Local exponents of primitive two-colored digraph with cycles of length $s$ and $2s - 1$

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Local exponents of primitive two-colored digraph with cycles of length \( s \) and \( 2s - 1 \)

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Abstract. Let \( D \) be a digraph on \( n \) vertices \( \{v_1, v_2, \ldots, v_n\} \). A two-colored digraph \( D(2) \) is a digraph \( D \) such that each of its arcs is colored by red or blue. An \((h, k)\)-walk is a walk with precisely \( h \) red arcs and \( k \) blue arcs. A strongly connected two-colored digraph is primitive if there are nonnegative integers \( h \) and \( k \) such that for each two vertices \( v_i \) and \( v_j \) there is a \((h, k)\)-walk from \( v_i \) to \( v_j \) and a \((h, k)\)-walk from \( v_j \) to \( v_i \). The local exponent of a two-colored digraph \( D(2) \) at the vertex \( v_i \), denoted by \( \exp(v_i, D(2)) \), is the least positive integer \( h_i + k_i \) over all nonnegative integers \( h_i \) and \( k_i \) such that for every vertex \( v_i, i = 1, 2, \ldots, n \), there is a \((h_i, k_i)\)-walk from \( v_i \) to \( v_j \). For some positive integer \( s \geq 6 \), we discuss the local exponent of each vertex \( v_i \) in \( D(2) \) where \( D(2) \) is a primitive two-colored digraph containing precisely a cycle of length \( s \) and a cycle of length \( 2s - 1 \).

1. Introduction

Let \( D \) be a digraph with vertex set \( V(D) = \{v_1, v_2, \ldots, v_n\} \). A walk of length \( \ell \) from a vertex \( v_i \) to a vertex \( v_j \) is a sequence of \( \ell \) arcs of the form \( v_i = u_0 \rightarrow u_1, u_1 \rightarrow u_2, \ldots, u_{\ell-1} \rightarrow u_{\ell} = v_j \). We denote this walk by \( v_i \rightarrow^\ell v_j \). A \( v_i \rightarrow v_j \) walk is open whenever \( v_i \neq v_j \) and is closed whenever \( v_i = v_j \). A path from \( v_i \) to \( v_j \) is a \( v_i \rightarrow v_j \) walk without repeated vertices except possibly \( v_i = v_j \). A cycle is closed path. The distance from a vertex \( v_i \) to a vertex \( v_j \), denoted by \( d(v_i, v_j) \), is the length of a shortest \( v_i \rightarrow v_j \) path. A digraph is strongly connected provided for each ordered pair of vertices \( v_i \) and \( v_j \) there is a \( v_i \rightarrow v_j \) walk.

A two-colored digraph \( D(2) \) is a digraph \( D \) where each of its arcs is colored by red or blue. An \((h, k)\)-walk from a vertex \( v_i \) to a vertex \( v_j \) in a two-colored digraph \( D(2) \), denoted by \( v_i \xrightarrow{(h,k)} v_j \), is a walk of length \( h + k \) consisting of \( h \) red arcs and \( k \) blue arcs. For a \( v_i \xrightarrow{} v_j \) walk \( W_{v_i,v_j} \) in \( D(2) \), \( r(W_{v_i,v_j}) \) denotes the number of red arcs in \( W_{v_i,v_j} \), and \( b(W_{v_i,v_j}) \) denotes the number of blue arcs in \( W_{v_i,v_j} \). The length of the walk \( W_{v_i,v_j} \) is \( \ell(W_{v_i,v_j}) = r(W_{v_i,v_j}) + b(W_{v_i,v_j}) \). The vector \( \begin{bmatrix} r(W_{v_i,v_j}) \\ b(W_{v_i,v_j}) \end{bmatrix} \) is the composition of \( W_{v_i,v_j} \).

A two-colored digraph \( D(2) \) is primitive whenever there are nonnegative integers \( h \) and \( k \) such that for each ordered pair of vertices \( v_i \) and \( v_j \) in \( D(2) \) there is a \( v_i \xrightarrow{(h,k)} v_j \) walk. The smallest positive integer \( h + k \) over all such pairs of nonnegative integers \( h \) and \( k \) is the exponent of \( D(2) \) \cite{2}. The local exponent of a vertex \( v_i \) in a primitive two-colored digraph \( D(2) \), denoted by \( \expin(v_i, D(2)) \), is the smallest positive integer \( h_i + k_i \) over all pairs of nonnegative integers

\( h, k \)
(h_t, k_t) such that for each vertex v_i in D(2) there is a v_i \((h_t, k_t)\) v_t walk. See a similar definition of local exponent on [3].

Gao and Shao [3] discussed local exponent of two-colored Wielandt digraph, that is a two-colored Hamiltonian two-cycles consisting of an n-cycle and an \((n-1)\)-cycle. Mardiningsih et al. [8] discussed local exponent of primitive two-colored digraphs consisting of two cycles whose lengths differ by one. More result on local exponents of primitive two-colored digraphs can be found on [4–7, 9].

For \( s \geq 6 \), we discuss local exponents of two-colored digraph consisting of two cycles of length \( s \) and \( 2s-1 \) whose uncolored digraph is shown in Figure 1.

![Figure 1. Two-colored digraphs with cycles of length s and 2s-1](image)

In section 2 we discuss primitivity of two-colored digraph and a way in setting up lower bounds for local exponents. In Section 3, we present our results that show for each \( t = 1, 2, \ldots, n \) the \( \exp(v_t, D^{(2)}) = \exp(v_1, D^{(2)}) + d(v_1, v_t) \).

2. Preliminary Background

Let \( D^{(2)} \) be a two-colored digraph and let \( C = \{C_1, C_2, \ldots, C_p\} \) be the set of all cycles in \( D^{(2)} \). A cycle matrix of \( D^{(2)} \) is a 2 by \( p \) matrix such that its \( i \)th column, for \( i = 1, 2, \ldots, p \), is the composition of the \( i \)th cycle \( C_i \), that is \( M = \begin{bmatrix} r(C_1) & r(C_2) & \cdots & r(C_p) \\ b(C_1) & b(C_2) & \cdots & b(C_p) \end{bmatrix} \). The content of a cycle matrix \( M \) is defined to be zero if the rank of \( M \) is 1, and the content of \( M \) is defined to be the greatest common divisor of the determinants of all \( 2 \times 2 \) submatrices of \( M \), otherwise. The following result presents an algebraic characterization of a primitive two-colored digraph.

**Theorem 1.** [1] Let \( D^{(2)} \) be a strongly connected two-colored digraph with at least one arc of each color and let \( M \) be a cycle matrix of \( D^{(2)} \). The two-colored digraph \( D^{(2)} \) is primitive if and only if the content of \( M \) is 1.

The following lemma present a way in setting up a lower bound for local exponent for two-colored digraph consisting of two cycles.

**Lemma 2.** Let \( D^{(2)} \) be a primitive two-colored digraph consisting of two cycles \( C_1 \) and \( C_2 \). If \( \det(M) = 1 \) and \( \exp(v_t, D^{(2)}) \) is obtained by \((h_t, k_t)\)-walks, then \( \begin{bmatrix} h_t \\ k_t \end{bmatrix} \geq M \begin{bmatrix} b(C_2)r(P_{v_t,v_1}) - r(C_2)b(P_{v_1,v_t}) \\ r(C_1)b(P_{v_t,v_1}) - b(C_1)r(P_{v_1,v_t}) \end{bmatrix} \) for some paths \( P_{v_t,v_1} \) and \( P_{v_1,v_t} \).

Let \( f_1 = b(C_2)r(P_{v_t,v_1}) - r(C_2)b(P_{v_1,v_t}) \) and \( f_2 = r(C_1)b(P_{v_t,v_1}) - b(C_1)r(P_{v_1,v_t}) \). Lemma 2 implies that

\[
\exp(v_t, D^{(2)}) \geq \ell(C_1)f_1 + \ell(C_2)f_2
\]

for some paths \( P_{v_t,v_1} \) and \( P_{v_1,v_t} \).
3. Results

For the rest of the paper we discuss the local exponents of two-colored digraph \( D^{(2)} \) consisting of two cycles \( C_1 : v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_c \rightarrow v_{c+1} \rightarrow \cdots \rightarrow v_s \rightarrow v_1 \) of length \( s \) and \( C_2 : v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_c \rightarrow v_{s+1} \rightarrow v_{s+2} \rightarrow \cdots \rightarrow v_n \rightarrow v_1 \) of length \( 2s - 1 \) as shown in Figure 1. Let the cycle matrix of \( D^{(2)} \) be \( M = \begin{bmatrix} r(C_1) & r(C_2) \\ b(C_1) & b(C_2) \end{bmatrix} \). As a consequence of Theorem 1 it is not hard to show following result.

**Corollary 3.** Let \( D^{(2)} \) be a two-colored digraph consisting of two cycles of length \( s \) and \( 2s - 1 \), respectively. If \( D^{(2)} \) is primitive and \( \det(M) = 1 \), then the cycle matrix of \( D^{(2)} \) is 
\[
M = \begin{bmatrix} s - 1 & 2s - 3 \\ 1 & 2 \end{bmatrix}.
\]

By Corollary 3 the two-colored digraph \( D^{(2)} \) either has two blue arcs or has three blue arcs. We first discuss the local exponents for the case where the two-colored digraph \( D^{(2)} \) has two blue arcs. We note, in this case, that the blue arcs of \( D^{(2)} \) are the arc of the form \( v_x \rightarrow v_{x+1} \) for some \( 1 \leq x \leq c - 1 \) and the arc of the form \( v_y \rightarrow v_{y+1} \) for some \( s + 1 \leq y \leq n \). Let \( d_1 = d(v_{x+1}, v_1) \) and \( d_2 = d(v_{y+1}, v_1) \).

**Theorem 4.** Let \( D^{(2)} \) be a primitive two-colored digraph consisting of two cycles of length \( s \) and \( 2s - 1 \), respectively. If \( D^{(2)} \) has two blue arcs \( v_x \rightarrow v_{x+1} \), for some \( 1 \leq x \leq c - 1 \) and \( v_y \rightarrow v_{y+1} \), for some \( s + 1 \leq y \leq n \), then 
\[
\exp(v_t, D^{(2)}) = \begin{cases} 
2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t), & \text{if } d_1 \leq d_2 \\
2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t), & \text{if } d_1 > d_2
\end{cases}
\]
for each \( t = 1, 2, \ldots, n \).

**Proof.** We employ Lemma 2 in order to get the lower bound. We split the proof into two cases where \( d_1 \leq d_2 \) and \( d_1 > d_2 \).

**Case 1.** \( d_1 \leq d_2 \)

To show the lower bounds, we consider paths \( P_{v_{y+1}, v_t} \) and \( P_{v_x, v_t} \). We define \( f_1 = b(C_2)r(P_{v_{y+1}, v_t}) - r(C_2)b(P_{v_{y+1}, v_t}) \) and \( f_2 = r(C_1)b(P_{v_x, v_t}) - b(C_1)r(P_{v_x, v_t}) \) and 
\[
M = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.
\]

The vertex \( v_t \) lies on the \( v_1 \rightarrow v_x \) path. There is a \((d_2 + d(v_1, v_t), 0)\)-path \( P_{v_{y+1}, v_t} \) from \( v_{y+1} \) to \( v_t \) using this path we have \( f_1 = (2)(d_2 + d(v_1, v_t)) - (2s - 3)(0) = 2d_2 + 2d(v_1, v_t) \). There are two paths \( P_{v_x, v_t} \) from \( v_x \) to \( v_t \) which are a \((d_1 + d(v_1, v_t), 1)\)-path and a \((2s - 2 + d_1 + d(v_1, v_t), 2)\)-path. Using the \((d_1 + d(v_1, v_t), 1)\)-path we have \( f_2 = s - 1 - d_1 - d(v_1, v_t) \). Using the \((s - 2 + d_1 + d(v_1, v_t), 2)\)-path we have \( f_2 = s - d_1 - d(v_1, v_t) \). We conclude that \( f_2 = s - d_1 - d(v_1, v_t) \). Therefore,
\[
\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 2s^2 + s(2d_2 - 2d_1 - 5) + 3 + 3d_1 - 2d_2 + d(v_1, v_t) \\ 2s - 2 - 2d_1 + 2d_2 \end{bmatrix}.
\]

By Lemma 2 we have \( \exp(v_t, D^{(2)}) \geq p + q = 2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t) \) for each vertex \( v_t \) that lies on the \( v_1 \rightarrow v_x \) path.

The vertex \( v_t \) lies on the \( v_{y+1} \rightarrow v_y \) or \( v_{x+1} \rightarrow v_x \) path. There is a \((d_2 + d(v_1, v_t) - 1, 1)\)-path from \( v_{y+1} \) to \( v_t \). Using this path we have \( f_1 = 2d_2 + 2d(v_1, v_t) + 1 - 2s \). There is a \((d_1 + d(v_1, v_t) - s, 1)\)-path from \( v_x \) to \( v_t \). Using this path we have \( f_2 = 2s - 1 - d_1 - d(v_1, v_t) \).
By (1) we have \( \exp(v_t, D^{(2)}) \geq 2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t) \) for each vertex \( v_t \) that lies on the \( v_{x+1} \rightarrow v_y \) or \( v_{x+1} \rightarrow v_x \) path.

The vertex \( v_t \) lies on the \( v_{y+1} \rightarrow v_n \) path. Using the \((d_2 - d(v_1, v_i), 0)-path from v_{y+1} to v_t\) we find that \( f_1 = 2d_2 - 2d(v_1, v_t) \). Using the \((s_2 - d(v_1, v_i), 2)-path from v_x to v_t\) we find that \( f_2 = s - d_1 + d(v_1, v_t) \). Hence \( s + f_1 + (2s_2 - 1)f_2 = 2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t) \). Since \( d(v_1, v_t) = 2s_1 - 1 - d(v_1, v_t) \), by (1) \( \exp(v_t, D^{(2)}) \geq 2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t) \) for each vertex \( v_t \) lies on the \( v_{y+1} \rightarrow v_n \) path.

Therefore, for each \( t = 1, 2, \ldots, n \), we have \( \exp(v_t, D^{(2)}) \geq 2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t) \).

We now show that \( \exp(v_t, D^{(2)}) \leq 2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t) \). We first show that, for each \( j = 1, 2, \ldots, n \), there is a \( v_j \rightarrow v_1 \) walk with \( h = 2s^2 + s(2d_2 - 2d_1 - 5) + 3 + d_1 - 2d_2 \) and \( \ell = 2s - 2 - d_1 - 2d_2 \). It suffices to show that the system

\[
Mz + \begin{bmatrix} r(P_{v_1, v_1}) \\ b(P_{v_1, v_1}) \end{bmatrix} = \begin{bmatrix} h \\ \ell \end{bmatrix}
\]

has a nonnegative integer solution for some path \( P_{v_j, v_1} \) from \( v_j \) to \( v_1 \).

The solution to the system (2) is \( z_1 = 2d_2 + (2s - 3)b(P_{v_1, v_1}) - 2r(P_{v_1, v_1}) \) and \( z_2 = s - 1 - d_1 + r(P_{v_1, v_1}) - (s - 1)b(P_{v_1, v_1}) \). If the vertex \( v_j \) lies on the \( v_1 \rightarrow v_x \) path, then there is a \((r(P_{v_1, v_1}), 1)-path from \( v_j \) to \( v_1 \) with \( d_1 \leq r(P_{v_1, v_1}) \leq s - 2 \). Using this path we find \( z_1 = 2d_2 - 2r(P_{v_1, v_1}) + 2s - 3 \) and \( z_2 = r(P_{v_1, v_1}) - d_1 \). Since \( r(P_{v_1, v_1}) \leq s - 2 \), we have \( z_1 \geq 2d_2 + 1 > 0 \). Since \( r(P_{v_1, v_1}) \geq d_1 \), we have \( z_2 \geq 0 \).

If the vertex \( v_j \) lies on the \( v_{y+1} \rightarrow v_y \) path, then there is a \((r(P_{v_1, v_1}), 1)-path from \( v_j \) to \( v_1 \) with \( d_2 \leq r(P_{v_1, v_1}) \leq 2s - 2 - c \). Using this path we have \( z_1 = 2s - 3 + 2d_2 - 2r(P_{v_1, v_1}) \) and \( z_2 = r(P_{v_1, v_1}) - d_1 \). Since \( r(P_{v_1, v_1}) \leq 2s - 2 - c \) and \( d_1 + c > s \), we have \( z_1 \geq 2(d_2 + c) - (2s - 1) \geq 2(d_1 + c) - (2s - 1) > 1 \). Since \( r(P_{v_1, v_1}) \geq d_2 \geq d_1 \), we have \( z_2 \geq 0 \).

If the vertex \( v_j \) lies on the \( v_{y+1} \rightarrow v_n \) path or on the \( v_{x+1} \rightarrow v_x \) path, then there is a \((r(P_{v_1, v_1}), 0)-path form \( v_j \) to \( v_1 \) with \( 0 \leq r(P_{v_1, v_1}) \leq d_2 \). Using this path we find that \( z_1 = 2d_2 - 2r(P_{v_1, v_1}) \) and \( z_2 = s - 1 - d_1 + r(P_{v_1, v_1}) \). Since \( r(P_{v_1, v_1}) \leq d_2 \), we have \( z_1 \geq 0 \). Since \( d_1 \leq s - 1 \) and \( r(P_{v_1, v_1}) \geq 0 \), we have \( z_2 \geq r(P_{v_1, v_1}) \geq 0 \).

Therefore, for each \( j = 1, 2, \ldots, n \), there is a path \( P_{v_j, v_1} \) such that the system (2) has a nonnegative integer solution. This implies the walk that starts at \( v_j \), moves to \( v_1 \) along the chosen path \( P_{v_j, v_1} \), and then moves \( z_1 \) and \( z_2 \) times around the cycle \( C_1 \) and \( C_2 \), respectively, is a \( v_1 \rightarrow v_1 \) walk. Notice that for each vertex \( v_t, t = 2, 3, \ldots, n \), there is a unique path \( P_{v_t, v_1} \) from \( v_t \) to \( v_1 \) of length \( d(v_1, v_t) \). This implies for each vertex \( v_j, j = 1, 2, \ldots, n \), there is a \((h + r(P_{v_1, v_1})) \), \( \ell + b(P_{v_1, v_1}) \)-walk from \( v_j \) to \( v_1 \). Thus, \( \exp(v_t, D^{(2)}) \leq h + \ell + d(v_1, v_t) = 2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t) \).

Therefore, we now conclude that for each \( t = 1, 2, \ldots, n \) we have \( \exp(v_t, D^{(2)}) = 2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t) \).

**Case 2.** \( d_1 > d_2 \)

To show the upper bounds, we consider paths \( P_{v_{x+1}, v_1} \) and \( P_{v_y, v_t} \). We define \( f_1 = b(C_2)r(P_{v_{x+1}, v_1}) - r(C_2)b(P_{v_{x+1}, v_1}) \) and \( f_2 = r(C_1)b(P_{v_y, v_1}) - b(C_1)r(P_{v_y, v_1}) \) and

\[
M = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
\]

The vertex \( v_t \) lies on the \( v_1 \rightarrow v_x \) path. Using the \((s_2 - d_1 + d(v_1, v_i), 1)-path from v_{x+1} to v_t\) we have \( f_1 = 2d_1 - 1 + d(v_1, v_t) \). Using the \((d_2 + d(v_1, v_i), 1)-path from v_y to v_t\) we have \( f_2 = s - 1 - d_2 - d(v_1, v_t) \). Therefore,

\[
\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 2s^2 + s(2d_1 - 2d_2 - 6) + 4 - 2d_1 + 3d_2 + d(v_1, v_t) \\ 2s - 3 + 2d_1 - 2d_2 \end{bmatrix}
\]
Lemma 2 implies $\exp(v_1, D^{(2)}) \geq 2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t)$ for each vertex $v_t$ that lies on the $v_1 \rightarrow v_s$ path.

The vertex $v_t$ lies on the $v_{x+1} \rightarrow v_s$ path or on the $v_{x+1} \rightarrow v_y$ path. Using the $(d_1 + d(v_1, v_t) - s, 0)$-path from $v_{x+1}$ to $v_t$ we have $f_1 = 2d_1 + 2d(v_1, v_t) - 2s$. Using the $(d_2 + d(v_1, v_t) - 1, 2)$-path form $v_y$ to $v_t$ we have $f_2 = 2s - 1 - d_2 - d(v_1, v_t)$. By (1) we have $\exp(v_1, D^{(2)}) \geq 2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t)$ for each vertex $v_t$ that lies on the $v_{x+1} \rightarrow v_s$ path or on the $v_{x+1} \rightarrow v_y$ path.

The vertex $v_t$ lies on the $v_{y-1} \rightarrow v_n$ path. Using the $(s - 2d_1 - 2d(v_1, v_t), 1)$-path from $v_{x+1}$ to $v_t$ we have $f_1 = 2d_1 - 2d(v_1, v_t) - 1$. Using the $(d_2 - d(v_1, v_t), 1)$-path from $v_y$ to $v_t$ we have $f_2 = 2s^2 + s(2d_1 - 2d_2 - 6) + 4 - 2d_1 + 3d_2 - d(v_1, v_t)$. Therefore, $[p \ q] = \left[ \begin{array}{c} 2d_1 - 2d_2 + 2s - 3 \\ 2 \end{array} \right]$.

We next examine the existence of the $v_y \rightarrow v_t$ walk in $D^{(2)}$. Since the path $P_{v_y, v_1}$ is a $(d_2 - d(v_1, v_t), 1)$-path and the solution to the system $Mz + \left[ \begin{array}{c} d_2 - d(v_1, v_t) \\ 1 \end{array} \right] = \left[ \begin{array}{c} p \\ q \end{array} \right]$ is $z_1 = 2s - 4 + 2d_1 - 2d_2$ and $z_2 = 0$, there is no $v_y \rightarrow v_t$ walk in $D^{(2)}$. Hence $\exp(v_1, D^{(2)}) > p + q$. Notice that the shortest $v_y \rightarrow v_t$ walk that contains at least $p$ red arcs and at least $q$ blue arcs is a $(p + 2s - 3, q + 2)$-walk. Hence we now conclude that $\exp(v_1, D^{(2)}) \geq p + q + 2s - 1$. Since $2s - 1 - d(v_1, v_t) = d(v_1, v_t)$, we conclude that $\exp(v_1, D^{(2)}) \geq 2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t)$ for each vertex $v_t$ that lies on the $v_{y+1} \rightarrow v_n$ path.

Therefore, we now conclude that the $\exp(v_1, D^{(2)}) \geq 2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t)$ for each $t = 1, 2, \ldots, n$.

We next show that $\exp(v_1, D^{(2)}) \leq 2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t)$ for each $j = 1, 2, \ldots, n$, we first show there is a $v_j \rightarrow v_1$ walk with $h = 2s^2 + s(2d_1 - 2d_2 - 6) + 4 - 2d_1 + 3d_2$ and $\ell = 2s - 3 + 2d_1 - 2d_2$. It suffices to show that the system

$$
Mz + \left[ \begin{array}{c} r(P_{v_j, v_1}) \\ b(P_{v_j, v_1}) \end{array} \right] = \left[ \begin{array}{c} h \\ \ell \end{array} \right]
$$

has a nonnegative integer solution for some path $P_{v_j, v_1}$ from $v_j$ to $v_1$.

The solution to the system (3) is $z_1 = 2d_1 - 1 + (2s - 3)b(P_{v_j, v_1}) - 2r(P_{v_j, v_1})$ and $z_2 = s - 1 - d_2 + r(P_{v_j, v_1}) - (s - 1)b(P_{v_j, v_1})$. If the vertex $v_j$ lies on the $v_1 \rightarrow v_s$ path, then there is a $(r(P_{v_j, v_1}), 1)$-path from $v_j$ to $v_1$ with $d_1 \leq r(P_{v_j, v_1}) \leq s - 2$. Using this path we find that $z_1 = 2s - 4 + 2d_1 - 2r(P_{v_j, v_1})$ and $z_2 = r(P_{v_j, v_1}) - d_2$. Since $r(P_{v_j, v_1}) \leq s - 2$, we have $z_1 \geq 2d_1$. Since $r(P_{v_j, v_1}) \leq d_1 > d_2$, we have $z_2 \geq 1$.

If the vertex $v_j = v_{x+1}$, there is a $(s - 2 + d_1, 1)$-path from $v_j$ to $v_1$. Using this path we have $z_1 = 0$ and $z_2 = s - 2 + d_1 - d_2$. Since $d_1 > d_2$, we have $z_2 \geq 1$.

If the vertex $v_j$ lies on the $v_{x+2} \rightarrow v_s$ path or $v_{y+1} \rightarrow v_n$ path, then there is a $(r(P_{v_j, v_1}), 0)$-path from $v_j$ to $v_1$ with $0 \leq r(P_{v_j, v_1}) \leq d_1 - 1$. Using this path we find $z_1 = 2d_1 - 1 - 2r(P_{v_j, v_1})$ and $z_2 = s - 1 - d_2 + r(P_{v_j, v_1})$. Since $r(P_{v_j, v_1}) \leq d_1 - 1$, we have $z_1 \geq 1$. Since $s - 1 \geq d_1 > d_2$ and $r(P_{v_j, v_1}) \geq 1$, we have $z_2 \geq s - 1 - d_2 \geq s - 1 - d_1 \geq 0$.

If the vertex $v_j$ lies on the $v_{y+1} \rightarrow v_n$ path, then there is a $(r(P_{v_j, v_1}), 1)$-path from $v_j$ to $v_1$ with $d_2 \leq r(P_{v_j, v_1}) \leq 2s - 2 - c$. Using this path we have $z_1 = 2s - 4 + 2d_1 - 2r(P_{v_j, v_1})$ and $z_2 = r(P_{v_j, v_1}) - d_2$. Since $d_1 + c > s$ and $r(P_{v_j, v_1}) \leq 2s - 2 - c$, we have $z_1 \geq 2$. Since $d_2 \leq r(P_{v_j, v_1})$, we have $z_2 \geq 0$.

Therefore, for each $j = 1, 2, \ldots, n$, there is a path $P_{v_j, v_1}$ such that the system (3) has a nonnegative integer solution. This implies there is a $(h + r(P_{v_j, v_1}), \ell + b(P_{v_j, v_1}))$-walk from $v_j$ to $v_1$. Hence, $\exp(v_1, D^{(2)}) \leq 2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t)$. Therefore, we now conclude that $\exp(v_1, D^{(2)}) = 2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t)$ for each $t = 1, 2, \ldots, n$. There is a $(h + r(P_{v_j, v_1}), \ell + b(P_{v_j, v_1}))$-walk from $v_j$ to $v_1$. Hence, $\exp(v_1, D^{(2)}) \leq 2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t)$.
Therefore, we now conclude that \( \exp(v_t, D^{(2)}) = 2s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_2 + d(v_1, v_t) \) for each \( t = 1, 2, \ldots, n \).

We now consider the case where the two-colored digraph \( D^{(2)} \) has three blue arcs. Notice that the blue arcs of \( D^{(2)} \) are the arcs of the form \( v_x \rightarrow v_{x+1}, c \leq x \leq s, v_y \rightarrow v_{y+1}, s + 1 \leq y \leq n - 1 \) and \( v_z \rightarrow v_{z+1}, y + 1 \leq z \leq n \). Let \( d_1 = d(v_{x+1}, v_1) \), \( d_3 = d(v_{y+1}, v_1) \) and \( d_2 = d(v_{z+1}, v_1) \). We note that \( d_2 < d_3 \). We consider three cases, they are when \( d_2 < d_3 \leq d_1 \), \( d_1 \leq d_2 < d_3 \), and \( d_2 < d_1 < d_3 \).

**Theorem 5.** Let \( D^{(2)} \) be a primitive two-colored digraph consisting of two cycles of length \( s \) and \( 2s - 1 \), respectively. If \( D^{(2)} \) has three blue arcs and \( d_2 < d_3 \leq d_1 \), then for each \( t = 1, 2, \ldots, n \), the \( \exp(v_t, D^{(2)}) = 4s^2 + s(2d_1 - 2d_2 - 4) + 1 + d_3 + d(v_1, v_t) \).

**Proof.** To show the lower bounds, we consider the paths \( P_{v_{x+1}, v_1} \) and \( P_{v_{y}, v_1} \). We define \( f_1 = b(C_2)^r(P_{v_{x+1}, v_1}) - r(C_2)b(P_{v_{x+1}, v_1}) \) and \( f_2 = r(C_1)b(P_{v_{y}, v_1}) - b(C_1)r(P_{v_{y}, v_1}) \) and \( \left[ \begin{array}{c} p \\ q \end{array} \right] = M \left[ \begin{array}{c} f_1 \\ f_2 \end{array} \right] \). The vertex \( v_1 \) lies on the \( v_1 \rightarrow v_x \) path or \( v_1 \rightarrow v_y \) path. Using the \( (d_1 + d(v_1, v_1), 0) \)-path from \( v_{x+1} \) to \( v_1 \) we have \( f_1 = 2d_1 + 2d(v_1, v_1) \). Using the \( (d_3 + d(v_1, v_1) - 1, 2) \)-path from \( v_y \) to \( v_1 \) we have \( f_2 = 2s - 1 - d_3 - d(v_1, v_1) \). Therefore, \( \left[ \begin{array}{c} p \\ q \end{array} \right] = \left[ \begin{array}{c} 4s^2 + s(2d_1 - 2d_3 - 8) + 3 - 2d_1 + 3d_3 + d(v_1, v_1) \\ 4s - 2 + 2d_1 - 2d_3 \end{array} \right] \). By Lemma 2 we have \( \exp(v_1, D^{(2)}) \geq 4s^2 + s(2d_1 - 2d_3 - 4) + 1 + d_3 + d(v_1, v_1) \) for each vertex \( v_t \) that lies on the \( v_1 \rightarrow v_x \) path or \( v_1 \rightarrow v_y \) path.

The vertex \( v_t \) lies on the \( v_{x+1} \rightarrow v_s \) path. Using the \( (d_1 - d(v_1, v_1), 0) \)-path from \( v_{x+1} \) to \( v_1 \) we have \( f_1 = 2d_1 - 2d(v_1, v_1) \). Using the \( (d_3 + s - d(v_1, v_1), 2, 3) \)-path from \( v_y \) to \( v_1 \) we have \( f_2 = 2s - d_3 + d(v_1, v_1) - 1 \). Therefore, \( \left[ \begin{array}{c} p \\ q \end{array} \right] = \left[ \begin{array}{c} 4s^2 + s(2d_1 - 2d_3 - 8) - 2d_1 + 3d_3 + 3 - d(v_1, v_1) \\ 4s - 2 + 2d_1 - 2d_3 \end{array} \right] \).

We next examine the existence of \( v_{x+1} \xrightarrow{(p,q)} v_t \) walk. Since the path \( P_{v_{x+1}, v_t} \) is a \( (d_1 - d(v_1, v_1), 0) \)-path and the solution to the system \( Mz = \left[ \begin{array}{c} d_1 - d(v_1, v_1) \\ 0 \end{array} \right] \) is \( z_1 = 0 \) and \( z_2 = 2s + d_1 - d_3 - 1 \), there is no \( v_{x+1} \xrightarrow{(p,q)} v_t \) walk in \( D^{(2)} \). Hence \( \exp(v_t, D^{(2)}) > p + q \). Notice that the shortest \( v_{x+1} \rightarrow v_t \) walk that contains at least \( p \) red arcs and \( q \) blue arcs is a \( (p + s - 1, 1) \)-walk. This implies \( \exp(v_t, D^{(2)}) \geq 4s^2 + s(2d_1 - 2d_3 - 4) + 1 + d_3 + d(v_1, v_t) \) for each vertex \( v_t \) that lies on the \( v_{x+1} \rightarrow v_s \) path.

The vertex \( v_t \) lies on the \( v_{y+1} \rightarrow v_z \) path. Using the \( (2s - 2 + d_1 - d(v_1, v_1), 1) \)-path from \( v_{x+1} \) to \( v_1 \) we have \( f_1 = 2s - 1 + 2d_1 - 2d(v_1, v_1) \). Using the \( (d_3 - d(v_1, v_1), 1) \)-path from \( v_y \) to \( v_1 \) we have \( f_2 = s - 1 - d_3 + d(v_1, v_1) \). Therefore, \( \left[ \begin{array}{c} p \\ q \end{array} \right] = \left[ \begin{array}{c} 4s^2 + s(2d_1 - 2d_3 - 8) + 4 - 2d_1 + 3d_3 - d(v_1, v_1) \\ 4s + 2d_1 - 2d_3 - 3 \end{array} \right] \). We examine the existence of \( v_{x+1} \xrightarrow{(p,q)} v_t \) walk. Since the path \( P_{v_{x+1}, v_t} \) is a \( (d_3 - d(v_1, v_1), 1) \)-path and the solution to the system \( Mz = \left[ \begin{array}{c} d_3 - d(v_1, v_1) \\ 1 \end{array} \right] \) is \( z_1 = 2d_1 - 2d_3 - 3 \) and \( z_2 = 0 \), there is no \( v_y \xrightarrow{(p,q)} v_t \) walk in \( D^{(2)} \). Hence \( \exp(v_t, D^{(2)}) > p + q \). Notice that the shortest \( v_y \rightarrow v_t \) walk that contains at least \( p \) red arcs and \( q \) blue arcs is a \( (p + 2s - 3, q + 2) \)-walk. This implies \( \exp(v_t, D^{(2)}) \geq p + q + 2s - 1 \). Since \( d(v_1, v_t) = 2s - 1 - d(v_1, v_1) \), we conclude that
We next examine the existence of the shortest $2$-path from $v_2$ to $v_1$ we have $f_1 = 2d_1 - 2d_2$. Using the $(d_3 - 1 - d(v_1, v_1), 2)$-path from $v_2$ to $v_1$ we have $f_2 = 2s - 1 - d_3 + d(v_1, v_1)$. Therefore, $\left[ \begin{array}{c} p \\ q \end{array} \right] = \left[ \begin{array}{c} 4s^2 + s(2d_1 - 2d_3 - 4) + 1 + d_3 + d(v_1, v_1) \\ 2 \end{array} \right].$

We next examine the existence of $v_y$ to $v_1$ walk. Since the path $P_{v_y, v_1}$ is a $(d_3 - 1 - d(v_1, v_1), 2)$-path and the solution to the system $Mz + \left[ \begin{array}{c} r(P_{v_j, v_1}) \\ b(P_{v_j, v_1}) \end{array} \right] = \left[ \begin{array}{c} h \\ \ell \end{array} \right]$ has a nonnegative integer solution.

The solution to the system (4) is $z_1 = 2d_1 + b(P_{v_j, v_1})(2s - 3) - 2r(P_{v_j, v_1})$ and $z_2 = 2s - 1 - d_3 + r(P_{v_j, v_1})$. If the vertex $v_j$ lies on the $v_1 \rightarrow v_y$ path, then there is a $(r(P_{v_j, v_1}), 2)$-path with $d_3 - 1 \leq r(P_{v_j, v_1}) \leq 2s - 3$. Using this path we find $z_1 = 2d_1 + 2(s - 3) - 2r(P_{v_j, v_1})$ and $z_2 = 2s + 1 - d_3 + r(P_{v_j, v_1})$. Since $r(P_{v_j, v_1}) \leq 2s - 3$, we have $z_1 \geq 2d_1$. Since $r(P_{v_j, v_1}) \geq d_3 - 1$, we have $z_2 \geq 2s$.

If the vertex $v_j$ lies on the $v_{y+1} \rightarrow v_x$, then there is a $(r(P_{v_j, v_1}), 1)$-path with $d_1 \leq r(P_{v_j, v_1}) \leq s - c - 1$. Using this path we find that $z_1 = 2s - d_3 + 3r(P_{v_j, v_1})$. Since $r(P_{v_j, v_1}) \leq s - c - 1$, we have $z_1 \geq 2d_1 - 1 + 2c \geq 2d_1$. Since $r(P_{v_j, v_1}) \geq d_1$, we have $z_2 \geq 2s - d_1 + d_2 > 0$.

If the vertex $v_j$ lies on the $v_{y+1} \rightarrow v_y$ path, then there is a $(r(P_{v_j, v_1}), 1)$-path from $v_j$ to $v_1$ with $d_2 \leq r(P_{v_j, v_1}) \leq d_3 - 1$. Using this path we find that $z_1 = 2s + 2d_1 - 3 - 2r(P_{v_j, v_1})$. Since $r(P_{v_j, v_1}) \leq d_3 - 1$, we have $z_1 \geq 2s - 1$. Since $r(P_{v_j, v_1}) \geq d_2$, we have $z_2 \geq s - d_1 + d_2 > 0$.

If the vertex $v_x$ is on the $v_{y+1} \rightarrow v_x$ path or on the $v_{y+1} \rightarrow v_n$ path, then there is a $(r(P_{v_j, v_1}), 0)$-path with $1 \leq r(P_{v_j, v_1}) \leq d_1$. Using this path we have $z_1 = 2d_1 - 2r(P_{v_j, v_1})$ and $z_2 = 2s - 1 - d_3 + r(P_{v_j, v_1})$. Since $r(P_{v_j, v_1}) \leq d_1$, we have $z_1 \geq 0$. Since $r(P_{v_j, v_1}) \geq 1$, we have $z_2 \geq 2s - d_3 \geq 2s - d_1 > 0$.

Therefore, for each $j = 1, 2, \ldots, n$, there is a path $P_{v_j, v_1}$ such that the system (4) has a nonnegative integer solution. This implies for each $j = 1, 2, \ldots, n$, there is a $v_j \rightarrow (h, \ell)$ walk in $D(2)$ and for any vertex $v_1$ in $D(2)$ there is a $(h + r(P_{v_1, v_1}), \ell + b(P_{v_1, v_1}), 0)$-walk from $v_2$ to $v_1$. Hence, for each $t = 1, 2, \ldots, n$, the $exp(v_1, D(2)) \leq h + \ell + d(v_1, v_1) = 4s^2 + s(2d_1 - 2d_3 - 4) + 1 + d_3 + d(v_1, v_1)$.

Theorem 6. Let $D(2)$ be a primitive two-colored digraph consisting of two cycles of length $s$ and $2s - 1$, respectively. Suppose $D(2)$ has three blue arcs and $d_1 \leq d_2 < d_3$, then for each
t = 1, 2, \ldots, n

\[
\exp(v_t, D^{(2)}) = \begin{cases}
2s^2 - 3s + 1 + d_3 + d(v_1, v_t) & \text{if } d_3 - d_1 \leq s \text{ and } 2s - 1 < 2d_3 - 2d_2, \\
4s^2 + s(2d_2 - 2d_3 - 4) + 1 + d_3 + d(v_1, v_t) & \text{if } d_3 - d_1 \leq s \text{ and } 2s - 1 > 2d_3 - 2d_2, \\
s(2d_3 - 2d_1 - 2) + 1 + d_3 + d(v_1, v_t) & \text{if } d_3 - d_1 > s \text{ and } 2s - 1 < 2d_3 - 2d_2, \\
2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_3 + d(v_1, v_t) & \text{if } d_3 - d_1 > s \text{ and } 2s - 1 > 2d_3 - 2d_2.
\end{cases}
\]

**Proof.** Suppose \(k(D^{(2)})\) is obtained by \((h, \ell)\)-walk. We employ Lemma 2 in order to set the lower bound for \(h + \ell\). To set the upper bound we consider the paths \(P^{(2)}\). To get the lower bound we consider the paths \(P^{(2)}\). There is a \(v_j \rightarrow v_1\) walk.

**Case 1.** \(d_3 - d_1 \leq s\) and \(2s - 1 < 2d_3 - 2d_2\)

To show that \(\exp(v_t, D^{(2)})\) has a nonnegative integer solution.

To show that, for each \(v_j\), there is a path \(P_{v_j, v_1}\) from \(v_j\) to \(v_1\) such that the system

\[
Mz + \begin{bmatrix}
r(P_{v_j, v_1}) \\
b(P_{v_j, v_1})
\end{bmatrix} = \begin{bmatrix}
2s^2 - 5s + 2 + d_3 \\
2s - 1
\end{bmatrix}
\]

has a nonnegative integer solution.

**Case 2.** \(2s - 1 > 2d_3 - 2d_2\) and \(d_3 - d_1 \leq s\)

To show that, for each \(v_j\), there is a path \(P_{v_j, v_1}\) from \(v_j\) to \(v_1\) such that the system

\[
Mz + \begin{bmatrix}
r(P_{v_j, v_1}) \\
b(P_{v_j, v_1})
\end{bmatrix} = \begin{bmatrix}
4s^2 + s(2d_2 - 2d_3 - 8) + 3 - 2d_2 + 3d_3 \\
4s - 2 + 2d_2 - 2d_3
\end{bmatrix}
\]

has a nonnegative integer solution.

**Case 3.** \(2s - 1 < 2d_3 - 2d_2\) and \(d_3 - d_1 > s\)

To show that, for each \(v_j\), there is a path \(P_{v_j, v_1}\) from \(v_j\) to \(v_1\) such that the system

\[
Mz + \begin{bmatrix}
r(P_{v_j, v_1}) \\
b(P_{v_j, v_1})
\end{bmatrix} = \begin{bmatrix}
2s(2d_3 - 2d_1 - 2) + 3 + 2d_1 - 2d_3 \\
2d_3 - 2d_1 - 1
\end{bmatrix}
\]

has a nonnegative integer solution.

**Case 4.** \(2s - 1 > 2d_3 - 2d_2\) and \(d_3 - d_1 > s\)

To show that, for each \(v_j\), there is a path \(P_{v_j, v_1}\) from \(v_j\) to \(v_1\) such that the system

\[
Mz + \begin{bmatrix}
r(P_{v_j, v_1}) \\
b(P_{v_j, v_1})
\end{bmatrix} = \begin{bmatrix}
r(C_2)b(P_{v_j, v_1}) - b(C_1)r(P_{v_j, v_1}) \\
f_2 = r(C_1)b(P_{v_j, v_1}) - b(C_1)r(P_{v_j, v_1})
\end{bmatrix}
\]

To show that, for each \(v_j\), there is a path \(P_{v_j, v_1}\) from \(v_j\) to \(v_1\) such that the system

\[
Mz + \begin{bmatrix}
r(P_{v_j, v_1}) \\
b(P_{v_j, v_1})
\end{bmatrix} = \begin{bmatrix}
r(C_2)b(P_{v_j, v_1}) - b(C_1)r(P_{v_j, v_1}) \\
f_2 = r(C_1)b(P_{v_j, v_1}) - b(C_1)r(P_{v_j, v_1})
\end{bmatrix}
\]
To show that, for each $t = 1, 2, \ldots, n$, the $\exp(v_t, D^{(2)}) \leq 2s^2 + s(2d_2 - 2d_1 - 3) + 1 + d_1 + d(v_1, v_t)$ it suffices to show that for each $j = 1, 2, \ldots, n$, there is a path $P_{v_j, v_1}$ from $v_j$ to $v_1$ such that the system

$$
Mz + \begin{bmatrix} r(P_{v_j, v_1}) \\ b(P_{v_j, v_1}) \end{bmatrix} = \begin{bmatrix} 2s^2 + s(2d_2 - 2d_1 - 5) + 3 + 3d_1 - 2d_2 \\ 2d_2 - 2d_1 + 2s - 2 \end{bmatrix}
$$

has a nonnegative integer solution.

**Theorem 7.** Let $D^{(2)}$ be a primitive two-colored digraph consisting of two cycles of length $s$ and $2s - 1$, respectively. Suppose $D^{(2)}$ has three blue arcs and $d_2 < d_1 < d_3$, then for each $t = 1, 2, \ldots, n$

$$
\exp(v_t, D^{(2)}) = \begin{cases} 
2s^2 - 3s + 1 + d_3 + d(v_1, v_t) & \text{if } d_3 - d_2 < s \text{ and } 2d_3 - 2d_1 > 2s - 1, \\
4s^2 + s(2d_1 - 2d_3 - 4) + 1 + d_3 + d(v_1, v_t) & \text{if } d_3 - d_2 < s \text{ and } 2d_3 - 2d_1 < 2s - 1, \\
2s^2 + s(2d_3 - 2d_2 - 2) + 1 + d_2 + d(v_1, v_t) & \text{if } d_3 - d_2 > s \text{ and } 2d_3 - 2d_1 > 2s - 1, \\
2s^2 + s(2d_3 - 2d_2 - 3) + 1 + d_2 + d(v_1, v_t) & \text{if } d_3 - d_2 > s \text{ and } 2d_3 - 2d_1 < 2s - 1.
\end{cases}
$$

**References**