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Two-point functions of random-length random walk on high-dimensional boxes

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Abstract. We study the two-point functions of a general class of random-length random walks (RLRWs) on finite boxes in $\mathbb{Z}^d$ with $d \geqslant 3$, and provide precise asymptotics for their behaviour. We show that in a finite box of side length L, the two-point function is asymptotic to the infinite-lattice two-point function when the typical walk length is $o(L^2)$, but develops a plateau when the typical walk length is $\Omega(L^2)$. We also numerically study walk length moments and limiting distributions of the self-avoiding walk and Ising model on five-dimensional tori, and find that they agree asymptotically with the known results for the self-avoiding walk on the complete graph, both at the critical point and also for a broad class of scaling windows/pseudocritical points. Furthermore, we show that the two-point function of the finite-box RLRW, with walk length chosen via the complete graph self-avoiding walk, agrees numerically with the two-point functions of the self-avoiding walk and Ising model on five-dimensional

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tori. We conjecture that these observations in five dimensions should also hold in all higher dimensions.

Keywords: upper critical dimension, finite-size scaling, Ising model, self-avoiding walk, two-point function

Contents

1. Introduction

The effects of boundary conditions on the finite-size scaling of statistical-mechanical lattice models in high dimensions has a rather long history [1–5], but has remained a very active area; see, for example, [6–14]. One particular topic of interest has been the scaling of the Ising susceptibility at the infinite volume critical point, where it has been observed numerically [5–14] that on boxes of side-length L, periodic boundary conditions produce a scaling $L^{d/2}$, in contrast to the L^2 behaviour observed with free boundary conditions.

Significant progress explaining this phenomenon mathematically was recently presented in [15], where a rigorous renormalisation group analysis was performed of the weakly coupled hierarchical $|\varphi|^4$ model in $d \geqslant 4$ dimensions, on finite boxes of volume L^d with both periodic and free boundary conditions. In particular, it was shown that while the effective critical point for the model with periodic boundary conditions coincides with the infinite volume critical point, the effective critical point for free boundary conditions is shifted away from the infinite-volume value by an amount of order L^{-2}. Moreover, it was shown for both boundary conditions that increasing the temperature above the effective critical point by an amount of order $L^{-\lambda}$ leads the susceptibility to scale as L^{λ}, for any $3/2 \leqslant \lambda < d/2$, but as $L^{d/2}$ for any $\lambda \geqslant d/2$. By universality, one would expect the same behaviour to hold for the self-avoiding walk (SAW) and Ising model on boxes in $\mathbb{Z}^d$, and indeed numerical evidence supporting this belief has been presented in [16].

Another striking, and related, feature of high-dimensional models with periodic boundary conditions is the so-called plateau which emerges in their two-point function, sufficiently close to the critical point, so that the initial simple random walk decay $|x|^{2-d}$ becomes subdominant to a term which is independent of x but decaying in L. Let $\mathbb{T}_L^d$ denote the d-dimensional discrete torus, of side length L. For the Ising model on $\mathbb{T}_L^d$ with $d > 4$, it was proved in [17] that at the (infinite volume) critical point the two-point function is bounded below by $c_1|x|^{2-d} + c_2 L^{-d/2}$, and it was conjectured that an upper bound of the same order should exist. This conjecture was extended in [16], where it was predicted that for the SAW and Ising model on $\mathbb{T}_L^d$ with $d > 4$, at temperatures shifted above the infinite volume critical point by $L^{-\lambda}$, the two-point function behaves as $c_1|x|^{2-d} + c_2 L^{\lambda-d}$ when $2 \leqslant \lambda \leqslant d/2$, and as $c_1|x|^{2-d} + c_2 L^{-d/2}$ when $\lambda \geqslant d/2$. The latter behaviour has been established rigorously [18] for the Domb–Joyce model with $d > 4$, for sufficiently weak interaction strength, and also very recently for SAW [19] with $d > 4$. Moreover, analogous behaviour is also now known for bond percolation [20] when $d \geqslant 11$ for the nearest-neighbour model, and $d > 6$ for spread-out models. We refer to the regime with $0 < \lambda < d/2$ as the high-temperature scaling window, and to the regime $\lambda > d/2$ as the critical window. It was observed numerically in [16] that for the SAW and Ising model with $\lambda < 2$, the two-point function decays faster than a power-law on large scales, but no conjecture for the precise nature of this behaviour was made.

The general plateau behaviour conjectured in [16] was supported by numerical simulations of the SAW and Ising model on five-dimensional tori, but was motivated by considering a model of simple random walk in which the walk length is chosen to be finite and random, and distributed as a SAW on the complete graph. The behaviour of the SAW on the complete graph has been recently studied [21–23]. Yet the behaviour of the two-point function of random-length random walk (RLRW), with arbitrary walk length distributions, appears not to have been studied in significant detail. This question was recently addressed for RLRW on $\mathbb{Z}^d$ in [24]. One contribution of the current work is to present sharp asymptotic results for the two-point function of RLRWs on finite boxes in $\mathbb{Z}^d$, for three distinct choices of boundary conditions, and with only modest assumptions on the walk length distribution. In summary, we find that if the walk length is concentrated on a scale $o(L^2)$, then the finite-box and infinite lattice two-point functions are asymptotic. This is to be expected, since simple random walks of length N

typically explore distances of order $\sqrt{N}$. By contrast, for walks whose expected length is $\Omega(L^2)$, we establish a plateau given by the ratio of the mean walk length to the system volume. We emphasise that these asymptotic results for the RLRW do not depend on the choice of boundary conditions, but only on the choice of walk length distribution. This provides concrete evidence for the suggestion made in [16] that for models such as the Ising model and SAW, the two-point function depends on the boundary conditions only via their influence on the walk length distribution.

Specialising our RLRW results to the case that the walk length distribution is that of the SAW on the complete graph, we find a universal model of high-dimensional torus two-point function behaviour, that agrees numerically with the SAW and Ising model on five-dimensional tori, in both the critical window and the high-temperature scaling window for any $0 < \lambda < d/2$. Moreover, we find that the high-temperature scaling window consists of two separate regimes: universal exponential decay in terms of the continuum Green function for $\lambda < 2$ and λ-dependent plateau for $2 < \lambda < d/2$.

We note that in the special case in which the walk length is geometrically distributed, the two-point function of RLRW on $\mathbb{Z}^d$ corresponds to the lattice Green function, which is very well studied; see [25] and references therein. In that case, RLRW is generally referred to as *killed* random walk [26]. We also note that plateau behaviour of the geometrically killed simple random walk two-point function on tori was recently established in [18, theorem 1.4], as a corollary of their weakly SAW result. The results we present here for RLRW are both sharper and more general than given in [18, theorem 1.4].

The key assumption underlying the use of the complete graph SAW length in the RLRW to describe the SAW and Ising two-point functions on $\mathbb{T}_L^d$, is that for $d > 4$ the large L behaviour of the length of the SAW or Ising walk on $\mathbb{T}_L^d$ should behave in the same way as SAW on the complete graph; see section 2 for a definition of the Ising walk. We therefore now summarise the known behaviour [21–23] of the SAW on the complete graph, K_n. At fugacities $1/n(1+an^{-p})$, the mean walk length scales as n^p for $p \in (0, 1/2)$ and $a > 0$, but scales as $\sqrt{n}$ for all $p \geqslant 1/2$ and $a \in \mathbb{R}$. Analogous behaviour has also been established [27] for SAW on the hypercube, $\mathbb{Z}_2^N$, and for weakly self-avoiding walk on the torus $\mathbb{T}_L^d$ with $d > 4$ when the interaction strength is sufficiently small [28]. Moreover, the variance and limiting distributions of the appropriately scaled/standardised length of the SAW on K_n are also known in detail [22, 23]. For $p \in (0, 1/2)$ with $a > 0$ the variance scales as n^{2p}, and the walk length divided by its mean converges to a mean-1 exponential distribution, while for $p > 1/2$ the variance scales as n and the standardised walk length converges to a half-normal distribution. In section 4.1 we provide strong numerical evidence that the same behaviours hold for the SAW and Ising walk length on $\mathbb{T}_L^d$ with $d = 5$, and we conjecture that they in fact hold for all $d \geqslant 5$.

1.1. Outline

Let us outline the remainder of this article. Section 1.2 lists some notational conventions. Section 2 defines the specific quantities of interest for the SAW and Ising model. Section 3 describes the RLRW models considered, and presents our main results for their two-point functions. Section 4 then presents numerical results for the SAW and Ising

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model on five-dimensional tori, both in the critical window and high-temperature scaling window. Specifically, section 4.1 presents numerical results for the SAW and Ising walk length, while section 4.2 considers their two-point functions. Finally, sections 5 and 6 present proofs of propositions 3.2 and 3.3, respectively, and section 7 presents a proof of lemma 3.1.

1.2. Notation

For integer $d \geqslant 1$ and $L > 2$, we let $\mathbb{B}_L^d := [-L/2, L/2)^d \cap \mathbb{Z}^d$. For each $x \in \mathbb{B}_L^d$, we denote its Euclidean norm by $|x| := \sqrt{x \cdot x}$. We let $\mathbb{T}_L^d$ denote the d-dimensional discrete torus, of linear size L. We view $\mathbb{T}_L^d$ both as a graph, whose vertex set is taken to be $\mathbb{B}_L^d$, and also, when convenient, as a module over the commutative ring $\mathbb{T}_L := [-L/2, L/2) \cap \mathbb{Z}$ in which addition and multiplication are defined modulo L.

The standard asymptotic symbols such as O, o etc will refer to large L asymptotics. The definition of RLRW requires a sequence of random walk lengths, $(\mathcal{N}_L)_L$. In the asymptotic results we present for RLRW, the implied constants may depend on d and the choice of the sequence of distributions corresponding to $(\mathcal{N}_L)_L$. Statements such as $f = O(g)$ in that context then mean that for any particular choice of d and the sequence of walk length distributions, there exists a constant $c > 0$ such that $f(L) \leqslant c\,g(L)$ for all sufficiently large L. In the case that constants depend on additional parameters, we will highlight this via subscripts; for example, if for fixed λ we have $f(L,\lambda) \leqslant c(d,\lambda) g(L,\lambda)$ for all $L \geqslant N(d,\lambda)$, then we will write $f = O_\lambda(g)$. If $f = O(g)$ and $g = O(f)$ we write $f \asymp g$. We find it convenient to also use the Vinogradov symbols, so that $f \ll g$ is equivalent to $f = O(g)$, and $f \gg g$ is equivalent to $g = O(f)$. We also find it convenient to write $f = \Omega(g)$ to denote $g = O(f)$ and $f = \omega(g)$ to denote $g = o(f)$.

The set of non-negative integers will be denoted by $\mathbb{N}$, and $\mathbb{Z}_+ := \mathbb{N} \setminus 0$. For any $n \in \mathbb{Z}_+$ we write $[n] := \{1, 2, \ldots, n\}$.

2. The SAW and Ising model

Let $G = (V, E)$ be a rooted graph, with root 0. For $n \in \mathbb{N}$, let Ω_G^n denote the set of all n-step walks on G which start at 0; i.e. all sequences $\omega_0, \ldots, \omega_n$ such that $\omega_i \in V$, $\omega_0 = 0$ and $\omega_i \omega_{i+1} \in E$. We set $\Omega_G := \bigcup_{n \in \mathbb{N}} \Omega_G^n$. For $\omega \in \Omega_G^n$, the notation $\omega : 0 \to v$ implies $\omega_n = v$, and we denote the end of ω by $e(\omega) = \omega_n$. In what follows, we let $|\omega|$ denote the number of steps, or *length*, of the walk $\omega \in \Omega_G$, so that

$$|\omega| = n \text{ iff } \omega \in \Omega_G^n. \tag{1}$$

A walk $\omega \in \Omega_G$ is *self-avoiding* if $\omega_i \neq \omega_j$ for all $i \neq j$. We consider the variable-length ensemble of SAWs on G, and let $\mathcal{S}$ denote a random SAW with this distribution, so that for all $\omega \in \Omega_G$

$$\mathbb{P}(\mathcal{S} = \omega) = \frac{\rho(\omega)}{\sum_{\omega' \in \Omega_G} \rho(\omega')}, \tag{2}$$

with

$$\rho(\omega) = z^{|\omega|}\mathbb{1}(\omega \text{ is self-avoiding}) \ . \tag{3}$$

The quantity $z > 0$ is the *fugacity*. We will be interested in the distribution of the walk length $|\mathcal{S}|$, and the two-point function defined by [29]

$$g(x) := \sum_{\substack{\omega\in\Omega_G \\ \omega:0\to x}} \rho(\omega) \ , \tag{4}$$

$$= \frac{\mathbb{P}[e(\mathcal{S}) = x]}{\mathbb{P}[|\mathcal{S}| = 0]} \ . \tag{5}$$

Our simulations of $\mathcal{S}$, discussed below, were performed using a lifted version [30] of the Berretti–Sokal algorithm [31].

We also consider analogous quantities for the Ising model. The zero-field ferromagnetic Ising model on finite graph $G = (V, E)$ at inverse temperature $\beta \geqslant 0$ is defined by the measure

$$\mathbb{P}(\sigma) \propto \exp\left(\beta \sum_{ij\in E} \sigma_i\sigma_j\right), \qquad \sigma \in \{-1, 1\}^V . \tag{6}$$

The corresponding two-point function is defined by

$$g(x) := \mathbb{E}(\sigma_0\sigma_x) \ . \tag{7}$$

The Ising two-point function can be conveniently re-expressed via the high-temperature expansion, as follows. For $v \in V \setminus 0$, let $\mathcal{C}_v$ denote the set of all $A \subseteq E$ such that the set of all vertices of odd degree in (V, A) is precisely $\{0, v\}$, and let $\mathcal{C}_0$ denote the set of all $A \subseteq E$ such that (V, A) has no vertices of odd degree. For a family of edge sets $S \subseteq 2^E$, let

$$\lambda(S) := \sum_{A\in S} [\tanh(\beta)]^{|A|} . \tag{8}$$

By analogy with the SAW case, we refer to $z = \tanh(\beta)$ as the Ising fugacity. The high-temperature expansion for the Ising model (see e.g. [32, equation (3.5)] or [33, lemma 2.1]) implies that we can re-express equation (7) so that for all $x \in V$

$$g(x) = \frac{\lambda(\mathcal{C}_x)}{\lambda(\mathcal{C}_0)} \ . \tag{9}$$

This high-temperature representation of the Ising model can also be used to provide a natural definition of the *Ising walk*, first discussed in [32, 34], and studied numerically in [24]. Fix an (arbitrary) ordering, $\prec$, of V. We define $\mathcal{T} : \cup_{v\in V}\mathcal{C}_v \to \Omega_G$ as follows. If $A \in \mathcal{C}_0$, then $\mathcal{T}(A) = 0$. If $A \in \mathcal{C}_v$ with $v \neq 0$, we recursively define the walk $\mathcal{T}(A) = v_0v_1 \dots v_k$ from $v_0 = 0$ to $v_k = v$, such that from v_i we choose v_{i+1} to be the smallest

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neighbour of v_i such that $v_i v_{i+1} \in A$ and $v_i v_{i+1}$ has not previously been traversed by the walk. It is clear that $\mathcal{T}(A)$ defines an edge self-avoiding trail from 0 to v.

Now let $\mathcal{A}$ denote a random element of $\cup_{v\in V}\mathcal{C}_v$ with distribution

$$\mathbb{P}(\mathcal{A}=A)=\frac{z^{|A|}}{\sum_{A'\in\cup_{v\in V}\mathcal{C}_v} z^{|A'|}}\,,\qquad A\in\cup_{v\in V}\mathcal{C}_v\,. \tag{10}$$

The distribution of $\mathcal{A}$ is precisely the stationary distribution of the Prokofiev–Svistunov worm algorithm [35], in which the worm tail is fixed at the root. Our simulations of $\mathcal{A}$, discussed below, were performed using such a worm algorithm. We will be interested in the induced distribution of $\mathcal{T}:=\mathcal{T}(\mathcal{A})$, and particularly in the distribution of its length, $|\mathcal{T}|$, which we refer to as the *Ising walk length*.

We note that by partitioning Ω_G in terms of $\mathcal{T}$, we can re-express the Ising two-point function of equation (7) in precisely the form of equation (4) but with

$$\rho(\omega)=\frac{\lambda\left(\mathcal{T}^{-1}(\omega)\right)}{\lambda(\mathcal{C}_0)}. \tag{11}$$

Moreover, it can also be re-expressed in the form of equation (5) with $\mathcal{S}$ replaced by $\mathcal{T}$.

The simulations of the SAW and Ising model to be presented in section 4 were performed on five-dimensional tori. As we will demonstrate, the asymptotic behaviour of both $|\mathcal{S}|$ and $|\mathcal{T}|$ appear to coincide with the known [22, 23] asymptotic behaviour of $|\mathcal{S}|$ on the complete graph, which we now summarise. Let G be the complete graph K_n, rooted at a fixed vertex, and suppose the fugacity z satisfies $1/z=n(1+an^{-p})$. Let $\mathcal{K}$ denote $\mathcal{S}$ in this setting. It is known [21] that the critical fugacity is $z=1/n$. Moreover, if $p<1/2$ and $a>0$ then we have for large n that

$$\mathbb{E}(|\mathcal{K}|)\sim\frac{n^p}{a}\,,\qquad\qquad \mathrm{var}(|\mathcal{K}|)\sim\left(\frac{n^p}{a}\right)^2 \tag{12}$$

while if $p>1/2$

$$\mathbb{E}(|\mathcal{K}|)\sim\sqrt{\frac{2}{\pi}}\sqrt{n}\,,\qquad\qquad \mathrm{var}(|\mathcal{K}|)\sim\left(1-\frac{2}{\pi}\right)n. \tag{13}$$

Furthermore, let X be a standard normal random variable, and let Y be an exponential random variable with mean 1. Then as $n\to\infty$ we have for $p<1/2$ and $a>0$ that

$$\frac{|\mathcal{K}|}{\mathbb{E}(|\mathcal{K}|)}\Longrightarrow Y\,, \tag{14}$$

while if $p>1/2$ then

$$\frac{|\mathcal{K}|-\mathbb{E}(|\mathcal{K}|)}{\sqrt{\mathrm{var}(|\mathcal{K}|)}}\Longrightarrow\frac{|X|-\mathbb{E}(|X|)}{\sqrt{\mathrm{var}(|X|)}}\,. \tag{15}$$

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For later reference, we shall denote by F the law of the standardised version of $|X|$, i.e. for $x \in \mathbb{R}$

$$F(x) := \mathbb{P}\left(\frac{|X| - \mathbb{E}(|X|)}{\sqrt{\operatorname{var}(|X|)}} \leqslant x\right). \tag{16}$$

2.1. Numerical details

Our simulations of the SAW and Ising model were performed on five-dimensional tori, at pseudocritical points $z_L = z_c(1 - L^{-\lambda})$ for various $\lambda > 0$, where z_c denotes the estimated location of the infinite-volume critical point. In the Ising case we used the estimate $z_c = 0.1134248(5)$ [14], while in the SAW case we used $z_c = 0.11314084(1)$ [30]. A detailed analysis of integrated autocorrelation time is presented in [36] for the worm algorithm and in [30] for the lifted Berretti–Sokal algorithm. Our fitting methodology and corresponding error estimation follow standard procedures, see for instance [37, 38].

3. Random walk models

3.1. Definitions and boundary conditions

Let $(C_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of uniformly random elements of $\{\pm e^1, \ldots, \pm e^d\}$, where $e^i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^d$ is the standard unit vector along the ith coordinate axis, and let $\mathcal{N}$ be an $\mathbb{N}$-valued random variable independent of $(C_n)_{n\in\mathbb{N}}$. The corresponding RLRW on $\mathbb{Z}^d$ is the process $\mathcal{Z} := (\mathcal{Z}_t)_{t=0}^{\mathcal{N}}$ defined so that $\mathcal{Z}_0 = 0$ and $\mathcal{Z}_t = \mathcal{Z}_{t-1} + C_t$ for each $1 \leqslant t \leqslant \mathcal{N}$. We also consider RLRWs $(\mathcal{X}_t^{\mathrm{P}})_{t=0}^{\mathcal{N}}$, $(\mathcal{X}_t^{\mathrm{R}})_{t=0}^{\mathcal{N}}$, and $(\mathcal{X}_t^{\mathrm{H}})_{t=0}^{\mathcal{N}}$ on $\mathbb{B}_L^d$, with periodic, reflecting and holding boundary conditions, respectively, defined so that $\mathcal{X}_0^* = 0$, and for all $1 \leqslant t \leqslant \mathcal{N}$ we have $\mathcal{X}_t^* = \mathcal{X}_{t-1}^* + C_t$ if $\mathcal{X}_{t-1}^* + C_t \in \mathbb{B}_L^d$, otherwise

$$\mathcal{X}_t^{\mathrm{P}} := \mathcal{X}_{t-1}^{\mathrm{P}} + C_t(1 - L) \tag{17}$$

$$\mathcal{X}_t^{\mathrm{R}} := \mathcal{X}_{t-1}^{\mathrm{R}} - C_t \tag{18}$$

$$\mathcal{X}_t^{\mathrm{H}} := \mathcal{X}_{t-1}^{\mathrm{H}} \tag{19}$$

when $\mathcal{X}_{t-1}^* + C_t \notin \mathbb{B}_L^d$, where $*$ denotes either P, R or H, as appropriate.

We define the two-point function of $\mathcal{Z}$ to be

$$g_{\mathcal{N}}(x) := \mathbb{E}\left(\sum_{t=0}^{\mathcal{N}} \mathbb{1}(\mathcal{Z}_t = x)\right), \qquad x \in \mathbb{Z}^d. \tag{20}$$

As noted in the Introduction, in the special case in which $\mathcal{N}$ is geometrically distributed, the two-point function of the RLRW on $\mathbb{Z}^d$ corresponds to the lattice Green function. Analogous definitions hold for $\mathcal{X}^{\mathrm{P}}$, $\mathcal{X}^{\mathrm{R}}$, $\mathcal{X}^{\mathrm{H}}$. Specifically, for such a process on $\mathbb{B}_L^d$ we set

$$g_{*,L,\mathcal{N}}(x) := \mathbb{E}\left(\sum_{t=0}^{\mathcal{N}} \mathbb{1}\left(\mathcal{X}_t^* = x\right)\right), \qquad x \in \mathbb{B}_L^d. \tag{21}$$

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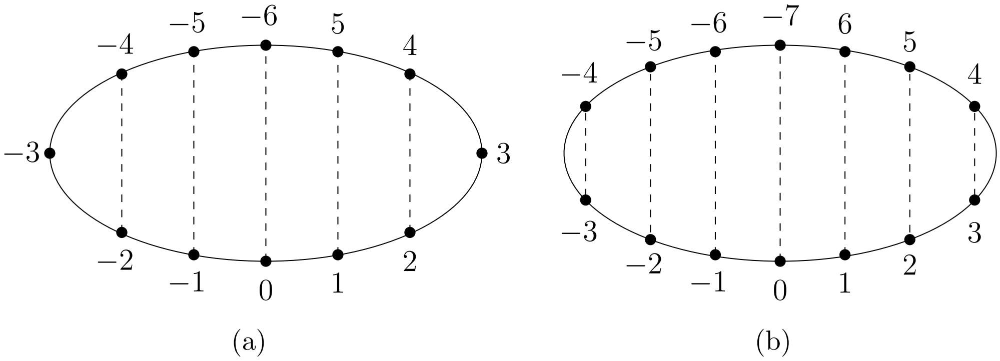

Figure 1. Illustration of the equivalence classes defined by (22) with $d = 1$. (a) Equivalence classes, $[x]_{L-1}$, on $\mathbb{T}^1_{2(L-1)}$ with $L = 7$. (b) Equivalence classes, $[x]_L$, on $\mathbb{T}^1_{2L}$ with $L = 7$. Note that in both cases the set of equivalence classes are in bijection with $\mathbb{B}^1_L$.

These two-point functions are closely related to one another, as the next lemma illustrates. Recall that we consider $\mathbb{T}^d_L$ as a module over the commutative ring T_L, with addition and scalar multiplication defined modulo L in each entry. For each $x \in \mathbb{T}^d_{2L}$, we can then define

$$[x]_L := \left\{y \in \mathbb{T}^d_{2L} \, : \, y_i \in \{x_i, -L - x_i\} \text{ for all } i \in [d]\right\}. \tag{22}$$

The partition of $\mathbb{T}^d_{2L}$ into the sets $[x]_L$ defines an equivalence relation on $\mathbb{T}^d_{2L}$, in which the sets $[x]_L$ are the equivalence classes. The case of $d = 1$, corresponding to projecting a cycle onto a path, is illustrated in figure 1.

Lemma 3.1. *Let $d, L \in \mathbb{Z}_+$ and let $x \in \mathbb{B}^d_L$. Then:*

(i) For any $L \geqslant 3$

$$g_{\mathrm{P},L,\mathcal{N}}(x) = \sum_{z \in \mathbb{Z}^d} g_{\mathcal{N}}(x + Lz)$$

(ii) For any odd $L \geqslant 3$

$$g_{\mathrm{R},L,\mathcal{N}}(x) = \sum_{x' \in [x]_{L-1}} g_{\mathrm{P},2(L-1),\mathcal{N}}(x')$$

(iii) For any odd $L \geqslant 3$

$$g_{\mathrm{H},L,\mathcal{N}}(x) = \sum_{x' \in [x]_L} g_{\mathrm{P},2L,\mathcal{N}}(x').$$

The proof of lemma 3.1, which is based on Markov chain projection arguments, is discussed in section 7.

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3.2. Main results for RLRW two-point functions

We now state our main results for the asymptotic behaviour of RLRW two-point functions on $\mathbb{B}_L^d$, for periodic, reflecting and holding boundary conditions. We defer proof of these results to sections 5 and 6. We provide numerical evidence of the connection of these results to the Ising and SAW models in section 4.2.

The RLRW model on $\mathbb{B}_L^d$ is most easily understood by relating it to the RLRW on the infinite lattice. The asymptotic behaviour of the latter was studied in detail in [24]. See also [25].

Suppose $\mathcal{N}_L$ is chosen so that its typical scale a_L grows with L. One would expect the behaviour of $g_{*,L,\mathcal{N}_L}(x)$ to differ qualitatively depending on whether or not a_L grows fast enough that the RLRW can explore distances from the origin of order L, so that the presence of the boundary can be felt. We therefore present two separate results relating $g_{*,L,\mathcal{N}_L}$ to $g_{\mathcal{N}_L}$, depending on the asymptotics of $\mathcal{N}_L$.

Let Δ denote the standard degenerate distribution function, i.e. Δ is the indicator function for $[0,\infty)$.

Proposition 3.2. *Consider a sequence of $\mathbb{N}$-valued random variables $\mathcal{N}_L$, for which there exists a sequence $a_L > 0$ satisfying:*

(i) $a_L \to \infty$.
(ii) $a_L = O(L^\lambda)$ *for some* $\lambda < 2$.
(iii) There exists $r, C > 0$ *such that* $\mathbb{E}(\mathrm{e}^{r\mathcal{N}_L/a_L}) \leqslant C$ *for all* L.
(iv) There exists a distribution function $G \neq \Delta$ *such that* $\mathbb{P}(\mathcal{N}_L/a_L \leqslant \cdot) \implies G$.

Fix $d \geqslant 3$, *and let* $(x_L)_{L\geqslant 3}$ *be a sequence in* $\mathbb{Z}^d$ *satisfying* $x_L \in \mathbb{B}_L^d$ *and* $|x_L|/\sqrt{a_L} \to \xi \in [0,\infty)$. *Then, with* $*$ *denoting* P, R *or* H, *as* $L \to \infty$ *we have:*

$$g_{*,L,\mathcal{N}_L}(x_L) = g_{\mathcal{N}_L}(x_L)\left[1 + o(1)\right].$$

The assumptions on $\mathcal{N}_L$ given in proposition 3.2 imply that typical $\mathcal{N}_L$ will have length of order $a_L = O(L^\lambda)$ with $\lambda < 2$. Since a simple random walk of length N typically explores distances from the origin of order $\sqrt{N}$, it then follows that a typical such RLRW will explore distances of order $o(L)$ from the origin, and will therefore be too short to feel the boundary. It is therefore unsurprising that the finite-box and infinite lattice two-point functions are asymptotic on such a scale, for any of the three choices of boundary conditions studied. The spatial scales probed by proposition 3.2 correspond to distances of the order of $\sqrt{a_L}$, where a_L is the typical scale of $\mathcal{N}_L$. Under the assumptions of proposition 3.2, it follows from proposition 3.2 and [24, proposition 3.1] that

$$\lim_{L\to\infty} \|x_L\|^{d-2} g_{*,L,\mathcal{N}_L}(x_L) = \frac{d}{2\pi^{d/2}} \int_0^\infty s^{d/2-2}\mathrm{e}^{-s}\left[1 - G\left(\frac{d}{2}\frac{\xi^2}{s}\right)\right]\mathrm{d}s. \tag{23}$$

We note that, in particular, if G corresponds to the distribution function of a mean-1 exponential random variable, then we have

$$\lim_{L\to\infty} \|x_L\|^{d-2} g_{*,L,\mathcal{N}_L}(x_L) = \mathcal{E}(\xi) \tag{24}$$

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with

$$\mathcal{E}(\xi) := \frac{d}{2\pi^{d/2}} \int_0^\infty s^{d/2-2} \exp\left(-s - \frac{d}{2}\frac{\xi^2}{s}\right) \mathrm{d}s$$
$$= \frac{2}{\pi^{d/2}} \left(\frac{d}{2}\right)^{\frac{d}{4}+\frac{1}{2}} \xi^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\left(\sqrt{2d}\xi\right) \tag{25}$$
$$\sim \frac{d^{d+1}4}{2^{\frac{d+1}{4}}\pi^{\frac{d-1}{2}}} \xi^{\frac{d-3}{2}} \mathrm{e}^{-\sqrt{2d}\xi}, \qquad \xi \to \infty$$

and where $K_\nu(\cdot)$ denotes the modified Bessel function of the second kind [39]. As discussed in [25], $\mathcal{E}$ is intimately related to the Green function of the continuum Laplacian on $\mathbb{R}^d$. As elaborated on in section 4, the exponential choice for G here is motivated by the behaviour of the SAW on the complete graph.

We now turn our attention to the case that $\mathbb{E}(\mathcal{N}_L) = \Omega(L^2)$.

Proposition 3.3. *Fix $d \geqslant 3$, and let $(x_L)_{L\geqslant 3}$ be a sequence in $\mathbb{Z}^d$ satisfying $x_L \in \mathbb{B}_L^d$. Let $(\mathcal{N}_L)_{L\in\mathbb{Z}_+}$ be a sequence of $\mathbb{N}$-valued random variables. Let $*$ denote P, R or H. Then as $L \to \infty$ we have:*

(i) If $\mathbb{E}(\mathcal{N}_L) = \Omega(L^2)$, then

$$g_{*,L,\mathcal{N}_L}(x_L) - g_{\mathcal{N}_L}(x_L) \asymp \mathbb{E}(\mathcal{N}_L)/L^d.$$

(ii) If $\mathbb{E}(\mathcal{N}_L) = \omega(L^2)$, then

$$g_{*,L,\mathcal{N}_L}(x_L) - g_{\mathcal{N}_L}(x_L) \sim \mathbb{E}(\mathcal{N}_L)/L^d.$$

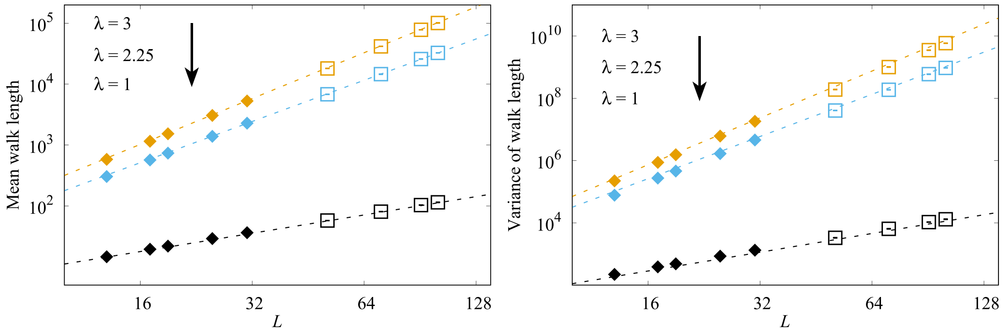
A similar result is given in [18, theorem 1.4] for the case of a geometric walk length distribution, giving upper and lower bounds for the difference between the torus and infinite lattice two-point functions in terms of the susceptibility, but without control of the constants, and with a lower bound that is weaker by a logarithmic factor when $d = 4$.

Suppose that $\mathcal{N}_L$ is such that $\mathcal{N}_L/\mathbb{E}(\mathcal{N}_L)$ converges in distribution, and that the limiting distribution function is continuous at the origin. For example, this occurs if $\mathcal{N}_L = |\mathcal{K}_{L^d}|$ in either the critical window or the high-temperature scaling window. Now suppose that $\mathbb{E}(\mathcal{N}_L) \asymp L^\lambda$ with $\lambda > 2$. It then follows[4] from [24, proposition 3.1] that for any sequence $(x_L)_{L\geqslant 3} \subset \mathbb{Z}^d$ satisfying $x_L \in \mathbb{B}_L^d$ and $|x_L| \to \infty$ we have

$$\lim_{L\to\infty} |x_L|^{d-2} g_{\mathcal{N}_L}(x_L) = \frac{d}{2\pi^{d/2}} \Gamma(d/2 - 1). \tag{26}$$

Therefore, the exponential decay displayed by (24) for $\lambda < 2$ cannot be observed when $\lambda > 2$. Consequently, proposition 3.3 implies that $g_{*,L,\mathcal{N}_L}(x_L)$ decays as a power-law for $|x_L| = o(L^{(d-\lambda)/(d-2)})$, but is then dominated by a term of order $L^{\lambda-d}$ for

[4] Although the statement of [24, proposition 3.1] specifies $\xi > 0$, its proof also holds when $\xi = 0$.



(a) (b)

Figure 2. Simulated mean and variance of the Ising (rhombi) and SAW (squares) walk lengths on five-dimensional tori. (a) Simulated $\mathbb{E}(|\mathcal{S}|)$ and $\mathbb{E}(|\mathcal{T}|)$ at fugacity $z_L = z_c(1-L^{-\lambda})$ with $\lambda = 1, 9/4, 3$, on a log–log scale. The dashed curves passing through the $\lambda = 1, 9/4$ data have slope λ, while the curve passing through the $\lambda = 3$ data has slope $d/2$. (b) Simulated $\mathrm{var}(|\mathcal{S}|)$ and $\mathrm{var}(|\mathcal{T}|)$ at fugacity $z_L = z_c(1-L^{-\lambda})$ with $\lambda = 1, 9/4, 3$. The dashed curves passing through the $\lambda = 1, 9/4$ data have slope 2λ, while the curve passing through the $\lambda = 3$ data has slope d.

$|x_L| = \omega(L^{(d-\lambda)/(d-2)})$. We note that the scale $L^{(d-\lambda)/(d-2)}$ is $o(L)$, and therefore realisable inside $\mathbb{B}_L^d$, iff $\lambda > 2$. To probe the crossover from power-law to plateau behaviour we choose $x_L \in \mathbb{B}_L^d$ such that $|x_L| = \varphi\left(L^d/\mathbb{E}(\mathcal{N}_L)\right)^{1/(d-2)}$ for $\varphi \in (0,\infty)$ to obtain

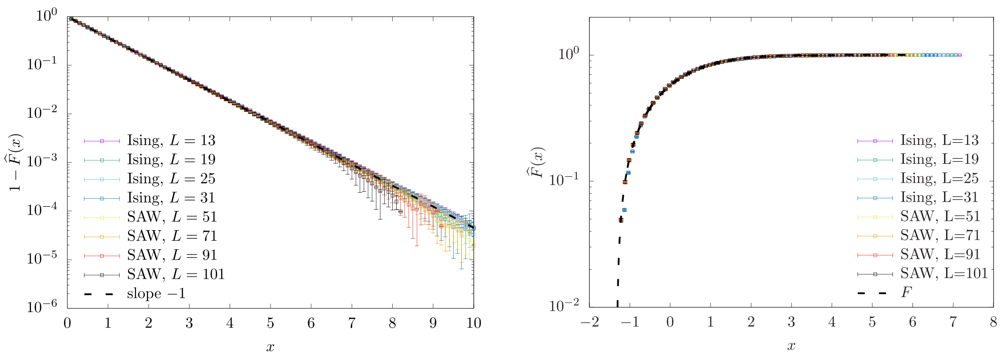
$$\lim_{L\to\infty} |x_L|^{d-2} g_{*,L,\mathcal{N}_L}(x_L) = \frac{d}{2\pi^{d/2}}\Gamma\left(d/2-1\right) + \varphi^{d-2}\,. \tag{27}$$

4. Numerical results

4.1. Universal walk length distributions

Figure 2(a) shows the simulated results for $\mathbb{E}(|\mathcal{S}|)$ and $\mathbb{E}(|\mathcal{T}|)$ at fugacity $z = z_c(1-L^{-\lambda})$ with $\lambda = 1, 9/4, 3$. If the boundary of the critical windows for the SAW and Ising model on $\mathbb{T}_L^d$ occurs at the square root of the volume, as occurs for the complete graph SAW, then for $d = 5$ the value $\lambda = 3 > d/2$ should lie inside the critical window while $\lambda = 1, 9/4$ should lie outside the critical window. Figure 2(a) is clearly consistent with the conjecture that $\mathbb{E}(|\mathcal{S}|)$ and $\mathbb{E}(|\mathcal{T}|)$ scale like L^λ when $\lambda < d/2$, but like $L^{d/2}$ for $\lambda > d/2$. Likewise, the results in figure 2(b) are consistent with the conjecture that $\mathrm{var}(|\mathcal{S}|)$ and $\mathrm{var}(|\mathcal{T}|)$ scale like $L^{2\lambda}$ when $\lambda < d/2$, but like L^d for $\lambda > d/2$. This behaviour is precisely analogous to the complete graph SAW behaviour shown in equations (12) and (13).

Similarly, figure 3 illustrates the appropriately scaled/standardised distribution functions of $|\mathcal{S}|$ and $|\mathcal{T}|$ for $\lambda = 1$ and $\lambda = 3$. Figure 3(a) strongly suggests for $\lambda = 1$ that $|\mathcal{S}|/\mathbb{E}(|\mathcal{S}|)$ and $|\mathcal{T}|/\mathbb{E}(|\mathcal{T}|)$ converge in distribution to a mean-1 exponential random variable, precisely as stated in equation (14) for the complete graph SAW in the high-temperature scaling window. Likewise, figure 3(b) strongly suggests for $\lambda = 3$ that $(|\mathcal{S}| - \mathbb{E}(|\mathcal{S}|))/\sqrt{\mathrm{var}(|\mathcal{S}|)}$ and $(|\mathcal{T}| - \mathbb{E}(|\mathcal{T}|))/\sqrt{\mathrm{var}(|\mathcal{T}|)}$ converge in distribution to a



(a) (b)

Figure 3. (a) Tail of the simulated distribution function, $\widehat{F}$, of the rescaled SAW length, $|\mathcal{S}|/\mathbb{E}(|\mathcal{S}|)$, and rescaled Ising walk length, $|\mathcal{T}|/\mathbb{E}(|\mathcal{T}|)$, on five-dimensional tori at fugacity $z_L = z_c(1-L^{-\lambda})$ with $\lambda = 1$. The dashed curve is e^{-x}, the tail of the mean-1 exponential distribution. (b) Simulated distribution function, $\widehat{F}$, of the standardised SAW length, $(|\mathcal{S}| - \mathbb{E}(|\mathcal{S}|))/\sqrt{\mathrm{var}(|\mathcal{S}|)}$, and standardised Ising walk length, $(|\mathcal{T}| - \mathbb{E}(|\mathcal{T}|))/\sqrt{\mathrm{var}(|\mathcal{T}|)}$, on five-dimensional tori, at fugacity $z_L = z_c(1-L^{-\lambda})$ and $\lambda = 3$. The dashed curve corresponds to the standardised half-normal distribution function, F, given in equation (16).

Table 1. Estimated μ values for $\mathbb{E}(|\mathcal{S}|)$ and $\mathbb{E}(|\mathcal{T}|)$ on $\mathbb{T}_L^d$ with $d = 5$ at fugacity $z_L = z_c(1-L^{-\lambda})$ at various values of λ.

λ	$\mathbb{E}(\lvert\mathcal{T}\rvert)$	$\mathbb{E}(\lvert\mathcal{S}\rvert)$
1	1.00(1)	0.998(2)
3/2	1.53(5)	1.499(2)
2	2.01(9)	2.01(1)
9/4	2.26(8)	2.28(6)
5/2	2.50(5)	2.46(4)
3	2.51(2)	2.5(1)

standardised half-normal distribution, precisely as stated in equation (15) for the complete graph SAW in the critical window.

We note that the same critical window behaviour for the Ising and SAW mean, variance and limit distribution were observed at the estimated infinite-volume critical fugacity, z_c, on five-dimensional tori in [24]. One can formally view this case as $\lambda = +\infty$.

To support the claim that the boundary of the critical window lies at $d/2$, table 1 provides estimates of the scaling exponent μ obtained by fitting $\mathbb{E}(|\mathcal{S}|)$ and $\mathbb{E}(|\mathcal{T}|)$ to an ansatz $a + bL^{\mu}$, for various values of λ. As expected from the complete graph SAW results, we indeed observe that $\mu = \lambda$ for each $\lambda \leqslant d/2$, but $\mu = d/2$ for all $\lambda \geqslant d/2$.

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Based on the above observations, we conjecture that the following holds for any $d \geqslant 5$ and $z_L = z_{\mathrm{c}} - a\,L^{-\lambda}$. If $\lambda < d/2$ and $a > 0$ then

$$|\mathcal{S}|/\mathbb{E}\,(|\mathcal{S}|)\,, \quad |\mathcal{T}|/\mathbb{E}\,(|\mathcal{T}|) \implies Y \tag{28}$$

and there exist constants $\mathcal{A}_{\mathcal{S},d,\lambda,a}, \mathcal{A}_{\mathcal{T},d,\lambda,a}, \mathcal{B}_{\mathcal{S},d,\lambda,a}, \mathcal{B}_{\mathcal{T},d,\lambda,a} > 0$ such that

$$\mathbb{E}\,(|\mathcal{S}|) \sim \mathcal{A}_{\mathcal{S},d,\lambda,a}\,L^{\lambda}\,, \qquad \mathbb{E}\,(|\mathcal{T}|) \sim \mathcal{A}_{\mathcal{T},d,\lambda,a}\,L^{\lambda}\,, \tag{29}$$

and

$$\mathrm{var}\,(|\mathcal{S}|) \sim \mathcal{B}_{\mathcal{S},d,\lambda,a}\,L^{2\lambda}\,, \qquad \mathrm{var}\,(|\mathcal{T}|) \sim \mathcal{B}_{\mathcal{T},d,\lambda,a}\,L^{2\lambda}\,. \tag{30}$$

While if $\lambda > d/2$, then for any $a \in \mathbb{R}$

$$(|\mathcal{S}| - \mathbb{E}\,(|\mathcal{S}|))\,/\sqrt{\mathrm{var}\,(|\mathcal{S}|)}, \quad (|\mathcal{T}| - \mathbb{E}\,(|\mathcal{T}|))\,/\sqrt{\mathrm{var}\,(|\mathcal{T}|)} \implies \frac{|X| - \mathbb{E}|X|}{\sqrt{\mathrm{var}\,(|X|)}} \tag{31}$$

and there exist constants $\mathcal{A}_{\mathcal{S},d}, \mathcal{A}_{\mathcal{T},d}, \mathcal{B}_{\mathcal{S},d}, \mathcal{B}_{\mathcal{T},d} > 0$ such that

$$\mathbb{E}\,(|\mathcal{S}|) \sim \mathcal{A}_{\mathcal{S},d}\,L^{d/2}\,, \qquad \mathbb{E}\,(|\mathcal{T}|) \sim \mathcal{A}_{\mathcal{T},d}\,L^{d/2}\,, \tag{32}$$

and

$$\mathrm{var}\,(|\mathcal{S}|) \sim \mathcal{B}_{\mathcal{S},d}\,L^{d}\,, \qquad \mathrm{var}\,(|\mathcal{T}|) \sim \mathcal{B}_{\mathcal{T},d}\,L^{d}\,. \tag{33}$$

4.2. Universal two-point functions

We now provide numerical evidence that in both the high-temperature scaling window and the critical window, the two-point functions of the SAW and Ising model display the same asymptotic behaviour as does a RLRW whose walk length distribution is chosen to be that of a corresponding complete graph SAW.

We first consider the high-temperature scaling window with $\lambda < 2$. Assuming the validity of the conjectures on $\mathcal{S}$ and $\mathcal{T}$ outlined in equations (28) and (29), it follows from standard convergence of types arguments (see e.g. [40, p 193]) that for all $y \in \mathbb{R}$

$$\lim_{L\to\infty} \mathbb{P}\left(\frac{|\mathcal{S}|}{L^{\lambda}} \leqslant y\right) = \left(1 - \mathrm{e}^{-y/\mathcal{A}_{\mathcal{S},d,\lambda,a}}\right) \mathbb{1}\,(y \geqslant 0)\,. \tag{34}$$

Now let $\mathcal{N}_L = |\mathcal{S}_L|$, $a_L = L^{\lambda}$ and for fixed $\xi \in (0,\infty)$ let x_L satisfy $|x_L| = L^{\lambda/2}\xi$. Assuming the validity of equation (34), and that the assumptions of proposition 3.2 hold for this choice of $\mathcal{N}_L$, it follows from equation (24) that as $L \to \infty$

$$|x_L|^{d-2}\,g_{\mathrm{P},L,\mathcal{N}_L}\,(x_L) \sim \mathcal{E}\left(\xi/\sqrt{\mathcal{A}_{\mathcal{S},d,\lambda,a}}\right). \tag{35}$$

Universality then makes it natural to conjecture that the asymptotics of $|x_L|^{d-2}g(x_L)$ for the SAW and Ising model on the torus should be given by

$$|x_L|^{d-2}\,g\,(x_L)\sim\alpha\,\mathcal{E}\,(\gamma\,\xi) \tag{36}$$

for suitable values of the model-dependent constants, α,γ, depending on d, λ and a. Figures 4(a) and (b) provide strong evidence in favour of these conjectures. In figure 4(a), the constants for the SAW are set to $\alpha=0.75$, $\gamma=0.83/\sqrt{A_{\mathcal{S},d,\lambda,a}}$, while in figure 4(b) the constants for the Ising model are set to $\alpha=0.75$, $\gamma=0.87/\sqrt{A_{\mathcal{T},d,\lambda,a}}$, where $\mathcal{A}_{\mathcal{S},d,\lambda,a}$ and $\mathcal{A}_{\mathcal{T},d,\lambda,a}$ were estimated by fitting the mean walk length; see equation (29).

We now consider $\lambda>2$. Assuming the validity of the conjecture given by equation (29), the discussion in section 3.2 suggests that we should observe a plateau in this case. Moreover, we expect the order of the plateau to be $L^{\lambda-d}$ for $\lambda<d/2$ and $L^{-d/2}$ for $\lambda>d/2$. More concretely, suppose $2<\lambda<d/2$ and let $\mathcal{N}_L=|\mathcal{S}_L|$, and for fixed $\varphi\in(0,\infty)$ let x_L satisfy $|x_L|=L^{(d-\lambda)/(d-2)}\,\varphi$. Assuming the validity of the conjecture given by equation (29), it follows from proposition 3.3 and equation (26) that

$$|x_L|^{d-2}\,g_{\mathrm{P},L,\mathcal{N}_L}\,(x_L)\sim\frac{d}{2\pi^{d/2}}\Gamma\,(d/2-1)+\mathcal{A}_{\mathcal{S},d,\lambda,a}\,\varphi^{d-2}\,. \tag{37}$$

Universality then makes it natural to conjecture that for both the SAW and Ising model on the torus

$$|x_L|^{d-2}\,g\,(x_L)\sim\alpha\,\frac{d}{2\pi^{d/2}}\Gamma\,(d/2-1)+\gamma\,\varphi^{d-2} \tag{38}$$

for suitable model-dependent parameters α,γ, depending on d,λ and a. Figure 4(c) plots the SAW and Ising cases on tori with $d=5$, $\lambda=9/4$ and $a=z_{\mathrm{c}}$, with the constants set to $\alpha=0.76$ and $\gamma=1.08\mathcal{A}_{\mathcal{S},d,\lambda,a}$ for SAW and $\alpha=0.79$ and $\gamma=1.05\mathcal{A}_{\mathcal{T},d,\lambda,a}$ for the Ising model.

Finally, we consider the case $\lambda>d/2$, which lies inside the critical window. Let $\mathcal{N}_L=|\mathcal{S}_L|$, and for fixed $\varphi\in(0,\infty)$ let x_L satisfy $|x_L|=L^{(d/2)/(d-2)}\,\varphi$. Then assuming the validity of the conjecture given by equation (29), it follows that equation (38) should hold for suitable model-dependent parameters α,γ, depending on d. Figure 4(d) plots the SAW and Ising cases on tori with $d=5$, $\lambda=3$ and $a=z_{\mathrm{c}}$, with the constants set to $\alpha=0.74$ and $\gamma=1.71\mathcal{A}_{\mathcal{S},d}$ for SAW, and $\alpha=0.71$ and $\gamma=1.59\mathcal{A}_{\mathcal{T},d}$ for the Ising model.

5. Proof of proposition 3.2

Let $(S_n)_{n=0}^{\infty}$ be a simple random walk on $\mathbb{Z}^d$, starting from the origin, and let

$$p_n\,(z):=\mathbb{P}\,(S_n=z)\,,\qquad z\in\mathbb{Z}^d. \tag{39}$$

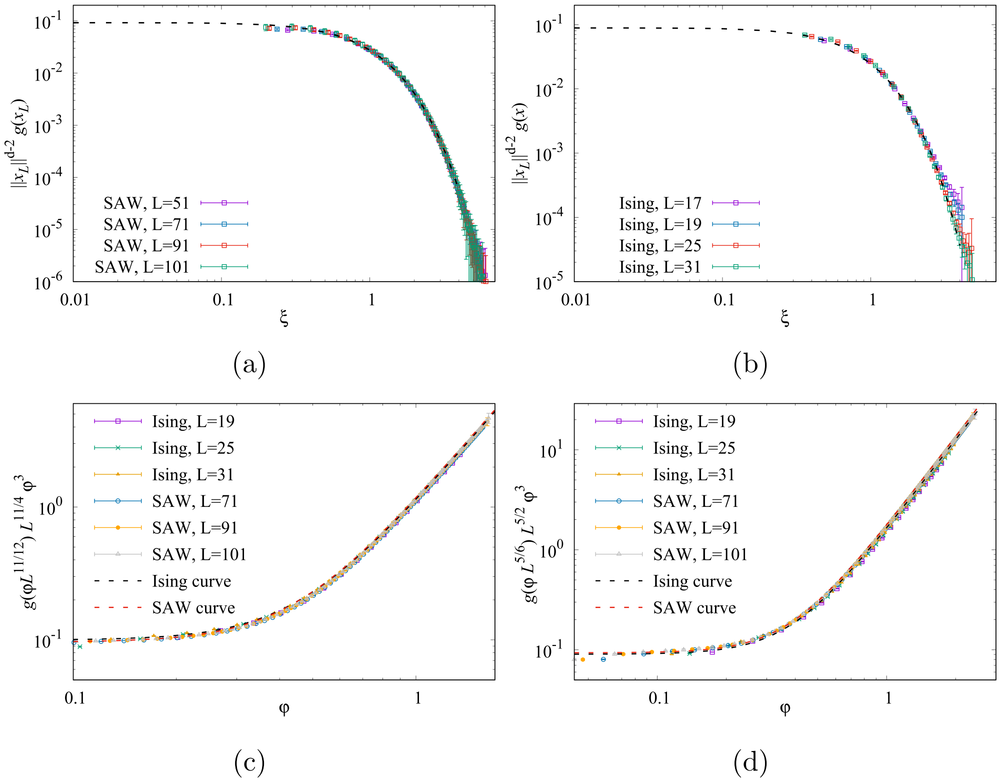

Figure 4. (a) Two-point functions on five-dimensional tori of SAW at fugacity $z_L = z_c(1 - L^{-\lambda})$ and $\lambda = 1$. The dashed curve corresponds to the ansatz given by equations (36) and (25), with constants α, γ set to the values described in the text, with $A_{\mathcal{S},d}$ estimated via simulation. (b) Analogous plot to (a) for Ising case. (c) Two-point functions on five-dimensional tori of SAW at fugacity $z_L = z_c(1 - L^{-\lambda})$ and $\lambda = 9/4$. The dashed curve corresponds to the plateau ansatz given by equation (38), with constants α, γ set to the values described in the text, with $B_{\mathcal{S},d}$ and $B_{\mathcal{T},d}$ estimated via simulation. (d) Analogous plot to (c), with $\lambda = 3$.

We say that $n \in \mathbb{N}$ and $z \in \mathbb{Z}^d$ have the same parity, and write $n \leftrightarrow z$, iff $n + |z|_1$ is even, where $|\cdot|_1$ denotes the ℓ^1 norm on $\mathbb{R}^d$. Clearly, $p_n(z) = 0$ if $n \not\leftrightarrow z$. Rearranging equation (20) we obtain

$$g_{\mathcal{N}}(z) = \sum_{n=0}^{\infty} p_n(z)\ \mathbb{P}(\mathcal{N} \geqslant n)\ . \tag{40}$$

The proof of proposition 3.2 will utilise the following three lemmas, whose proofs are deferred until the end of this section.

Lemma 5.1. *Let $d \in \mathbb{Z}_+$. Then for all $n \in \mathbb{Z}_+$ and all $x \in \mathbb{Z}^d$*

$$p_n(x) \ll \mathrm{e}^{-|x|/\sqrt{n}}\ .$$

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Lemma 5.2. *Let L and d be positive integers. For all $z\in\mathbb{Z}^d\setminus 0$ and all $x\in[-L/2,L/2]^d$*

$$\frac{1}{2}|z|\,L\leqslant|x+zL|\leqslant 2\sqrt{d}|z|\,L\,. \tag{41}$$

Lemma 5.3. *Consider $\mathbb{N}$-valued random variables $\mathcal{N}_L$ for which there exists $a_L>0$ satisfying $a_L\to\infty$ and a distribution function $G\neq\Delta$ such that $\mathbb{P}(\mathcal{N}_L/a_L\leqslant\cdot)\implies G$. Let $(x_L)_{L\geqslant3}$ be a sequence in $\mathbb{Z}^d$ satisfying $x_L\in\mathbb{B}_L^d$ and $|x_L|/\sqrt{a_L}\to\xi\in[0,\infty)$. Then*

$$g_{\mathcal{N}_L}(x_L)\gg_\xi\ a_L^{-d/2}\,.$$

Proof of proposition 3.2. Let $(x_L)_{L\geqslant3}$ be a sequence in $\mathbb{Z}^d$ satisfying $x_L\in\mathbb{B}_L^d$. In all that follows, any reference to reflecting or holding boundary conditions on $\mathbb{B}_L^d$ assumes L is odd.

First note that lemma 3.1 implies

$$\frac{g_{*,L,\mathcal{N}_L}(x_L)}{g_{\mathcal{N}_L}(x_L)}-1>0, \tag{42}$$

with $*=\mathrm{P,R,H}$. It therefore suffices to show that the left-hand side of equation (42) is $o(1)$ as $L\to\infty$.

We first consider the case of periodic boundary conditions. Part (i) of lemma 3.1 implies

$$g_{\mathrm{P},L,\mathcal{N}_L}(x_L)=g_{\mathcal{N}_L}(x_L)+\sum_{z\in\mathbb{Z}^d\setminus0}g_{\mathcal{N}_L}(x_L+L\,z) \tag{43}$$

and since $p_0(z)=0$ for $z\neq0$, equation (40) gives

$$g_{\mathrm{P},L,\mathcal{N}_L}(x_L)=g_{\mathcal{N}_L}(x_L)+\sum_{z\in\mathbb{Z}^d\setminus0}\sum_{n=1}^{\infty}p_n(x_L+L\,z)\,\mathbb{P}(\mathcal{N}_L\geqslant n)\,. \tag{44}$$

We begin by showing that the second term on the right-hand side of equation (44) is exponentially small in L. First consider the large n terms. Combining assumption (iii) with the Chernoff bound implies that $\mathbb{P}(\mathcal{N}_L\geqslant n)\leqslant C\mathrm{e}^{-rn/a_L}$. Letting $\zeta_L:=\lfloor L^{1+\lambda/2}\rfloor$, and recalling assumption (ii), it then follows that for any $S\subseteq\mathbb{Z}^d$ we have

$$\sum_{y\in S}\sum_{n=\zeta_L}^{\infty}p_n(y)\,\mathbb{P}(\mathcal{N}_L\geqslant n)\ll\sum_{n=\lfloor L^{1+\lambda/2}\rfloor}^{\infty}\mathrm{e}^{-rn/a_L}\ll\exp\left(-\kappa\,L^{\frac{2-\lambda}{2}}\right) \tag{45}$$

for some $\kappa>0$ which depends on d and on the specific sequence of distributions corresponding to $(\mathcal{N}_L)$.

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Now consider the small n terms. Fix $c>0$ and let $y\in\mathbb{Z}^d$ satisfy $|y|\geqslant cL$. Lemma 5.1 implies

$$\begin{aligned}\sum_{n=1}^{\zeta_L-1} p_n(y) &\ll \sum_{n=1}^{\zeta_L} \mathrm{e}^{-|y|/\sqrt{n}}\\ &\ll \sum_{n=1}^{\zeta_L} \mathrm{e}^{-cL/\sqrt{n}}\\ &\ll L^{1+\lambda/2}\exp\left(-cL^{\frac{2-\lambda}{4}}\right).\end{aligned}\tag{46}$$

But lemma 5.2 implies $|x_L+zL|\geqslant |z|L/2$ for all $z\in\mathbb{Z}^d\setminus 0$ and $x_L\in\mathbb{B}_L^d$, and so equation (46) implies

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d\setminus 0}\sum_{n=1}^{\zeta_L-1} p_n(x_L+Lz) &\ll L^{1+\lambda/2}\sum_{z\in\mathbb{Z}^d\setminus 0}\exp\left(-|z|L^{\frac{2-\lambda}{4}}/2\right)\\ &\ll L^{1+\lambda/2}\exp\left(-L^{\frac{2-\lambda}{4}}/4\right)\sum_{z\in\mathbb{Z}^d\setminus 0}\exp\left(-|z|/4\right).\end{aligned}\tag{47}$$

Therefore combining equations (45) and (47) with equation (44) shows that

$$g_{\mathrm{P},L,\mathcal{N}_L}(x_L)-g_{\mathcal{N}_L}(x_L)\ll L^{1+\lambda/2}\exp\left(-L^{\frac{2-\lambda}{4}}/4\right).\tag{48}$$

We now prove an analogous result for reflecting/holding boundaries. From parts (ii) and (iii) of lemma 3.1, we have for $*=\mathrm{R,H}$

$$\begin{aligned}g_{*,L,\mathcal{N}_L}(x_L) = g_{\mathcal{N}_L}(x_L) &+ \sum_{y\in[x_L]_{\tilde L}\setminus x_L}\sum_{n=1}^{\infty} p_n(y)\,\mathbb{P}(\mathcal{N}_L\geqslant n)\\ &+\sum_{y\in[x_L]_{\tilde L}}\sum_{z\in\mathbb{Z}^d\setminus 0}\sum_{n=1}^{\infty} p_n\left(y+2\tilde L z\right)\mathbb{P}(\mathcal{N}_L\geqslant n)\end{aligned}\tag{49}$$

where $\tilde L = L$ if $*=\mathrm{H}$, and $\tilde L = L-1$ if $*=\mathrm{R}$. The second and third terms on the right-hand side in equation (49) can be shown to be exponentially small by arguing analogously to the periodic case. Indeed, applying equation (45) immediately shows that

$$\begin{aligned}\sum_{y\in[x_L]_{\tilde L}\setminus x_L}\sum_{n=\zeta_L}^{\infty} p_n(y)\,\mathbb{P}(\mathcal{N}_L\geqslant n)+\sum_{y\in[x_L]_{\tilde L}}\sum_{z\in\mathbb{Z}^d\setminus 0}\sum_{n=\zeta_L}^{\infty} p_n\left(y+2\tilde L z\right)\mathbb{P}(\mathcal{N}_L\geqslant n)\\ \ll \exp\left(-\kappa L^{\frac{2-\lambda}{2}}\right).\end{aligned}\tag{50}$$

Now, $[x_L]_{\tilde{L}}$ is a subset of $\mathbb{B}^d_{2L}$ of fixed cardinality, 2^d, and so arguing analogously to (47) implies

$$\sum_{y\in[x_L]_{\tilde{L}}}\sum_{z\in\mathbb{Z}^d\setminus 0}\sum_{n=1}^{\zeta_L-1} p_n\left(y+2\tilde{L}\,z\right)\mathbb{P}\left(\mathcal{N}_L\geqslant n\right)\ll L^{1+\lambda/2}\exp\left(-L^{\frac{2-\lambda}{4}}/4\right). \tag{51}$$

Similarly, since $|y|\geqslant L/4$ for all $y\in[x_L]_{\tilde{L}}\setminus x_L$, it follows immediately from equation (46) that

$$\sum_{y\in[x_L]_{\tilde{L}}\setminus x_L}\sum_{n=1}^{\zeta_L-1} p_n\left(y\right)\mathbb{P}\left(\mathcal{N}_L\geqslant n\right)\ll L^{1+\lambda/2}\exp\left(-L^{\frac{2-\lambda}{4}}/4\right). \tag{52}$$

Summarising then, we have for $*=\mathrm{P,R,H}$ that

$$g_{*,L,\mathcal{N}_L}\left(x_L\right)-g_{\mathcal{N}_L}\left(x_L\right)\ll L^{1+\lambda/2}\exp\left(-L^{\frac{2-\lambda}{4}}/4\right). \tag{53}$$

But lemma 5.3 implies that $g_{\mathcal{N}_L}(x_L)\gg_\xi a_L^{-d/2}$, and so it follows from equation (53) that for $*=\mathrm{P,R,H}$

$$0\leqslant\frac{g_{*,L,\mathcal{N}_L}\left(x_L\right)}{g_{\mathcal{N}_L}\left(x_L\right)}-1\ll_\xi L^{1+(d+1)\lambda/2}\exp\left(-L^{\frac{2-\lambda}{4}}/4\right). \tag{54}$$

□

Proof of lemma 5.1. Let $a\geqslant 0$, and for $j\in[d]$ let S_n^j denote the jth coordinate of S_n. We begin with the elementary observation that if $(S_n^j)^2<a^2/d$ for all $j\in[d]$ then $\sum_{j\in[d]}(S_n^j)^2<a^2$. It then follows from the union bound that

$$\begin{aligned}\mathbb{P}\left(|S_n|\geqslant a\right)&\leqslant\sum_{j\in[d]}\mathbb{P}\left[|S_n^j|\geqslant a/\sqrt{d}\right]\\&=2d\,\mathbb{P}\left[S_n^1\geqslant a/\sqrt{d}\right]\\&\leqslant 2d\,\mathbb{E}\left(\mathrm{e}^{\lambda_n S_n^1}\right)\mathrm{e}^{-\lambda_n a/\sqrt{d}}\end{aligned} \tag{55}$$

for any $\lambda_n>0$, where in the last step we utilised the Chernoff bound. Consequently, for any $x\in\mathbb{Z}^d$ we have

$$p_n\left(x\right)\leqslant 2d\,\mathbb{E}\left(\mathrm{e}^{\lambda_n S_n^1}\right)\mathrm{e}^{-\lambda_n|x|/\sqrt{d}}. \tag{56}$$

A simple calculation shows that for all $t\in\mathbb{R}$

$$\mathbb{E}\left(\mathrm{e}^{tS_n^1}\right)=\left[1-\frac{1}{d}+\frac{\cosh\left(t\right)}{d}\right]^n. \tag{57}$$

Therefore, taking $\lambda_n=\sqrt{d/n}$ we have as $n\to\infty$ that

$$\log\mathbb{E}\left(\mathrm{e}^{\sqrt{d}S_n^1/\sqrt{n}}\right)=\frac{1}{2}+O\left(n^{-1}\right). \tag{58}$$

The stated result then follows from equation (58) and the specialisation of equation (56) to $\lambda_n=\sqrt{d/n}$. □

Proof of lemma 5.2. First note that $(a+y)^2 \geqslant y^2/4$ for all $a \in [-1/2, 1/2]$ and $y \in \mathbb{Z}$. It follows that for any $x \in [-L/2, L/2]^d$ and $z \in \mathbb{Z}^d \setminus 0$

$$\left|\frac{x}{L} + z\right|^2 = \sum_{i=1}^{d} \left(\frac{x_i}{L} + z_i\right)^2 \geqslant \sum_{i=1}^{d} \frac{z_i^2}{4} = \frac{|z|^2}{4},$$

which establishes the lower bound in equation (41).

Now, since $|x + zL|^2 = |x|^2 + L^2|z|^2 + 2L\, x \cdot z$, the Cauchy–Schwarz inequality implies

$$|x + zL|^2 \leqslant |z|^2 L^2 \left(1 + \frac{|x|}{L|z|}\right)^2. \tag{59}$$

Since $|x| \leqslant \sqrt{d}L/2$, we have

$$|x + zL| \leqslant \left(1 + \frac{\sqrt{d}}{2}\right) |z| L. \tag{60}$$

This establishes the upper bound in equation (41).

□

Proof of lemma 5.3. Since $\mathcal{N}_L, a_L \geqslant 0$ and $G \neq \Delta$, there must exist $s > 0$ such that $G(s) < 1$. It then follows that there exists $\varepsilon < 1$ and $\alpha \in (0, s)$ such that G is continuous at α and $G(\alpha) < \varepsilon$, which implies

$$\lim_{L\to\infty} \mathbb{P}(\mathcal{N}_L/a_L \leqslant \alpha) = G(\alpha) < \varepsilon. \tag{61}$$

Therefore there exists $\alpha > 0$ and $\varepsilon < 1$ such that for all sufficiently large L

$$\mathbb{P}(\mathcal{N}_L/a_L \leqslant \alpha) < \epsilon. \tag{62}$$

Now let $m_L := \lfloor \alpha a_L \rfloor - \mathbb{1}(x_L \nleftrightarrow \lfloor \alpha a_L \rfloor)$. Then $m_L \geqslant 0$ for all sufficiently large L, and

$$g_{\mathcal{N}_L}(x_L) \geqslant p_{m_L}(x_L)\, \mathbb{P}(\mathcal{N}_L \geqslant m_L) \geqslant (1 - \epsilon)\, p_{m_L}(x_L). \tag{63}$$

By construction, $x_L \leftrightarrow m_L$ and $m_L \to \infty$ as $L \to \infty$, and so it follows from [41, theorem 1.2.1, (1.10)] that there exists $C \in (0, \infty)$ such that for all sufficiently large L

$$p_{m_L}(x_L) \geqslant m_L^{-d/2} \left[2\left(\frac{d}{2\pi}\right)^{d/2} \exp\left(-\frac{d}{2}\frac{|x_L|^2}{m_L}\right) - \frac{C}{m_L}\right]. \tag{64}$$

But $|x_L|^2/m_L = (\xi^2/\alpha)[1 + o(1)]$ and so

$$p_{m_L}(x_L) \gg_\xi a_L^{-d/2}. \tag{65}$$

Combining equation (65) with equation (63) then yields the stated result.

□

6. Proof of proposition 3.3

For integer $n \geqslant 1$, let

$$\bar{p}_n(x) := 2\left(\frac{d}{2\pi n}\right)^{d/2} \exp\left(-\frac{d|x|^2}{2n}\right), \ \forall\, x \in \mathbb{Z}^d. \tag{66}$$

The local central limit theorem for simple random walk approximates p_n via $\bar{p}_n$.

We will make use of the following lemma, whose proof is deferred to the end of this section. Let

$$\Gamma(s,z) := \int_z^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad z \geqslant 0, \tag{67}$$

denote the (upper) incomplete gamma function, and let

$$\mathrm{erfc}(z) := \frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t, \qquad z \in \mathbb{R}, \tag{68}$$

denote the complementary error function.

Lemma 6.1. *Let $d \geqslant 1$ be an integer, let $a > 0$ and let $b \in \mathbb{R}^d$. Then,*

$$\frac{\pi^{d/2}}{a^{d/2}}\mathrm{erfc}^d\left(2\sqrt{a}\right) - 1 \leqslant \sum_{z\in\mathbb{Z}^d\setminus 0} \mathrm{e}^{-a|z+b|^2} \leqslant \frac{\pi^{d/2}}{a^{d/2}}\left(1+3\sqrt{\frac{a}{\pi}}\right)^d \tag{69}$$

and there exist $c_d, C_d > 0$ such that

$$c_d\, a^{-d/2}\Gamma(d/2, da) \leqslant \sum_{z\in\mathbb{Z}^d\setminus 0} \mathrm{e}^{-a|z|^2} \leqslant C_d\, a^{-d/2}. \tag{70}$$

Proof of proposition 3.3. Rearranging equation (44) yields

$$g_{\mathrm{P},L,\mathcal{N}_L}(x_L) = g_{\mathcal{N}_L}(x_L) + A(\mathcal{N}_L, x_L, L) + E(\mathcal{N}_L, x_L, L) + F(\mathcal{N}_L, x_L, L) \tag{71}$$

where

$$A(\mathcal{N}, x, l) := \frac{1}{2}\sum_{z\in\mathbb{Z}^d\setminus 0}\sum_{n=1}^\infty \bar{p}_n(x+l\,z)\,\mathbb{P}(\mathcal{N} \geqslant n) \tag{72}$$

$$E(\mathcal{N}, x, l) := \sum_{z\in\mathbb{Z}^d\setminus 0}\sum_{n=1}^\infty \mathbb{P}(\mathcal{N} \geqslant n)\,\frac{\bar{p}_n(x+l\,z)}{2}\left[\mathbb{1}(n \leftrightarrow x+l\,z) - \mathbb{1}(n \not\leftrightarrow x+l\,z)\right] \tag{73}$$

$$F(\mathcal{N}, x, l) := \sum_{z\in\mathbb{Z}^d\setminus 0}\sum_{n=1}^\infty \mathbb{P}(\mathcal{N} \geqslant n)\left[p_n(x+l\,z) - \bar{p}_n(x+l\,z)\right]\mathbb{1}(n \leftrightarrow x+l\,z). \tag{74}$$

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Similarly, rearranging equation (49) yields, for $* = \mathrm{R,H}$ and with $\tilde{L} = L$ if $* = \mathrm{H}$ and $\tilde{L} = L-1$ if $* = \mathrm{R}$,

$$\begin{aligned} g_{*,L,\mathcal{N}_L}(x_L) &= g_{\mathcal{N}_L}(x_L) + B(L) + \tilde{E}(L) + \tilde{F}(L) \\ &\quad + \sum_{y\in[x_L]_{\tilde{L}}} \left(A\left(\mathcal{N}_L, y, 2\tilde{L}\right) + E\left(\mathcal{N}_L, y, 2\tilde{L}\right) + F\left(\mathcal{N}_L, y, 2\tilde{L}\right)\right) \end{aligned} \tag{75}$$

where

$$B(L) := \frac{1}{2} \sum_{y\in[x_L]_{\tilde{L}}\backslash x_L} \sum_{n=1}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n)\, \bar{p}_n(y) \tag{76}$$

$$\tilde{E}(L) := \sum_{y\in[x_L]_{\tilde{L}}\backslash x_L} \sum_{n=1}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \frac{\bar{p}_n(y)}{2} \left[\mathbb{1}(n \leftrightarrow y) - \mathbb{1}(n \not\leftrightarrow y)\right] \tag{77}$$

$$\tilde{F}(L) := \sum_{y\in[x_L]_{\tilde{L}}\backslash x_L} \sum_{n=1}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \left[p_n(y) - \bar{p}_n(y)\right] \mathbb{1}(n \leftrightarrow y). \tag{78}$$

It is convenient in what follows to define the sequence l_L via

$$l_L = \begin{cases} L, & * = \mathrm{P}, \\ 2L, & * = \mathrm{H}, \\ 2(L-1), & * = \mathrm{R}. \end{cases} \tag{79}$$

and to then let y_L denote an arbitrary sequence in $\mathbb{Z}^d$ satisfying $y_L \in \mathbb{B}^d_{l_L}$. With this notation, we introduce the abbreviations

$$A(L) := A(\mathcal{N}_L, y_L, l_L), \quad E(L) := E(\mathcal{N}_L, y_L, l_L), \qquad F(L) := F(\mathcal{N}_L, y_L, l_L). \tag{80}$$

The first task is to show that $E(L)$, $\tilde{E}(L)$, $F(L)$ and $\tilde{F}(L)$ are all $o(\mathbb{E}(\mathcal{N}_L)/L^d)$. We begin by considering $E(L)$ and $\tilde{E}(L)$. Since $\mathbb{1}(w \not\leftrightarrow n) = \mathbb{1}(w \leftrightarrow n \pm 1)$, for any $w \in \mathbb{Z}^d$

$$\begin{aligned} &\sum_{n=1}^{\infty} \bar{p}_n(w) \mathbb{P}(\mathcal{N}_L \geqslant n) \mathbb{1}(n \leftrightarrow w) - \sum_{n=1}^{\infty} \bar{p}_n(w) \mathbb{P}(\mathcal{N}_L \geqslant n) \mathbb{1}(n \not\leftrightarrow w) \\ &\qquad \leqslant \bar{p}_1(w) + \sum_{n=1}^{\infty} \left|\bar{p}_{n+1}(w) - \bar{p}_n(w)\right| \mathbb{P}(\mathcal{N}_L \geqslant n), \end{aligned} \tag{81}$$

via a change of variables $n \mapsto n-1$ in the first sum. Similarly, changing variables in the second sum yields

$$\begin{aligned} &\sum_{n=1}^{\infty} \bar{p}_n(w) \mathbb{P}(\mathcal{N}_L \geqslant n) \mathbb{1}(n \leftrightarrow w) - \sum_{n=1}^{\infty} \bar{p}_n(w) \mathbb{P}(\mathcal{N}_L \geqslant n) \mathbb{1}(n \not\leftrightarrow w) \\ &\qquad \geqslant -\left(\bar{p}_1(w) + \sum_{n=1}^{\infty} \left|\bar{p}_{n+1}(w) - \bar{p}_n(w)\right| \mathbb{P}(\mathcal{N}_L \geqslant n)\right). \end{aligned} \tag{82}$$

It then follows that

$$|E(L)| \leqslant \frac{1}{2} \sum_{z\in\mathbb{Z}^d\setminus 0} \left(\bar{p}_1(y_L + l_L z) + \sum_{n=1}^{\infty} |\bar{p}_n(y_L + l_L z) - \bar{p}_{n+1}(y_L + l_L z)| \mathbb{P}(\mathcal{N}_L \geqslant n) \right) \tag{83}$$

$$|\tilde{E}(L)| \leqslant \frac{1}{2} \sum_{y\in[x_L]_{\tilde{L}}\setminus x_L} \left(\bar{p}_1(y) + \sum_{n=1}^{\infty} |\bar{p}_n(y) - \bar{p}_{n+1}(y)| \mathbb{P}(\mathcal{N}_L \geqslant n) \right). \tag{84}$$

Now define $\delta := (1+d/2)^{-1}$; the motivation for this will become clear following equation (91). We first consider the small n terms in equations (83) and (84). Let $w, z \in \mathbb{Z}^d \setminus 0$, with $|w| \geqslant |z|\, l_L/4$. If $1 \leqslant n \leqslant \lceil |z l_L|^{2-\delta} \rceil$ then for all $1 \leqslant j \leqslant n+1$ we have

$$\bar{p}_j(w) \ll \mathrm{e}^{-\kappa l_L^{\delta} |z|^{\delta}} \tag{85}$$

with $\kappa = \kappa(d) > 0$. The triangle inequality then implies that

$$\bar{p}_1(w) + \sum_{n=1}^{\lceil |z l_L|^{2-\delta} \rceil} |\bar{p}_n(w) - \bar{p}_{n+1}(w)| \mathbb{P}(\mathcal{N}_L \geqslant n) \ll |z\, l_L|^{2-\delta}\, \mathrm{e}^{-\kappa l_L^{\delta} |z|^{\delta}}. \tag{86}$$

Since $|y| \geqslant l_L/4$ for all $y \in [x_L]_{\tilde{L}} \setminus x_L$ and $x_L \in \mathbb{B}_L^d$, and since $[x_L]_{\tilde{L}}$ has only 2^d terms, it immediately follows from equation (86) with $z = 1$ that we can bound the small n terms in equation (84) via

$$\sum_{y\in[x_L]_{\tilde{L}}\setminus x_L} \left(\bar{p}_1(y) + \sum_{n=1}^{\lceil l_L^{2-\delta} \rceil} |\bar{p}_n(y) - \bar{p}_{n+1}(y)| \mathbb{P}(\mathcal{N}_L \geqslant n) \right) \ll l_L^{2-\delta} \mathrm{e}^{-\kappa l_L^{\delta}}. \tag{87}$$

But since $l_L^{\delta}|z|^{\delta} - l_L^{\delta} - |z|^{\delta} + 1 = (l_L^{\delta} - 1)(|z|^{\delta} - 1) \geqslant 0$ for any $z \in \mathbb{Z}^d \setminus 0$, we have that

$$\sum_{z\in\mathbb{Z}^d\setminus 0} |z\, l_L|^{2-\delta}\, \mathrm{e}^{-\kappa l_L^{\delta}|z|^{\delta}} \ll l_L^{2-\delta}\, \mathrm{e}^{-\kappa l_L^{\delta}}. \tag{88}$$

Therefore, combining the fact that lemma 5.2 implies $|y_L + l_L z| \geqslant |z| l_L/2$, with equations (86) and (88), it then follows that

$$\sum_{z\in\mathbb{Z}^d\setminus 0} \left(\bar{p}_1(y_L + l_L z) + \sum_{n=1}^{\lceil |z l_L|^{2-\delta} \rceil} |\bar{p}_n(y_L + l_L z) - \bar{p}_{n+1}(y_L + l_L z)| \mathbb{P}(\mathcal{N}_L \geqslant n) \right) \ll l_L^{2-\delta}\, \mathrm{e}^{-\kappa l_L^{\delta}}. \tag{89}$$

We therefore see that the sums of the small n terms in $E(L)$ and $\tilde{E}(L)$ are exponentially small in L.

We now consider the large n terms in equations (83) and (84). An elementary argument shows that there exists $c_d > 0$ such that for any $n \in \mathbb{Z}_+$ and $y \in \mathbb{Z}^d$

$$|\bar{p}_n(y) - \bar{p}_{n+1}(y)| \leqslant c_d\, n^{-1-d/2}. \tag{90}$$

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Using Markov's inequality, it then follows that for any $y, z \in \mathbb{Z}^d$

$$\begin{aligned}\sum_{n=\lceil |z\, l_L|^{2-\delta}\rceil+1}^{\infty} |\bar{p}_n(y) - \bar{p}_{n+1}(y)| \, \mathbb{P}(\mathcal{N}_L \geqslant n) &\ll \sum_{n=\lceil |z\, l_L|^{2-\delta}\rceil+1}^{\infty} \frac{1}{n^{d/2+1}} \mathbb{P}(\mathcal{N}_L \geqslant n) \\ &\ll \mathbb{E}(\mathcal{N}_L) \sum_{n=\lceil |z\, l_L|^{2-\delta}\rceil+1}^{\infty} \frac{1}{n^{d/2+2}} \\ &\ll \mathbb{E}(\mathcal{N}_L) \int_{|z\, l_L|^{2-\delta}}^{\infty} t^{-d/2-2}\, \mathrm{d}t \\ &\ll \frac{1}{|z|^{d+1}} \frac{\mathbb{E}(\mathcal{N}_L)}{l_L^{d+1}}.\end{aligned} \tag{91}$$

Since $|z|^{-d-1}$ is summable over $\mathbb{Z}^d \setminus 0$, combining equations (89), (87) and (91) shows that

$$\tilde{E}(L), \quad E(\mathcal{N}_L, x_L, L), \quad \sum_{y\in[x_L]_{\tilde{L}}} E\left(\mathcal{N}_L, y, 2\tilde{L}\right) = O\left(\frac{\mathbb{E}(\mathcal{N}_L)}{L^{d+1}}\right). \tag{92}$$

The bounds for $F(L)$ and $\tilde{F}(L)$ are obtained similarly, with the aid of the local central limit theorem. If $n \leqslant \lceil |z l_L|^{2-\delta} \rceil$ and $w, z \in \mathbb{Z}^d \setminus 0$ satisfy $|w| \geqslant |z|\, l_L/4$, then it follows from [26, proposition 2.1.2] that

$$p_n(w) \leqslant \mathbb{P}\left(\max_{0\leqslant i\leqslant n} |S_i| \geqslant |w|\right) \ll \mathrm{e}^{-\varphi |z\, l_L|^{\delta}}, \tag{93}$$

for some $\varphi = \varphi(d) > 0$. It then follows from equations (93), (85) and the triangle inequality that

$$\sum_{n=1}^{\lceil |z\, l_L|^{2-\delta}\rceil} |p_n(w) - \bar{p}_n(w)| \, \mathbb{P}(\mathcal{N}_L \geqslant n) \, \mathbb{1}(n \leftrightarrow w) \ll |z\, l_L|^{2-\delta} \mathrm{e}^{-(\kappa\wedge\varphi)\, l_L^{\delta} |z|^{\delta}}. \tag{94}$$

Just as equations (87) and (89) were obtained from equations (86) and (88), analogous arguments using equation (94) yield

$$\sum_{y\in[x_L]_{\tilde{L}}\setminus x_L} \sum_{n=1}^{\lceil l_L^{2-\delta}\rceil} |p_n(y) - \bar{p}_n(y)| \, \mathbb{P}(\mathcal{N}_L \geqslant n) \, \mathbb{1}(n \leftrightarrow y) \ll l_L^{2-\delta} \, \mathrm{e}^{-(\kappa\wedge\varphi)\, l_L^{\delta}}, \tag{95}$$

and

$$\begin{aligned}&\sum_{z\in\mathbb{Z}^d\setminus 0} \sum_{n=1}^{\lceil |z\, l_L|^{2-\delta}\rceil} |p_n(y_L + l_L z) - \bar{p}_n(y_L + l_L z)| \, \mathbb{P}(\mathcal{N}_L \geqslant n) \, \mathbb{1}(n \leftrightarrow y_L + l_L z) \\ &\quad \ll l_L^{2-\delta} \, \mathrm{e}^{-(\kappa\wedge\varphi)\, l_L^{\delta}}.\end{aligned} \tag{96}$$

The local central limit theorem for the simple random walk (see e.g. [41, theorem 1.2.1]) implies that that for all $y \in \mathbb{Z}^d$

$$|p_n(y) - \bar{p}_n(y)| \, \mathbb{P}(\mathcal{N}_L \geqslant n) \, \mathbb{1}(n \leftrightarrow y) \ll n^{-1-d/2}. \tag{97}$$

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Arguing as in equation (91), and combining with equation (95) and (96) then yields

$$\tilde{F}(L), \quad F(\mathcal{N}_L, x_L, L), \quad \sum_{y \in [x_L]_{\tilde{L}}} F\left(\mathcal{N}_L, y, 2\tilde{L}\right) = O\left(\frac{\mathbb{E}(\mathcal{N}_L)}{L^{d+1}}\right). \tag{98}$$

Next we consider $B(L)$. Let a_L be a positive sequence. For any $y \in \mathbb{Z}^d$ we have

$$\begin{aligned}\sum_{n=\lceil L^2 a_L \rceil}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \frac{\bar{p}_n(y)}{2} &\ll \frac{1}{a_L^{d/2} L^d} \sum_{n=\lceil L^2 a_L \rceil}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \\ &\ll a_L^{-d/2} \frac{\mathbb{E}(\mathcal{N}_L)}{L^d}.\end{aligned} \tag{99}$$

If now $|y| \geqslant \tilde{L}/2$, then since $\mathrm{e}^{-t} < t^{-\gamma}$ for all $t, \gamma > 0$, taking $\gamma = d/2$ we have

$$\frac{\bar{p}_n(y)}{2} \ll n^{-d/2} \, \mathrm{e}^{-d\tilde{L}^2/(8n)} \ll L^{-d} \tag{100}$$

and therefore

$$\sum_{n=1}^{\lceil L^2 a_L \rceil - 1} \mathbb{P}(\mathcal{N}_L \geqslant n) \frac{\bar{p}_n(y)}{2} \ll L^{2-d} a_L. \tag{101}$$

Consequently, if $\mathbb{E}(\mathcal{N}_L) = \Omega(L^2)$ then choosing $a_L \to 1$ in equations (99) and (101) implies

$$B(L) = O\left(L^{-d} \mathbb{E}(\mathcal{N}_L)\right), \tag{102}$$

while if $\mathbb{E}(\mathcal{N}_L) = \omega(L^2)$ then choosing $a_L \to \infty$ with $a_L L^2 = o(\mathbb{E}(\mathcal{N}_L))$ implies

$$B(L) = o\left(L^{-d} \mathbb{E}(\mathcal{N}_L)\right). \tag{103}$$

It now remains to study the asymptotic behaviour of $A(L)$. We first consider part (i), and therefore assume $\mathbb{E}(\mathcal{N}_L) = \Omega(L^2)$. It follows from lemma 5.2 and equation (70) of lemma 6.1 that

$$A(\mathcal{N}_L, y_L, l_L) \ll l_L^{-d} \sum_{n=1}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \ll \frac{\mathbb{E}(\mathcal{N}_L)}{L^d}. \tag{104}$$

Since $\mathbb{E}(\mathcal{N}_L) = \Omega(L^2)$, there exists $\gamma > 0$ such that $\mathbb{E}(\mathcal{N}_L) \geqslant \gamma L^2$ for all sufficiently large L. It follows again from lemma 5.2 and equation (70) of lemma 6.1 that

$$A(\mathcal{N}_L, y_L, l_L) \gg l_L^{-d} \sum_{n=1}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \, \Gamma\left(\frac{d}{2}, \frac{2d^3 l_L^2}{n}\right) \tag{105}$$

$$\gg L^{-d} \sum_{n=\lfloor \gamma L^2/2 \rfloor + 1}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \tag{106}$$

$$= L^{-d} \left(\mathbb{E}(\mathcal{N}_L) - \sum_{n=1}^{\lfloor \gamma L^2/2 \rfloor} \mathbb{P}(\mathcal{N}_L \geqslant n) \right) \tag{107}$$

$$\gg \frac{\mathbb{E}(\mathcal{N}_L)}{L^d}. \tag{108}$$

It follows in particular that

$$A(\mathcal{N}_L, x_L, L) \asymp \frac{\mathbb{E}(\mathcal{N}_L)}{L^d}, \qquad \sum_{y \in [x_L]_{\tilde{L}}} A\left(\mathcal{N}_L, y, 2\tilde{L}\right) \asymp \frac{\mathbb{E}(\mathcal{N}_L)}{L^d}. \tag{109}$$

Together with equations (102), (92), (98), (71) and (75) this establishes part (i).

We now focus on part (ii), and therefore assume $\mathbb{E}(\mathcal{N}_L) = \omega(L^2)$. Let $a_L > 0$ be a sequence satisfying $a_L \to \infty$ and $a_L = o(\mathbb{E}(\mathcal{N}_L)/L^2)$. From lemma 5.2 and equation (70) of lemma 6.1 we have

$$\begin{aligned} \frac{1}{2} \sum_{n=1}^{\lceil a_L L^2 \rceil - 1} \mathbb{P}(\mathcal{N}_L \geqslant n) \sum_{z \in \mathbb{Z}^d \setminus 0} \bar{p}_n(y_L + l_L z) &\ll l_L^{-d} \sum_{n=1}^{\lceil a_L L^2 \rceil - 1} \mathbb{P}(\mathcal{N}_L \geqslant n) \\ &\ll a_L L^{2-d} \\ &= o\left(\frac{\mathbb{E}(\mathcal{N}_L)}{L^d}\right). \end{aligned} \tag{110}$$

But equation (69) of lemma 6.1 implies

$$\begin{aligned} \frac{1}{2} \sum_{n=\lceil a_L L^2 \rceil}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \sum_{z \in \mathbb{Z}^d \setminus 0} \bar{p}_n(y_L + l_L z) &\leqslant l_L^{-d} \sum_{n=\lceil a_L L^2 \rceil}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \left(1 + 3\sqrt{\frac{d l_L^2}{2\pi a_L L^2}}\right)^d \\ &= l_L^{-d} [1 + o(1)] \sum_{n=\lceil a_L L^2 \rceil}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n), \end{aligned} \tag{111}$$

and, since erfc is decreasing and continuous and $\mathrm{erfc}(0) = 1$,

$$\begin{aligned} &\frac{1}{2} \sum_{n=\lceil a_L L^2 \rceil}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \sum_{z \in \mathbb{Z}^d \setminus 0} \bar{p}_n(y_L + l_L z) \\ &\quad \geqslant l_L^{-d} \sum_{n=\lceil a_L L^2 \rceil}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \left[\mathrm{erfc}^d\left(\sqrt{\frac{2 d l_L^2}{a_L L^2}}\right) - \left(\frac{d l_L^2}{2\pi a_L L^2}\right)^{d/2}\right] \\ &\quad = l_L^{-d} [1 + o(1)] \sum_{n=\lceil a_L L^2 \rceil}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n). \end{aligned} \tag{112}$$

But, by assumption on a_L,

$$\sum_{n=1}^{\lceil a_L L^2 \rceil - 1} \mathbb{P}(\mathcal{N}_L \geqslant n) \leqslant a_L L^2 = o(\mathbb{E}(\mathcal{N}_L)) \tag{113}$$

and so

$$\frac{1}{2} \sum_{n=\lceil a_L L^2 \rceil}^{\infty} \mathbb{P}(\mathcal{N}_L \geqslant n) \sum_{z \in \mathbb{Z}^d \setminus 0} \bar{p}_n(y_L + l_L z) = \frac{\mathbb{E}(\mathcal{N}_L)}{l_L^d} [1 + o(1)]. \tag{114}$$

Combining equations (110) and (114) we then have

$$A\left(\mathcal{N}_L, y_L, l_L\right) \sim \frac{\mathbb{E}\left(\mathcal{N}_L\right)}{l_L^d}\,. \tag{115}$$

In particular, for periodic boundary conditions we have $l_L = L$ and so equation (115) implies

$$A\left(\mathcal{N}_L, x_L, L\right) \sim \frac{\mathbb{E}\left(\mathcal{N}_L\right)}{L^d} \tag{116}$$

while for reflecting or holding boundary conditions we have $l_L = 2\tilde{L}$ and so equation (115) implies

$$\sum_{y\in[x_L]_{\tilde{L}}} A\left(\mathcal{N}_L, y, 2\tilde{L}\right) \sim \sum_{y\in[x_L]_{\tilde{L}}} \frac{\mathbb{E}\left(\mathcal{N}_L\right)}{2^d L^d} = \frac{\mathbb{E}\left(\mathcal{N}_L\right)}{L^d}\,. \tag{117}$$

Together with equations (103), (92), (98), (71) and (75) this establishes part (ii). □

Proof of lemma 6.1. Let $\epsilon \in (-1,1)$. For any $z \geqslant 2$ we have $\mathrm{e}^{-a(z+\epsilon)^2} \leqslant \mathrm{e}^{-a(t-1/2+\epsilon)^2}$ for all $t \in [z-1/2, z+1/2]$, and so

$$\sum_{z=1}^{\infty} \mathrm{e}^{-a(z+\epsilon)^2} \leqslant e^{-a(1+\epsilon)^2} + \sum_{z=2}^{\infty} \int_{z-1/2}^{z+1/2} \mathrm{e}^{-a(t-1/2+\epsilon)^2} \mathrm{d}t \leqslant 1 + \frac{1}{2}\sqrt{\frac{\pi}{a}}\,. \tag{118}$$

A similar argument also produces the lower bound

$$\sum_{z=1}^{\infty} \mathrm{e}^{-a(z+\epsilon)^2} \geqslant \sum_{z=1}^{\infty} \int_{z-1/2}^{z+1/2} \mathrm{e}^{-a(t+1/2+\epsilon)^2} \mathrm{d}t \geqslant \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\mathrm{erfc}\left(2\sqrt{a}\right)\,. \tag{119}$$

Now let $\alpha \in \mathbb{R}$, and define $\{\alpha\} := \alpha - \lfloor \alpha \rfloor \in [0,1)$. Since $z \mapsto z - \lfloor \alpha \rfloor$ is a bijection of $\mathbb{Z}$, we have

$$\sum_{z\in\mathbb{Z}} e^{-a(z+\alpha)^2} = \sum_{z\in\mathbb{Z}_+} e^{-a(z+\{\alpha\})^2} + \sum_{z\in\mathbb{Z}_+} e^{-a(z-\{\alpha\})^2} + e^{-a\{\alpha\}^2}. \tag{120}$$

It follows from equations (118) and (119) that

$$\sqrt{\frac{\pi}{a}}\,\mathrm{erfc}\left(2\sqrt{a}\right) \leqslant \sum_{z\in\mathbb{Z}} e^{-a(z+\alpha)^2} \leqslant \sqrt{\frac{\pi}{a}} + 3. \tag{121}$$

Equation (69) then follows by observing that

$$\sum_{z\in\mathbb{Z}^d\setminus 0} \mathrm{e}^{-a|z+b|^2} = \prod_{i=1}^{d} \sum_{z_i\in\mathbb{Z}} \mathrm{e}^{-a(z_i+b_i)^2} - 1\,. \tag{122}$$

We now consider equation (70). Let $n \in \mathbb{Z}_+$ and $\mathbb{A}_n = [-n,n]^d \cap \mathbb{Z}^d$. Let $\partial\mathbb{A}_n = \mathbb{A}_n \setminus \mathbb{A}_{n-1}$ be the set of vertices on the surface of the box $\mathbb{A}_n$. Since $\mathbb{A}_{n-1} \subset \mathbb{A}_n$, it follows that

$$|\partial\mathbb{A}_n| = |\mathbb{A}_n| - |\mathbb{A}_{n-1}| = (2n+1)^d - (2n-1)^d\,. \tag{123}$$

Therefore there exist $c_d, C_d > 0$ such that for all $n \in \mathbb{Z}_+$

$$c_d n^{d-1} \leqslant |\partial \mathbb{A}_n| \leqslant C_d n^{d-1}. \tag{124}$$

Now observe that if $|z|_\infty$ denotes the sup norm on $\mathbb{R}^d$, then $\partial \mathbb{A}_n = \{z \in \mathbb{Z}^d : |z|_\infty = n\}$ and $|z|_\infty \leqslant |z| \leqslant \sqrt{d}|z|_\infty$. Consequently,

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d\setminus 0} \mathrm{e}^{-a|z|^2} &\leqslant \sum_{n=1}^{\infty} \mathrm{e}^{-an^2} |\partial \mathbb{A}_n| \\ &\leqslant C_d \sum_{n=1}^{\infty} \int_n^{n+1} \mathrm{e}^{-at^2/4} t^{d-1}\,\mathrm{d}t \\ &\leqslant \left(2^{d-1}\Gamma(d/2)\,C_d\right) a^{-d/2}.\end{aligned} \tag{125}$$

A lower bound is obtained similarly:

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d\setminus 0} \mathrm{e}^{-a|z|^2} &\geqslant \sum_{n=1}^{\infty} \mathrm{e}^{-adn^2} |\partial \mathbb{A}_n| \\ &\geqslant c_d \sum_{n=1}^{\infty} \int_n^{n+1} \mathrm{e}^{-adt^2} \left(\frac{t}{2}\right)^{d-1} \mathrm{d}t \\ &\geqslant \left(\frac{c_d}{2^d\, d^{d/2}}\right) a^{-d/2}\Gamma(d/2, da).\end{aligned} \tag{126}$$

□

7. Proof of lemma 3.1

Let $(X_t)_{t\in\mathbb{N}}$ be a Markov chain on a countable set S with transition matrix P. Let $\mathcal{N}$ be an $\mathbb{N}$-valued random variable, independent of $(X_t)_{t\in\mathbb{N}}$. If $X_0 = x_0$ for some fixed $x_0 \in S$, then we define the corresponding two-point function to be

$$g(x_0, x) := \mathbb{E}_{x_0}\left(\sum_{t=0}^{\mathcal{N}} \mathbb{1}(X_t = x)\right) = \sum_{t=0}^{\infty} P^t(x_0, x)\,\mathbb{P}(\mathcal{N} \geqslant t); \tag{127}$$

the expected number of visits to $x \in S$ by time $\mathcal{N}$.

Now let $\sim$ denote an equivalence relation on S. For each $x \in S$, we let $[x] := \{x' \in S : x' \sim x\}$ denote its equivalence class, and denote the set of all equivalence classes on S by $S^\# := \{[x] : x \in S\}$. For each $x, y \in S$ define

$$P(x, [y]) := \sum_{y\in[y]} P(x, y). \tag{128}$$

We say P respects $\sim$ if $P(x, [y]) = P(x', [y])$ for all $x, y \in S$ and all $x' \sim x$. If P respects $\sim$, it is straightforward to show that the matrix $P^\#$ on $S^\#$ defined by

$$P^\#([x], [y]) := P(x, [y]) \tag{129}$$

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is stochastic, and that the process $([X_t])_{t\in\mathbb{N}}$ is a Markov chain on $S^\#$ with transition matrix $P^\#$. See, for example, [42, section 2.3.1].

Lemma 7.1. *Let S be a countable set, endowed with an equivalence relation respected by the stochastic matrix P. Let $(X_t)_{t\in\mathbb{N}}$ be a Markov chain with transition matrix P, and let $\mathcal{N}$ be an $\mathbb{N}$-valued random variable, independent of $(X_t)_{t\in\mathbb{N}}$. Let g be the corresponding two-point function of $(X_t)_{t\in\mathbb{N}}$, and $g^\#$ be the corresponding two-point function of $([X_t])_{t\in\mathbb{N}}$. Then for $[x],[y]\in S^\#$ and all $x'\in[x]$ we have*

$$g^\#\left([x],[y]\right)=\sum_{y'\in[y]}g\left(x',y'\right).$$

Proof. Let $x,y\in S$. A simple induction on t shows that $(P^\#)^t([x],[y])=P^t(x',[y])$ for all integers $t\geqslant 0$ and all $x'\in[x]$. Therefore if $x'\in[x]$, it follows that

$$\begin{aligned}g^\#\left([x],[y]\right)&=\sum_{t=0}^{\infty}P^t\left(x',[y]\right)\mathbb{P}\left(\mathcal{N}\geqslant t\right)=\sum_{y'\in[y]}\sum_{t=0}^{\infty}P^t\left(x',y'\right)\mathbb{P}\left(\mathcal{N}\geqslant t\right)\\&=\sum_{y'\in[y]}g\left(x',y'\right).\end{aligned}\tag{130}$$

□

7.1. Proof of lemma 3.1

The proof of lemma 3.1 relies on the following two results, whose proofs are deferred to section 7.2. Recall the definition of $[x]_L$ given in equation (22).

Lemma 7.2. *Let $L\geqslant 2$, and let $P_{\mathrm{P},2L}$ denote the transition matrix of the simple random walk on $\mathbb{B}^d_{2L}$ with periodic boundary conditions. If $x,y\in\mathbb{B}^d_{2L}$, then for all $x'\in[x]_L$*

$$P_{\mathrm{P},2L}\left(x,[y]_L\right)=P_{\mathrm{P},2L}\left(x',[y]_L\right).$$

Lemma 7.3. *Let $L\geqslant 3$, and let $P_{\mathrm{P},L}$, $P_{\mathrm{H},L}$, and $P_{\mathrm{R},L}$ denote the transition matrices of the simple random walk on $\mathbb{B}^d_L$ with periodic, holding and reflective boundary conditions, respectively. For any odd L we have*

$$P_{\mathrm{R},L}\left(x,y\right)=P^\#_{\mathrm{P},2(L-1)}\left([x]_{L-1},[y]_{L-1}\right)\tag{131}$$

$$P_{\mathrm{H},L}\left(x,y\right)=P^\#_{\mathrm{P},2L}\left([x]_L,[y]_L\right)\tag{132}$$

for all $x,y\in\mathbb{B}_L$.

Proof of lemma 3.1. Let P denote the transition matrix of the simple random walk on $\mathbb{Z}^d$, and let $P_{*,L}$ denote the transition matrix of the simple random walk on $\mathbb{B}^d_L$ with * boundary conditions, where * can denote P, R or H, for periodic, reflecting or holding, respectively. Fix a random variable $\mathcal{N}$, and let g denote the two-point function defined in equation (127) corresponding to P and $\mathcal{N}$. Likewise, let $g_{*,L}$ denote the two-point function corresponding to $P_{*,L}$ and $\mathcal{N}$. The corresponding two-point functions defined

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in section 3 are the specialisations of g and $g_{*,L}$ to the case $x_0 = 0$. We therefore freely omit the first argument of equation (127) when convenient, with the understanding that in such instances it takes the value 0.

We begin by proving part (i). Consider the equivalence relation on $\mathbb{Z}^d$ defined so that to each $x \in \mathbb{B}_L^d$ there corresponds the equivalence class

$$[x] := \left\{x + L\,z : z \in \mathbb{Z}^d\right\}. \tag{133}$$

It is straightforward to verify that P respects this equivalence relation, and a simple calculation shows that $P^{\#}([x],[y]) = P_{\mathrm{P},L}(x,y)$ for all $x, y \in \mathbb{B}_L^d$. We therefore have for any $x \in \mathbb{T}_L^d$ that

$$\begin{aligned} g_{\mathrm{P},L}(x) &:= \sum_{t=0}^{\infty} P_{\mathrm{P},L}^t(0,x)\,\mathbb{P}(\mathcal{N} \geqslant t) \\ &= \sum_{t=0}^{\infty} \left(P^{\#}\right)^t([0],[x])\,\mathbb{P}(\mathcal{N} \geqslant t) \\ &= g^{\#}([0],[x]) \\ &= \sum_{x' \in [x]} g(0,x') \\ &= \sum_{z \in \mathbb{Z}^d} g(x + zL), \end{aligned} \tag{134}$$

where the penultimate step follows from lemma 7.1. This establishes part (i).

Similarly, since $\mathbb{B}_L^d \subset \mathbb{T}_{2(L-1)}^d$ we have for all $x \in \mathbb{B}_L^d$ that

$$\begin{aligned} g_{\mathrm{R},L}(x) &:= \sum_{t=0}^{\infty} P_{\mathrm{R},L}^t(0,x)\,\mathbb{P}(\mathcal{N} \geqslant t) \\ &= \sum_{t=0}^{\infty} \left(P_{\mathrm{R},2(L-1)}^{\#}\right)^t \left([0]_{L-1},[x]_{L-1}\right)\mathbb{P}(\mathcal{N} \geqslant t) \\ &= g_{\mathrm{P},2(L-1)}^{\#}\left([0]_{L-1},[x]_{L-1}\right) \\ &= \sum_{x' \in [x]_{L-1}} g_{\mathrm{P},2(L-1)}(x') \end{aligned} \tag{135}$$

where the second step follows from lemmas 7.2 and 7.3, and the last step follows from lemma 7.1. This establishes part (ii). Part (iii) is proved similarly. $\square$

7.2. Proof of lemmas 7.2 and 7.3

Recall the definition of $[x]_L$ given in equation (22). If $x, y \in \mathbb{T}_{2L}^d$ and $i \in [d]$, we will write $y_i \sim x_i$ iff $y_i \in \{x_i, -L - x_i\}$, where we emphasise that addition and multiplication are modulo $2L$. If $y_i \sim x_i$ for all $i \in [d]$, then we will also write $y \sim x$, so that $y \sim x$ iff $y \in [x]_L$. To avoid confusion, in this section we will denote adjacency between two vertices $x, y \in \mathbb{T}_L^d$ by $x \leftrightarrow y$, where the value of L will be clear from the context.

Proof of lemma 7.2. A simple random walk on $\mathbb{B}_{2L}^d$ with periodic boundary conditions corresponds to a simple random walk on $\mathbb{T}_{2L}^d$. Let $x, y \in \mathbb{T}_{2L}^d$. Then

$$P_{\mathrm{P},2L}(x,[y]_L) = \sum_{y' \in [y]_L} P_{\mathrm{P},2L}(x,y') = \frac{1}{2d}|N(x) \cap [y]_L| \tag{136}$$

where $N(x) := \{y \in \mathbb{T}_{2L}^d : x \leftrightarrow y\}$ is the set of neighbours of x in $\mathbb{T}_{2L}^d$.

Let $x' \sim x$. Suppose $N(x') \cap [y]_L$ is nonempty, and let $z' \in N(x') \cap [y]_L$. Then $z' = x' + \delta e^k$ for some $\delta \in \{-1,1\}$ and $1 \leqslant k \leqslant d$, where e^k denotes the standard unit vector along the kth coordinate axis. Consider

$$z = \begin{cases} x + \delta e^k, & x'_k = x_k \\ x - \delta e^k, & x'_k \neq x_k. \end{cases} \tag{137}$$

Clearly, $z \in N(x)$. Moreover, for all $i \neq k$ we have $z_i = x_i \sim x'_i = z'_i \sim y_i$ since $z' \in [y]_L$, so that $z_i \sim y_i$. Furthermore, if $x_k = x'_k$ then $z_k = x_k + \delta = x'_k + \delta = z'_k \sim y_k$, while if $x_k \neq x'_k$ then $z_k = x_k - \delta = (-L - x'_k) - \delta = -L - (x'_k + \delta) \sim x'_k + \delta = z'_k \sim y_k$, so that in either case $z_k \sim y_k$. It then follows that $z \sim y$ and so $z \in N(x) \cap [y]_L$. In particular, we see that $N(x') \cap [y]_L$ is nonempty iff $N(x) \cap [y]_L$ is nonempty.

Suppose now that $N(x) \cap [y]_L$ is nonempty, and define $f : N(x) \cap [y]_L \to N(x') \cap [y]_L$ via

$$f\left(x + \delta e^k\right) = \begin{cases} x' + \delta e^k, & x'_k = x_k \\ x' - \delta e^k, & x'_k \neq x_k. \end{cases} \tag{138}$$

The above argument implies that for any $z' = x' + \delta e^k \in N(x') \cap [y]_L$, if we define $z \in N(x) \cap [y]_L$ as in equation (137) then

$$\begin{aligned} f(z) &= \begin{cases} f\left(x + \delta e^k\right), & x'_k = x_k \\ f\left(x - \delta e^k\right), & x'_k \neq x_k \end{cases} \\ &= x' + \delta e^k \\ &= z' \end{aligned} \tag{139}$$

and so f is surjective.

Now suppose $z, z' \in N(x) \cap [y]_L$ satisfy $f(z) = f(z')$. Without loss of generality, suppose $z = x + \delta e^k$, so that

$$(f(z))_k = \begin{cases} x'_k + \delta, & x'_k = x_k \\ x'_k - \delta, & x'_k \neq x_k. \end{cases} \tag{140}$$

If $z' = x + \delta' e^{k'}$ with $k' \neq k$, then $\left(f(z')\right)_k = x'_k \neq x'_k \pm \delta$. So $f(z) = f(z')$ implies $k' = k$, and it follows that

$$(f(z'))_k = \begin{cases} x'_k + \delta', & x'_k = x_k \\ x'_k - \delta', & x'_k \neq x_k \end{cases} \tag{141}$$

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and $\big(f(z)\big)_k = \big(f(z')\big)_k$ implies $\delta = \delta'$. Therefore, $z' = x + \delta e^k = z$, which implies f is injective. We conclude that since f bijective, $|N(x) \cap [y]_L| = |N(x') \cap [y]_L|$ for all $x, y \in \mathbb{T}_{2L}^d$ and $x' \in [x]_L$. The stated result then follows from equation (136).

□

Proof of lemma 7.3. Let $l \in \mathbb{Z}_+$ and set $L = 2l+1$. We first consider equation (131). By construction, $\mathbb{B}_{2l+1}^d \subset \mathbb{T}_{4l}^d$. Let $S_{4l}^{\#}$ denote the set of equivalence classes on $\mathbb{T}_{4l}^d$ defined by equation (22). Since the map $\mathbb{B}_{2l+1}^d \to S_{4l}^{\#}$ defined by $x \mapsto [x]_{2l}$ is a bijection, and since both $P_{\mathrm{R},2l+1}$ and $P_{\mathrm{P},4l}^{\#}$ are stochastic, in order to show that $P_{\mathrm{R},2l+1}(x,y) = P_{\mathrm{P},4l}^{\#}([x]_{2l},[y]_{2l})$ for all $x, y \in \mathbb{B}_{2l+1}^d$, it suffices to consider only pairs $x, y \in \mathbb{B}_{2l+1}^d$ with $P_{\mathrm{R},2l+1}(x,y) > 0$. By definition, $P_{\mathrm{R},2l+1}(x,y) > 0$ only if $y = x + \delta e^k$ for some $\delta \in \{-1,1\}$ and $k \in [d]$.

Let $x \in \mathbb{T}_{4l}^d$ with $-l \leqslant x_i \leqslant l$ for all $i \in [d]$. Suppose $y = x + \delta e^k$ for $\delta \in \{-1,1\}$ and $k \in [d]$. Clearly, $y \in N(x) \cap [y]_{2l}$, where $N(x)$ is the set of neighbours of x in $\mathbb{T}_{4l}^d$. Suppose $y' \in [y]_{2l}$. Then either $y'_k = y_k = x_k + \delta$ or $y'_k = -2l - y_k = -2l - (x_k + \delta)$. Clearly $x_k + \delta \neq x_k$, and $-2l - (x_k+\delta) = x_k$ iff $-2l - \delta = 2x_k$, but the latter cannot hold, since the left-hand side is odd, the right-hand side is even, and addition is modulo $4l$. We therefore see that $y'_k \neq x_k$. In order for y' to belong to $N(x)$, it is therefore necessary that $y'_i = x_i$ for all $i \neq k$. Defining $\tilde{y}$ via $\tilde{y}_i = x_i$ for $i \neq k$ and $\tilde{y}_k = -2l - (x_k + \delta)$ we conclude that $\{y\} \subset N(x) \cap [y]_{2l} \subset \{y, \tilde{y}\}$. Since $\tilde{y} \in [y]_{2l}$ by construction, we have

$$N(x) \cap [y]_{2l} = \begin{cases} \{y, \tilde{y}\}, & \tilde{y} \in N(x) \\ \{y\}, & \tilde{y} \notin N(x). \end{cases} \tag{142}$$

But $\tilde{y} \in N(x)$ if and only if $\tilde{y}_k = x_k + \epsilon$ for $\epsilon \in \{-1,1\}$. And $\tilde{y} \neq y$ if and only if $y'_k \neq y_k$. So $|N(x) \cap [y]| = 2$ if and only if $\tilde{y}_k = x_k - \delta$, which holds iff $-2l - (x_k + \delta) = x_k - \delta$ modulo $4l$, which in turn holds iff $x_k \in \{-l, l\}$. It follows that if $y = x + \delta e^k$ then

$$\begin{aligned} P_{\mathrm{P},4l}^{\#}([x]_{2l},[y]_{2l}) = P_{\mathrm{P},4l}(x,[y]) = \frac{1}{2d}|N(x) \cap [y]| &= \begin{cases} 1/(2d), & x_k \neq \pm l \\ 1/d, & x_k = \pm l \end{cases} \\ &= P_{\mathrm{R},2l+1}(x,y). \end{aligned} \tag{143}$$

We next consider equation (132). Note that $\mathbb{B}_{2l+1}^d \subset \mathbb{T}_{2(2l+1)}^d$, and let $S_{2(2l+1)}^{\#}$ denote the set of equivalence classes on $\mathbb{T}_{2(2l+1)}^d$ corresponding to equation (22). Since the map $\mathbb{B}_{2l+1}^d \to S_{2(2l+1)}^{\#}$ defined by $x \mapsto [x]_{2l+1}$ is a bijection, arguing as above it suffices to show $P_{\mathrm{P},2(2l+1)}^{\#}([x]_{2l+1},[y]_{2l+1}) = P_{\mathrm{H},2l+1}(x,y)$ for all $x, y \in \mathbb{B}_{2l+1}^d$ with $P_{\mathrm{H},2l+1}(x,y) > 0$. By definition, $P_{\mathrm{H},2l+1}(x,y) > 0$ only if $y = x + \delta e^k$ for some $\delta \in \{-1,1\}$ and $k \in [d]$ or if $y = x$.

Let $x \in \mathbb{B}_{2l+1}^d$. It is straightforward to show that $x + \delta e^k \sim x$ if and only if $x_k = \delta l$ which implies $|N(x) \cap [x]_{2l+1}| = \sum_{i=1}^d \mathbb{1}\big(|x_i| = l\big)$, where $N(x)$ is the set of neighbours of x in $\mathbb{T}_{2(2l+1)}^d$, and therefore

$$P_{\mathrm{P},2(2l+1)}^{\#}\left([x]_{2l+1},[x]_{2l+1}\right) = P_{\mathrm{P},2(2l+1)}\left(x,[x]_{2l+1}\right) = \frac{1}{2d}\sum_{i=1}^{d} \mathbb{1}\left(|x_i| = l\right) = P_{\mathrm{H},2l+1}(x,x). \tag{144}$$

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Suppose instead that $y = x + \delta e^k$ with $x_k \neq \delta l$. Then $y \not\sim x$ and so if $y' \in [y]_{2l+1}$ then $y' \neq x$. If $y' \in [y]_{2l+1}$ then either $y'_k = y_k = x_k + \delta$ or $y'_k = -2l - 1 - y_k$. But since $x_k \in \{-(l-1), \ldots, (l-1)\}$, we have $x_k + \epsilon \in \{-l, \ldots, l\}$ for $\epsilon \in \{-1, 1\}$ and

$$-2l - 1 - y_k \in \{-2l-1, \ldots, -l-1\} \cup \{l+1, \ldots, 2l\} \, . \tag{145}$$

It follows that $-2l - 1 - y_k \neq x_k, x_k + \epsilon$. But in order for $y' \in N(x)$, it is necessary that $y'_k \in \{x_k, x_k \pm 1\}$. So we conclude that if $y' \in N(x)$ then $y'_k \neq -2l - 1 - y_k$, which then implies that $y'_k = y_k = x_k + \delta$. It also then follows that $y'_i = x_i = y_i$ for all $i \neq k$, so that $y' = y$. We have therefore established that $N(x) \cap [y]_{2l+1} = \{y\}$ when $y = x + \delta e^k$ with $x_k \neq \delta l$, and it follows that

$$P^{\#}_{\mathrm{P},2(2l+1)}\left([x]_{2l+1}, [y]_{2l+1}\right) = P_{\mathrm{P},2(2l+1)}\left(x, [y]_{2l+1}\right) = 1/(2d) = P_{\mathrm{H},2l+1}(x, y) \, . \tag{146}$$

$\square$

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