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Approximate Solution of Riccati Differential Equations and DNA Repair Model with Adomian Decomposition Method

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Abstract. This paper investigates the approximate solution of Riccati differential equations, and DNA repair model which is formed by Riccati equation. The existence result of the solution is obtained by Adomian decomposition method (ADM). Some initial value problems are solved by the method to demonstrate the main results. The obtained results are compared with the Runge-Kutta method to verify the validity of the ADM. It shows that the ADM has better results than the Runge-Kutta method. This results confirm that the ADM is a suitable method for approaching the exact solution of Riccati differential equations and the proposed model.

1. Introduction

Riccati differential equation, formulated by the Italian mathematician Jacopo Francesco Riccati, is a first-order nonlinear ordinary differential equations that arise in different areas of mathematics and physics. Moreover Riccati differential equation appears on the DNA repair problem, which the solution of this equation interpreted the average number of DNA double-strand breaks (DSBs) per cell at a certain time [1]. Analytically, some forms of this differential equation are difficult to solve exactly and explicitly by elementary methods. However there is a good numerical method that can be used to obtain the approximate solution of this equation, i.e. Adomian Decomposition Method (ADM), presented by George Adomian in 1988 [2]. The method has been examined to solve some class of linear and nonlinear equations which the obtained solution rapidly closed to the exact solution, as in [3-5]. In this paper, we will discuss the use of ADM to approximate the exact solution of Riccati differential equation and the DNA repair model, and perform some numerical simulations.

2. Analysis of Method

2.1. Adomian Decomposition Method

ADM is a method to resolve various types of nonlinear differential equations without requiring a linearization or prerequisite nonlinearity [2]. ADM generates a solution in the form of series whose terms are determined by a recursive relation of the Adomian polynomial. The ADM will be described by following the procedures [2]. To introduce this method, we consider the operator equation $F[u(t)] = g(t)$, where F represents a general nonlinear ordinary differential operator and g is a given function. The linear part of F is decomposed into $L + R$, where L is an invertible operator and R is the reminder of F. Thus the equation can be written as

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$$(L+R+N)u = g$$

or

$$Lu = g - Ru - Nu \tag{1}$$

where N is a nonlinear part of F.

Applying the operator L^{-1} formally to the Eq. (1), we obtain $u = L^{-1}g - L^{-1}Ru - L^{-1}Nu$. Suppose h is the solution of the homogeneous equation $Lu = 0$, with the given initial/boundary conditions. Then the general solution of Eq. (1) is

$$u = h + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{2}$$

The next problem is the decomposition of the nonlinear term Nu. Adomian developed a very elegant technique, such that the approximate solution of Eq. (2) can be represented as

$$u = u_0 + \lambda u_1 + \lambda^2 u_2 + \ldots = \sum_{n=0}^{\infty} \lambda^n u_n \tag{3}$$

with λ is constant whereas $u_0, u_1, u_2, \ldots, u_n$ are sought. If the nonlinear operator N is attempted to Eq. (3) then

$$N(u) = N(u_0 + \lambda u_1 + \lambda^2 u_2 + \ldots)$$

Expanding $\mathrm{N(u)}$ to Maclaurin series with respect to λ we obtain

$$N(u) = A_0 + \lambda A_1 + \lambda^2 A_2 + \ldots = \sum_{n=0}^{\infty} \lambda^n A_n \tag{4}$$

with $A_n = \dfrac{1}{n!}\dfrac{d^n}{d\lambda^n} N(\sum_{k=0}^{n} \lambda^k u_k)$, where the components of A_n are called Adomian polynomials that are generated for each non-linearity Nu [1].

As an example, if $N(u) = f(u)$, the Adomian polynomials are given as:

$$A_0 = f(u_0)$$
$$A_1 = u_1 f'(u_0)$$
$$A_2 = \frac{u_1^{\;2}}{2!} f''(u_0) + u_2 f'(u_0)$$
$$A_3 = \frac{u_1^{\;3}}{3!} f'''(u_0) + u_1 u_2 f''(u_0) + u_3 f'(u_0)$$
$$\vdots$$

Now suppose $\mathrm{L^{-1}Ru}$ and $\mathrm{L^{-1}Nu}$ have an λ order, then Eq. (2) can be written as

$$u = h + L^{-1}g - \lambda L^{-1}Ru - \lambda L^{-1}Nu \tag{5}$$

If Eq. (3) and (4) are substituted for Eq. (5), then we obtain

$$\sum_{n=0}^{\infty} \lambda^n u_n = h + L^{-1}g - \lambda L^{-1}(R(\sum_{n=0}^{\infty} \lambda^n u_n)) - \lambda L^{-1}(\sum_{n=0}^{\infty} \lambda^n A_n) \tag{6}$$

Equating the coefficients of equal powers λ on both sides of Eq. (6), we obtain

$$u_0 = h + L^{-1}g$$
$$u_1 = -L^{-1}R(u_0) - L^{-1}(A_0)$$
$$u_2 = -L^{-1}R(u_1) - L^{-1}(A_1)$$
$$\vdots$$

In general can be expressed by the recursive relations

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$$u_n = -L^{-1}R(u_{n-1}) - L^{-1}(A_{n-1}), n \geq 1$$

Therefore, the approximate solution is given by

$$u = u_0 + u_1 + u_2 + \dots$$

3. Application and Numerical Simulations

3.1. ADM for Solving Riccati Differential Equations

With reference to [6], Riccati differential equations form is given by the equation below

$$\frac{du}{dx} = A(x)u^2 + B(x)u + C(x) \tag{7}$$

where A, B, and C are the real functions of the real argument x.

Now, let consider the Riccati differential Eq. (7), with the initial condition

$$u(0) = D(x)$$

The ADM requires that Eq. (7) be expressed in terms of operator form as

$$Lu - B(x)u - A(x)N_1(u) = C(x) \tag{8}$$

where $L = \frac{d}{dx}$ and $N_1(u) = u^2$.

By ADM, the approximate solution of Eq. (7) is the infinite series $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and the nonlinear term is $N_1(u) = u^2 = \sum_{n=0}^{\infty} A_n$, where A_n is the Adomian polynomial of u^2. The first component of Adomian polynomial is given by

$$\begin{aligned} A_0 &= N_1(u_0) = u_0^{\;2} \\ A_1 &= u_1 N_1^{'}(u_0) = 2u_0u_1 \\ A_2 &= \frac{u_1^{\;2}}{2!}N''(u_0) + u_2N'(u_0) = u_1^{\;2} + 2u_0u_2 \\ A_3 &= \frac{u_1^{\;3}}{3!}N'''(u_0) + u_1u_2f''(u_0) + u_3f'(u_0) = 2u_0u_3 + 2u_1u_2 \\ &\vdots \end{aligned} \tag{9}$$

Now if both side of Eq. (8) are attempted by the operator L^{-1}, we obtain

$$u - L^{-1}[B(x)u + A(x)N_1(u)] = h + L^{-1}C(x) \tag{10}$$

If $u(x) = \sum_{n=0}^{\infty} u_n(x)$ and $N_1(u) = \sum_{n=0}^{\infty} A_n$ are substituted for Eq. (10), we get

$$\sum_{n=0}^{\infty} u_n(x) - L^{-1}[B(x)\sum_{n=0}^{\infty} u_n(x) + A(x)\sum_{n=0}^{\infty} A_n] = h + L^{-1}C(x) \tag{11}$$

If equating the coefficients of equal powers of λ on both side of Eq. (11), yields the recursive relation

$$\begin{aligned} u_0 &= h + L^{-1}C(x) \\ u_n &= L^{-1}[B(x)u_{n-1} + A(x)A_{n-1}], n \geq 1 \end{aligned} \tag{12}$$

Hence, the approximate solution of Eq. (7) is

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \dots$$

with $u_0, u_1, u_2, \ldots$ is given by (12).

To give a better understanding of the methodology, the following two examples will be discussed.

Example 1: Consider the following Riccati differential equation

$$\frac{du}{dx} = u^2 - 2xu + x^2 + 1 \tag{13}$$

subject to the initial condition

$$u(0) = \frac{1}{2}$$

The exact solution of the initial condition problem (13) is given by

$$u(x) = \frac{1}{2-x} + x$$

Substituting the initial condition problem (13) into (12) and using (9) to calculate the Adomian polynomials, yields the following recursive relation

$$\begin{aligned} u_0 &= \frac{1}{2} + L^{-1}(x^2 + 1) \\ u_n &= L^{-1}[-2xu_{n-1} + A_{n-1}], n \geq 1 \end{aligned} \tag{14}$$

By using (14), the first three terms of the approximate solution $u(x)$ is given by

$$u_0 = \frac{1}{2} + \frac{1}{3}x^3 + x$$

$$u_1 = \frac{x^7}{63} + \frac{x^4}{12} - \frac{x^3}{3} + \frac{x}{4}$$

$$u_2 = \frac{2x^{11}}{2079} + \frac{x^8}{112} - \frac{2x^7}{63} + \frac{x^5}{20} - \frac{x^4}{12} + \frac{x^2}{8}$$

Hence, the third-term approximate solution for the initial condition problem (13) is given by

$$u(x) = u_0 + u_1 + u_2 = \frac{2x^{11}}{2079} + \frac{x^8}{112} - \frac{x^7}{63} + \frac{x^5}{20} + \frac{x^2}{8} + \frac{5x}{4} + \frac{1}{2}$$

The graphs of the approximate solution of (13) by ADM and Runge-Kutta method, and the error comparison between both methods are given in Fig. 1 and Fig. 2 respectively.

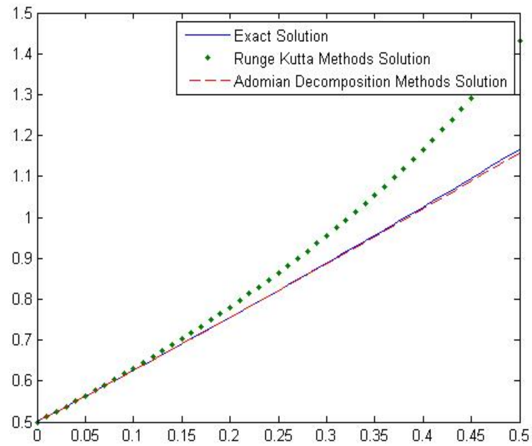

Figure 1. Comparison of the exact solution, Runge-Kutta, and ADM.

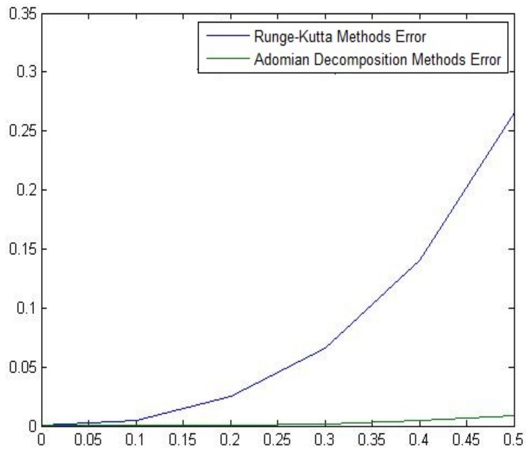

Figure 2. Error comparison between ADM and Runge-Kutta.

It can be seen in Fig. 1, that the ADM has a better approximation than the Runge-Kutta method. Moreover Fig. 2 shows clearly that the accuracy gained by ADM is much better. The error value of both methods shows an enhancement on every iteration. However, the error values of the ADM are smaller than the Runge-Kutta method.

The following table is given to see the accuracy of both methods on the initial value problem (13).

Table 1. The absolute error of the approximate solution with ADM and Runge-Kutta

n	x_n	Exact Solution	Runge-Kutta		ADM	
			Solution u	Error	Solution u	Error
0	0.0	0.5000	0.5000	0.0000	0.5000	0.0000
1	0.1	0.6263	0.6314	0.0051	0.6263	0.0001
2	0.2	0.7556	0.7805	0.0250	0.7550	0.0005
3	0.3	0.8882	0.9548	0.0665	0.8864	0.0019
4	0.4	1.0250	1.1657	0.1407	1.0205	0.0045
5	0.5	1.1667	1.4323	0.2656	1.1577	0.0089

Example 2: Consider the following Riccati differential equation

$$\frac{du}{dx} = xu^2 + (1-2x)u + x - 1 \tag{15}$$

subject to the initial condition

$$u(0) = -1$$

The exact solution of the initial condition problem (15) is

$$u(x) = 1 + \frac{1}{-\frac{3}{2}e^{-x} - x + 1}$$

Substituting the initial condition problem (15) into (12) and using (9) to calculate the Adomian polynomials, yields the following recursive relation

$$\begin{aligned} u_0 &= -1 + L^{-1}(x-1) \\ u_n &= L^{-1}[(1-2x)u_{n-1} + xA_{n-1}], n \geq 1 \end{aligned} \tag{16}$$

By using (16), the first three terms of the approximate solution $u(x)$ is given by

$$u_0 = \frac{x^2}{2} - x - 1$$

$$u_1 = \frac{x^6}{24} - \frac{x^5}{5} - \frac{x^4}{4} + \frac{3x^3}{2} + x^2 - x$$

$$u_2 = \frac{x^{10}}{240} - \frac{17x^9}{540} - \frac{x^8}{480} + \frac{341x^7}{840} - \frac{x^6}{5} - \frac{37x^5}{20} - \frac{x^4}{8} + \frac{5x^3}{3} - \frac{x^2}{2}$$

Hence, the third-term approximate solution for the initial condition problem (15) is given by

$$u(x) = u_0 + u_1 + u_2 = \frac{x^{10}}{240} - \frac{17x^9}{540} - \frac{x^8}{480} + \frac{341x^7}{840} - \frac{19x^6}{120} - \frac{41x^5}{20} - \frac{3x^4}{8} + \frac{19x^3}{6} + x^2 - 2x - 1$$

The graphs of the approximate solution of (15) by ADM and Runge-Kutta method, and the error comparison between both methods are given in Fig. 3 and Fig. 4 respectively.

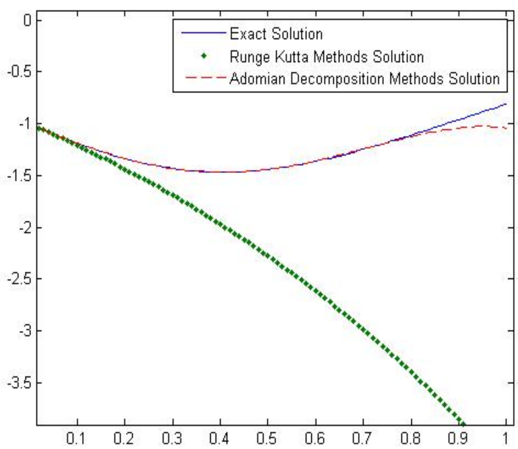

Figure 3. Comparison of the exact solution, Runge-Kutta, and ADM.

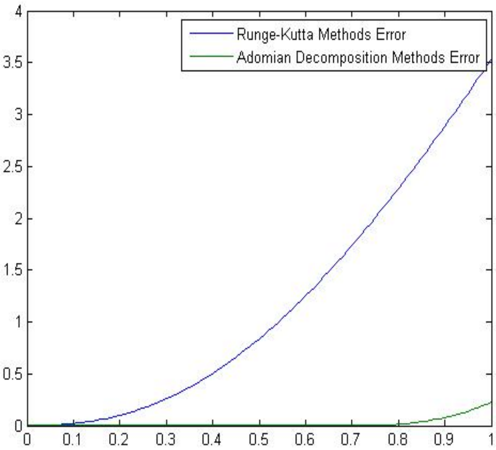

Figure 4. Error comparison between ADM and Runge-Kutta.

It is clear, as in the previous case, that the ADM has a better approximation than the Runge-Kutta method (See Fig. 3). The error value of both methods shows an enhancement on every iteration, however the error values of the ADM are smaller than the Runge-Kutta method (See Fig. 4).

The following table is given to see the accuracy of both methods on the initial value problem (15).

Table 2. The absolute error of the approximate solution with ADM and Runge-Kutta

n	x_n	Exact Solution	Runge-Kutta		ADM	
			Solution u	Error	Solution u	Error
0	0	-1.0000	-1.0000	0.0000	-1.0000	0.0000
1	0.2	-1.3359	-1.4374	0.1015	-1.3359	0.0000
2	0.4	-1.4662	-1.9690	0.5028	-1.4679	0.0017
3	0.6	-1.3629	-2.6145	1.2516	-1.3604	0.0025
4	0.8	-1.1097	-3.3971	2.2874	-1.1245	0.0148
5	1	-0.8122	-4.3446	3.5324	-1.0401	0.2279

3.2. ADM for Solving DNA Repair Model

Riccati differential equation has been used very heavily, one of the applications is in radiotherapy modeling and in studying damage to cells by ionizing radiation. The most important radiation damage is to chromatin, e.g. DNA double-strand breaks (DSBs). DNA DSBs are one of the most toxic of these lesions and must be repaired to preserve chromosomal integrity. Most DBSs are repaired during the next half hour or so, and a few are not repaired. Suppose U be the number of DSBs, then the average number of DSB per cell at time t is given by

$$\frac{dU}{dt} = \delta R - \frac{1}{\tau}U - \gamma U^2 \tag{17}$$

$$\delta, \tau, \gamma = const. \geq 0$$

where δ is the average number of DSBs induced per unit dose, R is radiation dose rate, τ is a repair time constant, and γ is a binary reaction rate constant in the sense of mass-action chemical kinetics [1].

The following illustration of using ADM for solving Eq. (17) is given. Consider the following DSBs repair model:

$$\frac{dU}{dt} = 150 - 5U - U^2 \tag{18}$$

If there are 50 number of DSBs repair at time $t = 0$, then yields the initial condition

$$U(0) = 50$$

The exact solution of the initial condition problem (18) is given by

$$U(t) = \frac{130e^{25t} + 120}{13e^{25t} - 8}$$

The recursive relation of the initial condition problem (18) is

$$\begin{aligned} U_0 &= 50 + L^{-1}(150) \\ U_n &= L^{-1}[-5u_{n-1} - A_{n-1}], n \geq 1 \end{aligned} \tag{19}$$

By using (19), the first three terms of the approximate solution $\mathrm{U(t)}$ is given by

$$U_0 = 50 + 150\mathrm{t}$$

$$U_1 = -7500\mathrm{t}^3 - 7875\mathrm{t}^2 - 2750\mathrm{t}$$

$$U_2 = -450000\mathrm{t}^5 + 393750\mathrm{t}^4 + 550625\mathrm{t}^3 + 144375\mathrm{t}^2$$

Hence, the third-term approximate solution for the initial condition problem (18) is given by

$$U(t) = U_0 + U_1 + U_2 = -450000\mathrm{t}^5 + 393750\mathrm{t}^4 + 543125\mathrm{t}^3 + 136500\mathrm{t}^2 - 2600\mathrm{t} + 50$$

The other components can be determined similarly by using the recursive relation (19). The following graph shows the approximate solution of (18) with ADM for $n = 2$, $n = 3$, and $n = 4$. The validity of this method shown by comparing the approximate solutions of the ADM, and the exact solution.

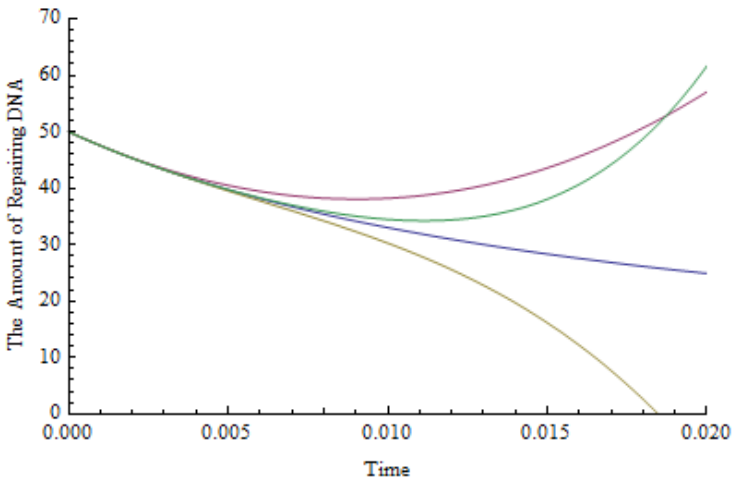

Figure 5. Comparison between the exact solution and the approximate solution with ADM. Blue line: the exact soluton ; purple line: n=2 ; yellow line: n=3 ; green line: n=4

It can be seen, the solution of the ADM is closer to the exact solution for greater order. Mathematically, the Eq. (17) can be written as

$$\frac{dU}{dt} = (\delta - U)(\gamma U + R)$$

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where $\tau = \frac{1}{R - \delta\gamma}$. It means, Eq. (18) can be expressed as

$$\frac{dU}{dt} = (10 - U)(U + 15)$$

The following figure shows the phase plane of the DNA repair model (18).

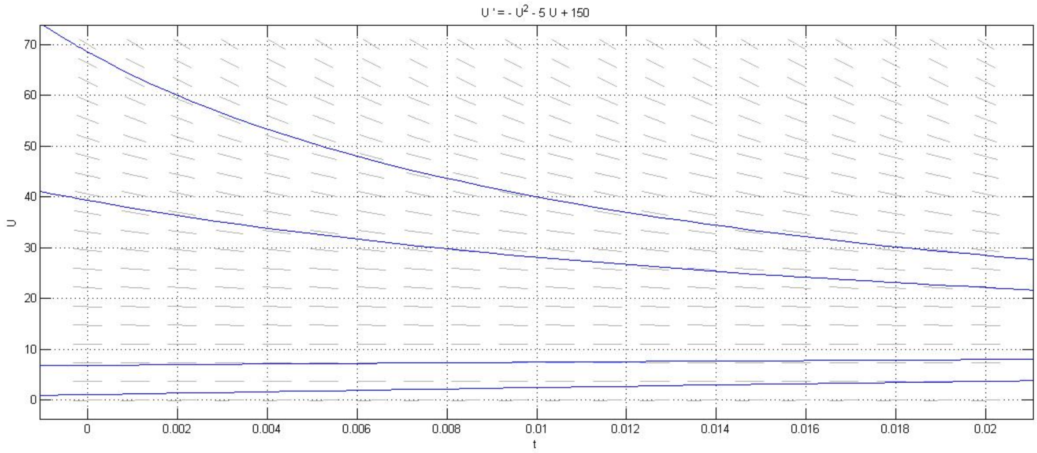

Figure 6. Phase plane of DNA repair problem (19) with $\delta = 10$, $R = 15$, and $\gamma = 1$

Based on Fig. 6, it can be analyzed that the number of repairing DNA will increase and approach to ten, if the number of induced dose DNA is below ten. If the number of induced dose DNA are in between forty and seventy, then the amount of repairing DNA will decrease and approach to twenty.

4. Conclusion

In this paper, the Adomian decomposition method has been applied to finding the approximate solution of the Riccati differential equations and the DNA repair model. All of the examples show that the numerical results of the proposed method are better than the Runge-Kutta method. The method was clearly very efficient in approaching the exact solution of the proposed equation. The flexibility of the method and reduction in the size of computational components give this method wider applicability.

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