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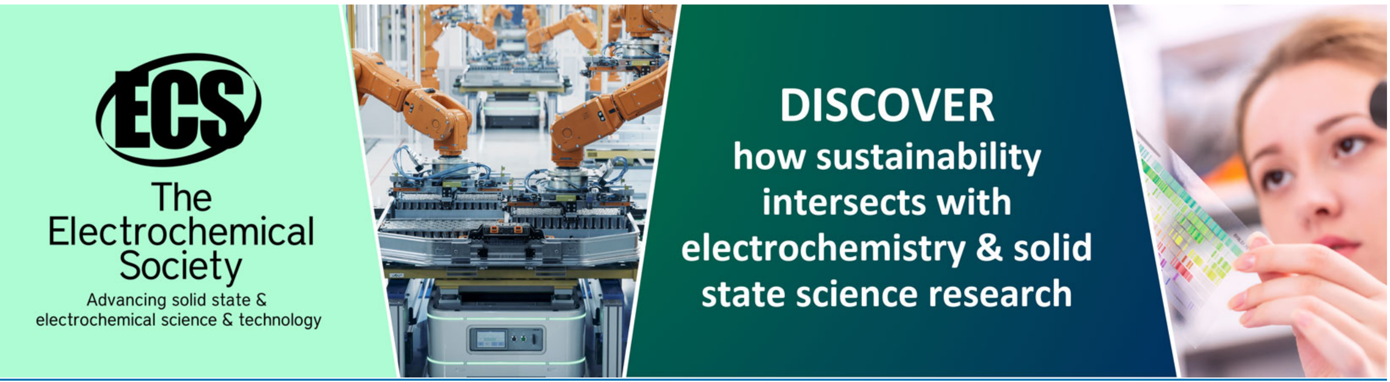

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A new aftertreatment for improving adomian modified decomposition method for solving nonlinear differential equations

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Abstract. The theory of the Adomian modified decomposition method (AMDM) for solving linear and nonlinear differential equations is well established. However, the solutions obtained by using the current AMDM are valid only for a very small region. In this paper, a new aftertreatment technique is proposed to improve the accuracy of the AMDM during a wide region. Based on the proposed aftertreatment technique, the truncated series solution obtained by the AMDM can be expressed as another series in terms of the independent sine and cosine trigonometric functions. Two numerical examples are presented and compared to those obtained from the numerical 4th-order Runge-Kutta algorithm. It is shown that the AMDM with the proposed aftertreatment technique offers an accurate and effective method for solving nonlinear differential equations in a wide applicable region.

1. Introduction

The Adomian decomposition method (ADM) [1–3] is a useful and powerful method for solving linear and nonlinear differential equations. The goal of ADM is to find the solution of linear and nonlinear, ordinary or partial differential equation without depending on any small parameter, such as the case with the perturbation method. The main advantages of ADM are computational simplicity and do not involve any linearization or smallness assumptions [1].

However, the result by using ADM is usually considered as a truncated series solution which only gives a good approximation to the accurate solution in a small region and has very small convergence radius. To overcome this drawback, two aftertreatment techniques have been proposed. One method is called as multistage ADM [4–6], which divide the solution space into M small regions. The solution is found an approximate solution in a closed and recursive form in a sequence of intervals $[0, t_1]$, $[t_1, t_2]$, …, $[t_M; T]$ such that the initial condition in sub-division interval $[t_i, t_i+1]$ is taken to be the condition at t_i. The accuracy of this aftertreatment technique is strongly dependence on the sub-division interval length. Another aftertreatment technique is based on Padé approximants, Laplace transform and its inverse to deal with the truncated series solution obtained by ADM. The combining Laplace ADM with Padé approximants have been investigated by many researchers and obtained plentiful achievement. No attempt will be made here to present a bibliographical account of previous work in this area. A few selective recent papers are quoted [7–10] which provide further references on the subject.

In this paper, a modified type of ADM, termed as the Adomian modified decomposition method (AMDM), proposed by Wazwaz and EI-Sayed [11] is used to solve nonlinear differential equations. And a new aftertreatment technique for improving the accuracy of the AMDM is proposed. The truncated series solution obtained by the AMDM can be expressed as another series in terms of the independent sine and cosine trigonometric functions based on the proposed aftertreatment technique. Finally, two numerical examples are given to demonstrate the feasibility of the proposed aftertreatment technique.

2. A brief review of Adomian modified decomposition method (AMDM)

In this section, the concept of Adomian modified decomposition method (AMDM) is briefly introduced. Consider the general nonlinear functional equation:

$$Lu(x)+Ru(x)+Nu(x)=g(x) \tag{1}$$

where L is a linear invertible operator of highest-order derivative with respect to x. R is the remainder linear operators. N is the nonlinear operator, and $g(x)$ is the source term.

As L is invertible, and applying the inverse operator L^{-1} to both sides of equation (1), we obtain

$$L^{-1}Lu(x)=L^{-1}g(x)-L^{-1}Ru(x)-L^{-1}Nu(x) \tag{2}$$

And
$$u(x)=\Phi_0+L^{-1}g(x)-L^{-1}Ru(x)-L^{-1}Nu(x) \tag{3}$$

where L^{-1} can be an integral operator defined from 0 to x. And Φ_0 is the kernel of the inverse operator L^{-1} and satisfies $L\Phi_0 = 0$.

According to the standard Adomian modified decomposition method [11–13], it defines the solution $u(x)$ by an infinite series of the form

$$u(x)=\sum_{m=0}^{\infty}C_m x^m \tag{4}$$

where the unknown coefficients C_m will be determined recurrently.

And for the non-linear term $Nu(x)$, it defines [11]

$$Nu(x)=\sum_{m=0}^{\infty}A_m(C_0,C_1,\cdots,C_m)x^m \tag{5}$$

where the A_m are the classical Adomian's polynomials [1,14].

Notice that $g(x)$ in equation (1) is also required to decompose as an infinite series

$$g(x)=\sum_{m=0}^{\infty}G_m x^m \tag{6}$$

Substituting equations (4), (5) and (6) into equation (3), we obtain

$$\begin{aligned} u(x)&=\sum_{m=0}^{\infty}C_m x^m \\ &=\Phi_0+L^{-1}\left(\sum_{m=0}^{\infty}G_m x^m\right)-L^{-1}\left(R\sum_{m=0}^{\infty}C_m x^m\right)-L^{-1}\left(\sum_{m=0}^{\infty}A_m(C_0,C_1,\cdots,C_m)x^m\right) \end{aligned} \tag{7}$$

The recurrence relation with respect to initial condition is applied to determine the coefficient C_k in equation (7). However, in practice, all series in equation (7) cannot be obtained exactly. The solution of $u(x)$ is expressed by a truncated series and will be rewritten as

$$u(x)=\sum_{m=0}^{M}C_m x^m \tag{8}$$

3. Aftertreatment technique

It is well-known that the main disadvantage of the AMDM is that solution's series may only have very small convergence radius and the truncated series solution may be inaccurate in some regions [12].

The main goal of this paper is to extend the convergence radius of the AMDM with a new proposed aftertreatment technique.

Assume that the truncated series solution in equation (8) can be expressed as another series in terms of the independent sine and cosine trigonometric functions:

$$u(x)=\sum_{k=1}^{K}\left[S_k\cos(p_k x)+W_k\sin(q_k x)\right] \tag{9}$$

where S_k, W_k, p_k, and q_k are the unknown values.

Notice that the both side of equation (9) can be expanded as a power series, yields

$$\begin{aligned}
&C_0+C_1x+C_2x^2+\ldots+C_Mx^M\\
&=\sum_{k=1}^{K}\left[S_k\left(1-\frac{p_k^2}{2!}x^2+\ldots+(-1)^k\frac{p_k^{2n}}{(2n)!}x^{2n}+\ldots\right)\right]\\
&+\sum_{k=1}^{K}\left[W_k\left(q_kx-\frac{q_k^3}{3!}x^3+\ldots+(-1)^n\frac{p_k^{2n-1}}{(2n-1)!}x^{2n-1}+\ldots\right)\right]\\
&=\sum_{k=1}^{K}S_k+\left(\sum_{k=1}^{K}W_kq_k\right)x-\left(\sum_{k=1}^{K}\frac{S_kp_k^2}{2!}\right)x^2-\left(\sum_{k=1}^{K}\frac{W_kq_k^3}{3!}\right)x^3+\ldots
\end{aligned} \tag{10}$$

From equation (10), it can be found that the unknown values S_k, W_k, p_k, and q_k in equation (9) can be determined by the following nonlinear algebraic equations, such as

$$C_{2n}=(-1)^n\sum_{k=1}^{K}\frac{S_kp_k^{2n}}{(2n)!},\ C_{2n+1}=(-1)^n\sum_{k=1}^{K}\frac{W_kq_k^{2n+1}}{(2n+1)!}\qquad n=0,1,2,3,\ldots \tag{11}$$

By using the MATLAB functions sovle or MATHEMATICA function *Nsolve*, equation (11) can be directly solved. Then the solution $u(x)$ in equation (9) can be obtained.

4. Numerical calculations

In order to verify the proposed aftertreatment technique for the AMDM, two numerical examples will be discussed in this section.

4.1. Example 1: Consider a nonlinear differential equation

$$\frac{d^2y(t)}{dt^2}+y(t)+[y(t)]^3=0 \tag{12}$$

with the initial conditions

$$y(0)=1,\qquad \frac{dy(0)}{dt}=0 \tag{13}$$

According to AMDM presented in Section 2, the solution $y(t)$ and nonlinear term $[y(t)]^3$ for this example can be expressed as

$$y(t)=\sum_{m=0}^{M}C_mt^m,\qquad [y(t)]^3=\sum_{m=0}^{M}A_mt^m \tag{14}$$

The Adomian's polynomials for nonlinear term $[y(t)]^3$ can be found in Ref. [14], and the first several terms are listed below for the convenience of the reader.

$$
\begin{aligned}
A_0 &= C_0^3 \\
A_1 &= 3C_0^2 C_1 \\
A_2 &= 3C_0^2 C_2 + 3C_1^2 C_0 \\
A_3 &= C_1^3 + 3C_0^2 C_3 + 6C_0 C_1 C_2 \\
A_4 &= 3C_0^2 C_4 + 3C_1^2 C_2 + 3C_2^2 C_0 + 6C_0 C_1 C_3 \\
A_5 &= 3C_0^2 C_5 + 3C_1^2 C_3 + 3C_2^2 C_1 + 6C_0 C_1 C_4 + 6C_0 C_2 C_3 \\
A_6 &= C_2^3 + 3C_0^2 C_6 + 3C_1^2 C_4 + 3C_3^2 C_0 + 6C_0 C_1 C_5 + 6C_0 C_2 C_4 + 6C_1 C_2 C_3 \\
A_7 &= 3C_0^2 C_7 + 3C_1^2 C_5 + 3C_2^2 C_3 + 3C_3^2 C_1 + 6C_0 C_1 C_6 + 6C_0 C_2 C_5 + 6C_0 C_3 C_4 \\
&\ldots
\end{aligned}
$$

To determine the unknown coefficients C_m, a linear operator $L = \dfrac{d^2}{dX^2}$ is imposed, then the inverse operator of L is therefore a 2-fold integral operator defined by

$$L^{-1} = \int_0^t \int_0^t (\ldots) dt dt \tag{15}$$

and

$$L^{-1} L[y(t)] = y(t) - C_0 - C_1 X \tag{16}$$

Applying equation (12) with L^{-1} and using equations (14) and (16), we get

$$
\begin{aligned}
y(t) &= C_0 + C_1 t + L^{-1}\left[-\sum_{m=0}^{M} C_m t^m - \sum_{m=0}^{M} A_m t^m \right] \\
&= C_0 + C_1 t + \frac{-C_m - A_m}{(m+1)(m+2)} t^{m+2}
\end{aligned}
\qquad m = 0,1,2,3,\ldots,M \tag{17}
$$

Finally, using equations (13), (15) and (17), the coefficients C_m in equation (11) can be determined by using the following recurrence relations

$$C_0 = 1,\ C_1 = 0 \tag{18}$$

$$C_{m+2} = \frac{-C_m - A_m}{(m+1)(m+2)} \qquad m \geq 0 \tag{19}$$

Substituting the solved C_m into equation (11) and solving the system of nonlinear algebraic equations, the values S_k, W_k, p_k, and q_k in equation (11) can be determined. In this case, we set the series summation limit M = 7 and K = 2 in equations (8) and (9), respectively. Based on proposed aftertreatment technique, the solution for equation (12) can be written as

$$y(t) = 0.988094\cos(1.334337t) + 0.01191\cos(4.496615t) \tag{20}$$

In order to check the accuracy of the proposed aftertreatment technique, the numerical result based on the four order Runge-Kutta solution (RK4) is also calculated. Figure 1 shows the comparisons between the AMDM solution, the AMDM with aftertreatment solution and the RK4 solution of equation (12). From figure 1, it can be found that the AMDM solution diverges rapidly when $t > 0.8$. However, the AMDM with aftertreatment solution shows a good convergence in comparison with the RK4 solution for $t < 10$. If more terms of C_m are used for aftertreatment, the accuracy of the solution can be improved. Figure 2 shows the solution when M = 11 and K = 3 in equations (8) and (9) are used. It illustrates that the AMDM with aftertreatment solution coincides with RK4 solution for $t < 30$.

For the sake of simplicity, the series summation limit M and K in equations (8) and (9) will be simply truncated to M = 7 and K = 2 in all the subsequent calculations.

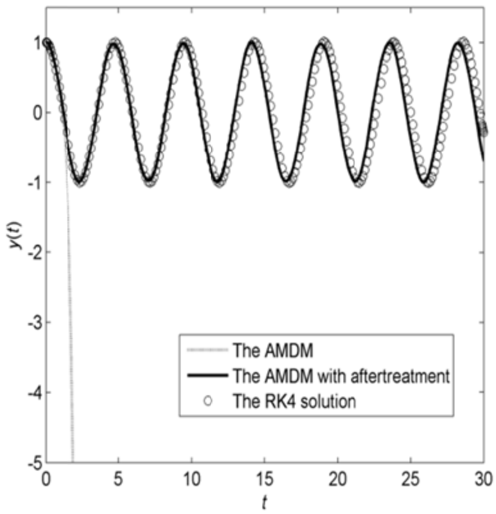

Figure 1. The AMDM solution, the AMDM with aftertreatment solution and the RK4 solution for example 1 when $M = 7$ and $K = 2$.

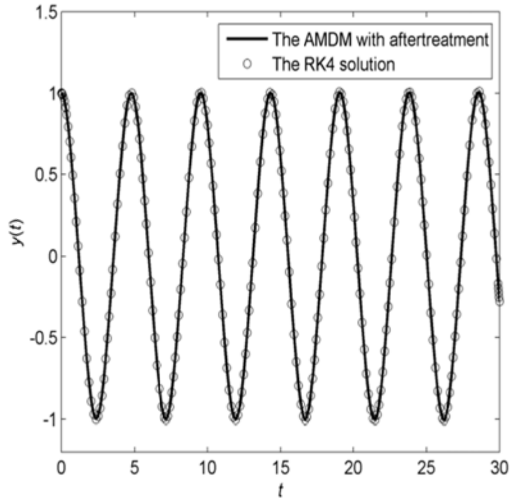

Figure 2. The AMDM solution, the AMDM with aftertreatment solution and the RK4 solution for example 1 when $M = 11$ and $K = 3$.

4.2. Example 2: Consider a nonlinear undamped pendulum

The free vibration of the nonlinear undamped pendulum problem can be expressed as a dimensionless differential equation, yields

$$\ddot{\theta}(t)+\sin[\theta(t)]=0 \tag{21}$$

with initial conditions

$$\theta(0)=\alpha\,,\ \dot{\theta}(0)=0 \tag{22}$$

where θ is the angular displacement. α is the initial amplitude.

By using the AMDM proceeding as above, the solution of equation (21) can be expressed as $\theta(t)=\sum_{m=0}^{M} C_m t^m$ and the first few coefficients C_m are

$$C_0=\theta(0)=\alpha\,,\ C_1=\dot{\theta}(0)=0\,,\ C_2=\frac{-\sin(\alpha)}{2}\,,\ C_3=0\,,\ C_4=\frac{\sin(2\alpha)}{48} \tag{23}$$

$$C_5=0\,,\ C_6=\frac{2\sin(\alpha)-\sin(3\alpha)}{720}\,,\ C_7=0\,,\ C_8=-\frac{\sin(2\alpha)}{5040}-\frac{17\sin(4\alpha)}{161280} \tag{24}$$

First, a small amplitude simple pendulum with $\theta(0) = 0.1745$ and $\alpha = 1$ studied by Ref. [10] is restudied to further check the accuracy and effectiveness of the proposed method. Substituting the solved C_m in equations (23) and (24) into equation (11), then the AMDM with aftertreatment solution can be obtained as follows,

$$\theta(t)=0.174501\cos(0.997561t)-6.653862\times10^{-7}\cos(7.237361t) \tag{25}$$

The linearized solution [10] for small amplitude pendulum is

$$\theta(t)=0.1745\cos(t) \tag{26}$$

From equations (25) and (26), it can be found that the solution obtained by proposed method agrees will to the linearized solution for small initial amplitude θ_0, and figure 3 shows the AMDM solution, the AMDM with aftertreatment and the linearized solution when $\theta_0 = 0.1745$. Form figure 3, it can be found that AMDM solution diverges rapidly when time $t > 3$, while the aftertreatment technique can significantly improve the accuracy of the AMDM in a wide applicable region.

Finally, a large amplitude nonlinear pendulum with $\theta_0 = \pi/3$ studied by Ref. [10] is restudied. Substituting the solved C_m with $\theta_0 = \pi/3$ and $\alpha = 1$ in equations (23, 24) into equation (11), then the AMDM with aftertreatment solution can be obtained as follows,

$$\theta(t) = 1.056052\cos(0.935205t) - 8.854025\times10^{-3}\cos(2.550738t) \tag{27}$$

Comparison equation (25) to Ref. [10], it can be found that the results obtained by the proposed method agrees well to the solution based on Laplace-ADM with Padé approximant.

Figure 4 shows the AMDM solution, the AMDM with aftertreatment and the RK4 solution when $\theta_0 = \pi/3$. From figure 4, it can be found that AMDM solution diverges rapidly when time $t > 2.5$. When the aftertreatment technique is used, the accuracy of results is improved significantly over the AMDM in a wide applicable region.

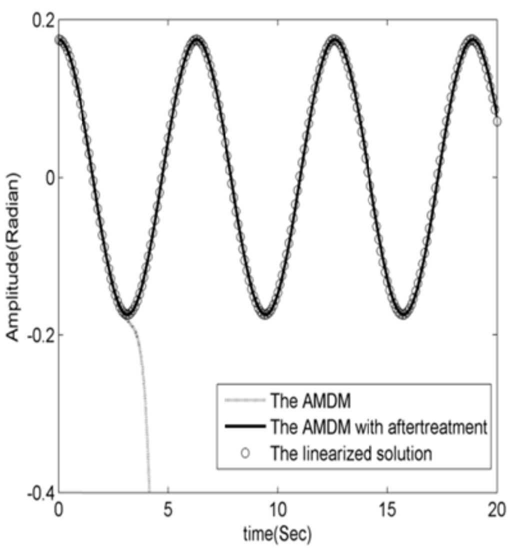

Figure 3. The AMDM solution, the AMDM with aftertreatment and the linearized solution for example 2 when $\theta_0 = 0.1745$ and $\alpha = 1$.

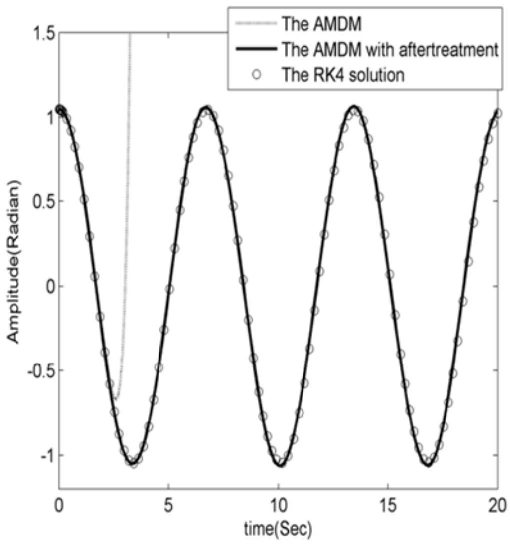

Figure 4. The AMDM solution, the AMDM with aftertreatment and the RK4 solution for example 2 when $\theta_0 = \pi/3$ and $\alpha = 1$.

5. Conclusions

In this paper, the new aftertreatment technique is proposed to improve the accuracy of the Adomian modified decomposition method (AMDM). Based on proposed aftertreatment technique, the solution using the AMDM can be expressed as another series in terms of the independent sine and cosine trigonometric functions. Two numerical examples are investigated to demonstrate the accuracy of the proposed approach. Comparison to traditional AMDM, the main advantage of the proposed aftertreatment technique is that the applicable region of the solution can be greatly extended.

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