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Explicit Computation of Cheeger Constants for some Classes of Graphs

S Opiyo, Y Soeharyadi and M W Setyabudhi

Department of Mathematics, Institut Teknologi Bandung

samie.opiyo@gmail.com

Abstract. This article deals with the explicit computation of the Cheeger constant and its connection to understanding the properties of some classes of graphs. Cheeger constant is a measure of the connectivity or disconnectivity of graphs and gives the best possible way to cut a graph. More precisely, we deal with the problem of splitting the graph into two large components of approximately equal volumes by making a small cut, which is the idea of Cheeger constant of a graph. We computed the Cheeger constants for simple classes of graphs such as 2-comb graphs, cycle graphs, complete graphs, and cube graphs. We further analyzed the dynamics of Cheeger constants of the graphs under vertex-edge transformation.

1. Introduction

One of the earliest problems in geometry was the isoperimetric problem, which was considered by the ancient Greeks [3]. The basic isoperimetric problem for graphs is essentially removing as little of the graph as possible to separate out a subset of vertices of some desired "size". Isoperimetric problems examine optimal relations between the size of the cut and the sizes of the separated parts. Roughly speaking, isoperimetric problems involving edge-cuts correspond in a natural way to Cheeger constants in spectral geometry. Cheeger constants were studied in the thesis of Cheeger [2], but they can be traced back to Polya and Szego [6]. Related works in isoperimetric/problem are found in [7], [8], [9], [10].

Cheeger constant (sometimes called isoperimetric constant) is the basic idea of cutting a set or graph into two equal volume sub-manifolds. It is defined for n -dimensional Riemannian manifold as well as for a graph. In either case, Cheeger constant measures how well the set/graph can be split into two pieces and its magnitude is a numerical measure of the degree of connectedness of a set/graph. The *Cheeger constant* of the n -manifold M , is defined by:

$$h(M) := \inf_{S_1, S_2} \frac{\text{Area}(M \cap H)}{\min \{vol(S_1), vol(S_2)\}} \quad (1)$$

where $M \cap H$ is any smooth $(n - 1)$ -dimensional submanifold of M which divides M into two disjoint sub-manifolds S_1, S_2 . The infimum may be restricted to all open subsets S of M such that $\partial S \cap M$ (relative boundary) is a smooth sub-manifold of co-dimension 1. The estimation of the Cheeger constant in this case is consistent with the minimization defining the conductance of the neighborhood graph to subsets associated with subsets of M with controlled sliding of the hypersurface [11]. This quantity was introduced by Cheeger [3] in order to bound the eigen-gap of the spectrum of the Laplacian on a manifold. A *Cheeger set* is a subset $S \subseteq M$ such that $h(S; M) = h(M)$. The Cheeger set always exists and it is unique under some conditions on the domain M [4]. Since M is a continuous



domain, the area of $M \cap H$ and volume of the subsets of M are easily be determined by integration over the appropriate domains.

Cheeger constant on a compact metric graph, X , is defined for a non-empty open subset $S \subset X$ as.

$$h(X) := \inf_{S \subset X} \frac{|\partial S|}{\min \{vol(S), vol(S^c)\}} \quad (2)$$

where ∂S = set of all edges with one vertex in S and the other in S^c , $vol(S) = \sum_{v \in S} d_v$ is the of degree of vertex v , and $S^c = X - S$ is the complement of S . The quantity $|\partial S|$ denotes the number of points in the boundary and $vol(S) = \int_X 1_S dx$ the total length of S . The infimum runs over all open subset $S \subset X$, $S \neq X$ and $S \neq \emptyset$ [5].

Suppose that X is connected and that each vertex is standard, then the corresponding (standard or Kirchhoff) Laplacian (denoted by Δ_X^{std}) has discrete spectrum [5]. Therefore, given a sample X_1, \dots, X_n in X , it is possible to translate X inside a manifold, M , however, the translation does not share a boundary with M , while its discrete volume and perimeter are left equal to those of X . It is suffices to replace the infimum in the minimization problem by minimum in the Cheeger ratio. The Cheeger ratio for a vertex set S is defined as

$$h(S) = \frac{|\partial S|}{\min \{vol(S), vol(S^c)\}} \quad (3)$$

Let $\Gamma = (V, E)$ be a simple graph and let S be a nonempty subset of vertices V of the graph and S^c the complement of S . The Cheeger Constant of $\Gamma = (V, E)$, is defined as

$$h(\Gamma) = \min \frac{|\partial S|}{\min (\sum_{v \in S} d_v, \sum_{u \in S^c} d_u)} \quad (4)$$

where $\partial S = \{\{u, v\} \in E(\Gamma) | u \in S \text{ and } v \notin S\}$, $vol(S) = \sum_{v \in S} d_v$, and d_v is the degree of the vertex v .

Although Cheeger constant is well-established, it still forms active field of research. The definitions of Cheeger constant given in equations (1), (2) and (4) are generalisation of the subject. In this paper, our focus is establishment of an alternative approach to Cheeger constant of some classes of simple finite graphs such as 2-comb graphs, cycle graphs, complete graphs, and cube graphs. Our results provide simple formulae one would require to compute Cheeger constants of the classes of graphs stated herein.

2. Method

Cheeger constant analysis depends directly on nature of graph someone is dealing with; simple graph, metric graph or a surface. In our approach, we apply first principles to analyse Cheeger constants of some simple classes of graphs. The measure we will use here takes into consideration the degree of a vertex, the subset S of the vertices of a graph Γ , and the partition of the volume of Γ , in term of the sum of the degrees of the vertices in Γ , into the volume of S and S^c respectively. Equation (4) has been used to explicitly compute the Cheeger constant of the graphs followed by analytic approach to obtain formula for each of the classes of the graphs (2-comb, cycle, complete and cube graphs).

A graph $\Gamma = (V, E)$ is called a *comb graph* if it is a tree, all vertices are of degree at most 3, and all the vertices of degree 3 lie on a single simple path. The maximal path which starts and ends in degree 3 vertices is called the *spine* of the comb graph. A path from a degree 3 vertex to a leaf which contains exactly one degree 3 vertex is called a *tooth*. Comb is a graph obtained by joining a single pendant edge to each vertex of a path. We define the *spine length*, denoted by ρ , as the number of edges it contains. To compute Cheeger constant, we partition the given graph into two subsets S and S^c such that $|S| \leq \frac{|V|}{2}$. The Cheeger ratio of the respective subsets are determined from which Cheeger constant obtain by taking the minimum. In a 2-comb graph, Γ_n with n vertices, $\sum_{v \in V} d_v = n(n-1)$. Therefore, the Cheeger constant

$$h(\Gamma_n) = \min \frac{1}{\min ("split" n(n-1))} \quad (5)$$

For instance, consider Γ_6 below

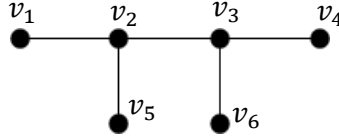


Figure 1. 2-Comb graphs with spine length 3

There are two distinct cuts; $s_1 = \{v_1\}$ with Cheeger ratio $h(S_1) = \frac{1}{\min(1,9)} = 1$ and $s_2 = \{v_1, v_2, v_5\}$ with Cheeger ratio, $h(S_2) = \frac{1}{\min(5,5)} = \frac{1}{5}$. Hence, $h(\Gamma_n) = \min\left(1, \frac{1}{5}\right) = \frac{1}{5}$.

A *cycle graph* C_n , sometimes simply known as an n -cycle, is a graph on n nodes containing a single cycle through all nodes. In a cycle graph, C_n with n vertices, $\sum_{v \in V} d_v = 2n$. Denote the subsets of the graph by S_i , $1 \leq i \leq n$ and complement S_i by S_i^c . Consider a cycle graph, C_4 , below:

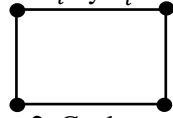


Figure 2. Cycle graphs 4

Here there are also two distinct cuts with Cheeger ratios $h(S_1) = \frac{2}{\min(2,6)} = \frac{2}{2} = 1$ and $h(S_2) = \frac{2}{\min(4,4)} = \frac{2}{4} = \frac{1}{2}$ respectively. Hence, $h(C_4) = \frac{1}{2}$.

A *cubic graph*, C^8 , is a graph in which all vertices have degree three. In other words, a cubic graph is a 3-regular graph. Cubic graphs are also called trivalent graphs. The Cheeger constants of cubic graphs were computed in the same way as in the examples above and the result revealed that

$$h(C^8) = \frac{1}{3}. \quad (6)$$

A *complete graph* denoted by K_n is a graph in which each pair of graph vertices is connected by an edge. K_n has n vertices and $\binom{n}{2} = \frac{n(n-1)}{2}$ undirected edges. We computed the Cheeger constants of complete graphs and analysed the results to generate a simple formula as in Lemma 2 here below.

3. Statement of Main Results

The formulae giving exact value of Cheeger constants of cycle graph, complete graph, cubic graph or 2-comb graph are not yet been stated explicitly. In order to determine the Cheeger constant, we used to apply equations (1), (2) or (4), which are sometimes not easy to interpret. Here, we shall give simple formulae of Cheeger constants of cycle graph, complete graph, cubic graph and 2-comb graph, which are very easy to interpret and apply.

3.1 Lemma 1

Let C_n denotes a cycle graph on n vertices. The Cheeger constant of a C_n is given by

$$h(C_n) = \begin{cases} \frac{2}{n-1}, & n \text{ odd} \\ \frac{2}{n}, & n \text{ even} \end{cases} \quad (7)$$

Proof

Let C_n be a cycle graph with n vertices, then the sum of the degree of the vertices, $\sum_{v \in V} d_v = 2n$. Cut a cycle graph into two disjoint sets, then $|\partial S| = 2$ for all possible cuts.

By definition of Cheeger constant,

$$h(C_n) = \min \frac{2}{\min(\sum_{x \in S} d_x, \sum_{y \in S^c} d_y)} = \min \frac{2}{\min("split" 2n)} = \min \frac{2}{\max_k \min(2k, 2(n-k))}$$

If n even, then $n = 2k$ for some $k \in \mathbb{N}$, then $\max_k \min(2k, 2(n-k))$ is attained for the largest $\min(2k, 2(n-k))$. This occurs when $k = \frac{n}{2}$, hence,

$$h(C_n) = \min \frac{2}{\max_k \min(2k, 2(n-k))} = \min \frac{2}{\min\left(2\left(\frac{n}{2}\right), 2\left(\frac{n}{2}\right)\right)} = \frac{2}{n}$$

Similarly if n odd, then $n = 2r + 1$ for some $r \in \mathbb{N}$, $\max_r \min(2r, 2(n-r))$ is attained for the largest $\min(2r, 2(n-r))$, which occurs when, $r = \frac{\lfloor n \rfloor}{2} = \frac{n-1}{2}$, hence,

$$h(C_n) = \min \frac{2}{\max_r \min(2r, 2(n-r))} = \min \frac{2}{\min\left(2\left(\frac{n-1}{2}\right), 2\left(\frac{n+1}{2}\right)\right)} = \frac{2}{n-1} \quad \blacksquare$$

3.2 Lemma 2

Let K_n denotes a cycle graph on n vertices. The Cheeger constant of a complete graph, K_n , is given by

$$h(K_n) = \begin{cases} \frac{n+1}{2(n-1)}, & n \text{ odd} \\ \frac{n}{2(n-1)}, & n \text{ even} \end{cases} \quad (8)$$

Proof

Let K_n be a complete graph. Then the degree of each vertex is $n-1$ and $\sum_{v \in V} d_v = n(n-1)$.

By induction steps, for $n \geq 2$, the Cheeger constant K_n is the minimum of:

Cutting 1 vertex gives,

$$h(S_1) = \frac{n-1}{\min((n-1), (n-1)^2)}$$

With 2 vertices cut, we have

$$h(S_2) = \frac{2(n-2)}{\min(2(n-1), (n-1)(n-2))}$$

With 3 vertices cut, we have

$$h(S_3) = \frac{3(n-3)}{\min(3(n-1), (n-1)(n-3))}$$

\vdots

With k steps, $k \in \mathbb{N}$ we have

$$h(S_k) = \frac{k(n-k)}{\min(k(n-1), (n-1)(n-k))}$$

We stop when $k = \frac{n}{2}$ for n even. Thus,

$$\begin{aligned} h(K_n) &= \frac{k(n-k)}{\min(k(n-1), (n-1)(n-k))} = \frac{\frac{n}{2}\left(n - \frac{n}{2}\right)}{\min\left(\frac{n}{2}(n-1), \frac{n}{2}(n-1)\right)} \\ &= \frac{\frac{n}{2}\left(\frac{n}{2}\right)}{\min\left(\frac{n}{2}(n-1), \frac{n}{2}(n-1)\right)} = \frac{n}{2(n-1)}. \end{aligned}$$

For odd n , we stop when $k = \left\lfloor \frac{n}{2} \right\rfloor$. Here $n = 2k + 1$. Thus,

$$h(K_n) = \frac{\frac{n-1}{2}(n - \frac{n-1}{2})}{\min\left(\frac{n-1}{2}(n-1), (n-1)(n - \frac{n-1}{2})\right)} = \frac{n+1}{2(n-1)} \quad \blacksquare$$

3.3 Lemma 3

Let C^8 denote a cubic graph on 8 vertices. Then the Cheeger constant of C^8 is given by

$$h(C^8) = \frac{1}{3}. \quad (9)$$

Proof

For a cubic graph,

$$\sum_{v \in V} d_v = 3n.$$

Any cubic graph has 8 vertices and is 3-regular. Thus, Cheeger constant of a cubic graph is given by

$$h(C^8) = \min \frac{\frac{n}{2}}{\min(\text{"split"} 3n)} = \min \frac{\frac{n}{2}}{\min \frac{3n}{2}} = \frac{1}{3} \quad \blacksquare$$

3.4 Conjecture 1

Let Γ_n be a 2-comb graph on n vertices. Then the Cheeger constant of Γ_n is given by

$$h(\Gamma_n) = \begin{cases} \frac{1}{\rho}, & n \text{ even} \\ \frac{1}{\rho+1}, & n \text{ odd} \end{cases} \quad (10)$$

where ρ denotes the spine length of the 2-comb graph with one end fixed to only one vertex while the other end has increasing number of vertices. More works are yet being done on this conjecture in order to provide a stronger result and proof.

4. Conclusion

In this paper, we considered the problem of cutting a graph into two disjoint subsets of approximately equal volumes, followed by analysis of the values of the Cheeger constant of the computation to generalize a simple formulae for each of the graphs named above. The results presented in this article give the easiest way to compute Cheeger constant of the graphs with only minimal efforts as compared to applying the first principles.

Cheeger constant relates directly to the algebraic connectivity of the graph. This signifies that the larger the Cheeger constant, the more difficult is to cut the graph into two disjoint subsets and vice versa. Cheeger constant is not only important in the studying of eigenvalues of Laplacians but also plays very important role in understanding class-structure of data abstracted in terms of graphs as well as graph clustering structures.

5. Further work

Compute Cheeger constant and analyse its relation with graph spectrum properties.

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