 0 \tag{21}$$

and so on.
From the above Eqns. (18) – (21), we can easily obtain the following components:

$$g_1 = c g_0 - g_0 g_0' - c g_0^2 + h(x,0) \tag{22}$$

$$2g_2 = c g_1 - g_0 g_1' - g_0' g_1 - 2c g_0 g_1 + h_T(x,0) \tag{23}$$

$$3g_3 = c g_2 - g_1 g_1' - g_0 g_2' - g_0' g_2 - c(2 g_0 g_2 + g_1^2) + h_{TT}(x,0) \tag{24}$$

$$4g_4 = c g_3 - (g_0 g_3' + g_1 g_2' + g_1' g_2 + g_0' g_3) - 2c(g_0 g_3 + g_1 g_2) + h_{TTT}(x,0) \tag{25}$$

5. Numerical case studies

Example 5.1 **C**onsider the differential equation (1) with c = 1, $g_0(x) = e^{-x}$ and $h(x,t) = 0$
According to RPS method described in section 4, by applying the Eqns. (22) – (25), the first few coefficients of power series expansion are given by

$$g_1(x) = e^{-x},\ \ g_2(x) = \frac{e^{-x}}{2},\ \ g_3(x) = \frac{e^{-x}}{6},\ \ g_4(x) = \frac{e^{-x}}{24},..$$

Substituting the above values in to the Eqn. (12), we have

$$v(x,t) = e^{-x} + e^{-x}\left(\frac{t^\beta}{\Gamma(1+\beta)}\right) + \frac{e^{-x}}{2!}\left(\frac{t^\beta}{\Gamma(1+\beta)}\right)^2 + \frac{e^{-x}}{3!}\left(\frac{t^\beta}{\Gamma(1+\beta)}\right)^3 + \frac{e^{-x}}{4!}\left(\frac{t^\beta}{\Gamma(1+\beta)}\right)^4 + \cdots$$

which converges to the closed form solution

$$v(x,t) = e^{-x+\frac{t^\beta}{\Gamma(1+\beta)}} \tag{26}$$

which is exactly same as the results obtained by DTM[4], NHPM[8], HPM[9], FHAT[10], NIM[13], FRDTM[16] when $\beta = 1$.
Example 5.2 **C**onsider the differential equation (1) with $c = logb$, $g_0(x) = b^{-x}$ and $h(x,t) = 0$
According to RPS method described in section 4, we are utilizing the Eqns. (22) – (25), the first few coefficients of power series expansion are given by

$$g_1(x) = b^{-x}(\mathrm{logb}),\ \ g_2(x) = \frac{b^{-x}(\mathrm{logb})^2}{2},\ \ g_3(x) = \frac{b^{-x}(\mathrm{logb})^3}{6},\ \ g_4(x) = \frac{b^{-x}(\mathrm{logb})^4}{24},..$$

Substituting the above values in to the Eqn. (12), we have
which converges to the closed form solution

$$v(x,t) = b^{-x} + b^{-x}(\log b)\left(\frac{t^{\beta}}{\Gamma 1+\beta}\right) + \frac{b^{-x}}{2!}(\log b)^2\left(\frac{t^{\beta}}{\Gamma 1+\beta}\right)^2 + \frac{b^{-x}(\log b)^3}{3!}\left(\frac{t^{\beta}}{\Gamma 1+\beta}\right)^3 + \frac{b^{-x}}{4!}(\log b)^4\left(\frac{t^{\beta}}{\Gamma 1+\beta}\right)^4 + \cdots \quad (27)$$

$$v(x,t) = a^{-x+\frac{t^{\beta}}{\Gamma(1+\beta)}} \quad (28)$$

which is exactly same as the results by FHAT[10], FRDTM[16] when $\beta = 1$.

Example 5.3 Consider the differential equation (1) with $c = 1$, $g_0(x) = 1 - e^{-x}$ and $h(x,t) = -e^{-x+t}$

According to RSP method, we can obtain the values of first few power series coefficients :

$$g_1(x) = -e^{-x},\ \ g_2(x) = -\frac{e^{-x}}{2},\ \ g_3(x) = -\frac{e^{-x}}{6},\ \ g_4(x) = -\frac{e^{-x}}{24}, ..$$

Substituting the above values in to the Eqn. (12), we have

$$v(x,t) = 1 - e^{-x} - e^{-x}\left(\frac{t^{\beta}}{\Gamma 1+\beta}\right) - \frac{e^{-x}}{2!}\left(\frac{t^{\beta}}{\Gamma 1+\beta}\right)^2 - \frac{e^{-x}}{3!}\left(\frac{t^{\beta}}{\Gamma 1+\beta}\right)^3 - \frac{e^{-x}}{4!}\left(\frac{t^{\beta}}{\Gamma 1+\beta}\right)^4 - \cdots$$

which converges to the closed form solution

$$v(x,t) = 1 - e^{-x+\frac{t^{\beta}}{\Gamma(1+\beta)}} \quad (29)$$

which is exactly same as the results by LTNHPM [8], FRDTM[16] when $\beta = 1$.

6. Conclusion

In the present study, the RPS method via FCT has been successfully applied to obtain the solution of time fractional gas dynamics equationin the closed form. Thiscoupling techniqueneed not require any discretization, small perturbation and unrealistic assumptions. We can extend this approach to similar type of space time fractional differential equationsalso.Therefore, the proposed method is a promising tool for solvingfractional differential equations effectively.

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