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Introduction to the Special Issue on Emergent Hydrodynamics in Integrable Many-Body Systems

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Contents

References **6**

Hydrodynamics is arguably one of the most successful theories of physics. Although it is extremely old, it is still nowadays as relevant as it was at its inception. With the 20th-century atomistic understanding of heath phenomena via statistical physics, hydrodynamics is now most fruitfully seen as a theory for emergent dynamical behaviours of interacting many-body systems. Much like thermodynamics, it is a set of laws for a reduced number of degrees of freedom that are relevant for describing observations at large scales of space-time, transcending the high-dimensional space of trajectories or wave functions into a lower-dimensional space of aggregated, coarse-grained quantities. Hydrodynamics has been applied to a wide variety of systems, classical and quantum. Perhaps in the view of explicitly implementing Boltzmann's idea of 'molecular chaos', which is at the basis of irreversibility, many applications of hydrodynamics have successfully concentrated on stochastic systems [1–3]. It is, however, generally accepted that stochasticity is not necessary. Instead, either classical ergodicity, or the build-up of quantum entanglement and the resulting local decoherence, are phenomena that are believed to generate some sort of molecular chaos.

While providing full proofs of these expectations remains undoubtedly an important challenge, the ideas expressed here—molecular chaos, stochasticity, ergodicity, local decoherence—nevertheless suggest, naively, that any many-body system which possesses a hydrodynamic description must be of 'generic enough' type: perhaps, if it is deterministic, it must be chaotic.

The development of the theory of generalised hydrodynamics (GHD), the subject of this Special Issue, is based on an important realisation: chaos is, in fact, *not necessary* for the emergence of a predictive hydrodynamic theory. A sufficient condition is *extensivity*—namely, the interaction should be short-range enough, and distributed over the whole volume of the system so that conserved charges are extensive. As a consequence, there is also a hydrodynamic theory for systems that are *integrable* and thus, according to usual classifications, non-chaotic.

In order to understand how this can be true, it is useful to start with a more fundamental concept: the 'ergodic principle'—somewhat related to, but different from, the eigenstate thermalisation hypothesis [4–7]. Suppose that a state $\langle \cdots \rangle$ is both space-time translation invariant, and clustering at large spatial separations; states with such properties are sometimes referred to as 'ergodic states' [8, 9]. The ergodic principle says that if a state is ergodic, then it must be of the Gibbs form for the evolution Hamiltonian H, i.e.

$$\langle A \rangle = \frac{\mathrm{Tr}\left(\mathrm{e}^{-\beta H} A\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right)}. \tag{1}$$

J. Stat. Mech. (2022) 014001

Here, we adopted a quantum mechanical language, where, for classical systems, the trace is replaced by an integral over an invariant measure, such as the standard phase-space measure. More formally: the state is a Kubo–Martin–Schwinger state for the evolution generated by H. If the particle number or momentum are conserved, then the Gibbs state is accordingly modified, with associated chemical potentials. These are referred to as thermodynamic equilibrium states: within the co-moving frame, where the momentum chemical potential vanishes, the state is time-reversal symmetric (if H is) and thus at 'equilibrium'. Ergodic states are supposed to be the relevant states occurring after relaxation. The basic physical principle of hydrodynamics is that locally, in mesoscopic 'fluid cells', the system can be assumed to be in, or near to, ergodic states, as a result of local relaxation: this is the local thermodynamic equilibrium assumption.

This simple principle naturally breaks down in integrable systems. Indeed, in infinite volume integrable systems admit an infinity of short-range conserved quantities $H_1, H_2, H_3, \ldots$ of the type of the Hamiltonian H. Therefore, in order to describe ergodic states, we must include chemical potentials, or 'inverse temperatures', for *all* such conserved quantities:

$$\langle A \rangle = \frac{\mathrm{Tr}\left(\mathrm{e}^{-\sum_i \beta^i H_i} A\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\sum_i \beta^i H_i}\right)}, \tag{2}$$

where β^i are an infinite set of independent parameters. These are the so-called generalised Gibbs ensembles (GGEs) [10–12], which have been extremely successful at determining the result of relaxation in integrable systems, or 'generalised thermalisation' [13–17]. In fact, relaxation in integrable models was the main focus of the 2016 JSTAT Special Issue [18], which can be seen as the predecessor of the current one.

One may wonder if this generalised ergodic principle is of use at all. It is easy to accept that states of the form (2) be admissible, but, besides describing the relaxation from very special initial conditions in certain quench protocols, one might naively expect such states to be extremely rare and of little use in real physical systems. Compelling experimental evidence suggests that the situation is not that simple and that thermal states are not enough to describe the relaxation of certain real physical systems, such as cold atom gases confined in one dimension. Most famously, the quantum Newton's cradle experiment [19] exhibited a one-dimensional cold-atom cloud displaying very slow thermalisation and strong memory of the initial condition. A quantitative description of this phenomenon, however, remained missing for almost a decade. One of the most important achievements of GHD has been to provide such a quantitative description, showing that (2) are exactly the states needed to describe the emergent hydrodynamics of integrable systems. Effectively, GHD simply arises as the application of basic hydrodynamic principles—local thermodynamic relaxation and conservation laws—to systems whose thermalisation properties are generalised as such. Real isolated physical systems, with generic and realistic inhomogeneous initial conditions, locally relax, under reversible time evolution, to GGEs and not to thermal states on experimentally accessible timescales.

We note that the (generalised) ergodic principle is conceptually a powerful tool. One may in fact conjecture that in order to develop the hydrodynamic theory of a given

J. Stat. Mech. (2022) 014001

extensive system, the required information is the manifold of ergodic states. There is no need to impose any strong notion of ergodicity or chaos: the same set of fundamental ideas applies equally well to chaotic and integrable systems; although, naturally, very different phenomenologies emerge. This puts in stark relief the difficulty in deriving conventional hydrodynamic equations for generic, realistic models: any rigorous derivation will have to grapple with the difficult issue of showing that the system is not integrable, and that the manifold of ergodic states is finite-dimensional. But it also suggests a new, potentially fruitful path, where in a first step universal equations are proven based on the abstract manifold of ergodic states, and the particularities of the manifold are analysed in a second step.

With this universal understanding of hydrodynamics, the development of GHD necessitated two technical ingredients. The first is an efficient characterisation of the space of ergodic states for integrable systems (the GGEs). Crucially, this had recently become available: results from studies of quantum quenches [20, 21] showed that one can describe stationary states by Bethe ansatz [22–24]. In fact, it was realised that the notion of an 'asymptotic state' of many-body systems plays a pivotal role. In integrable systems, every scattering event is purely elastic (preserving all momenta), and further can be factorised into a product of two-body scattering events (leading to the Yang–Baxter equation). These two facts have two important consequences. Consider a many-body state where energy is concentrated on a large but finite volume, and its associated asymptotic momenta from scattering theory. Then, not only is the (coarse-grained) density of asymptotic momenta conserved,

$$Q_{\theta,\Delta\theta} = \frac{1}{(\Delta\theta)} N_{\theta,\Delta\theta}, \tag{3}$$

where $N_{\theta,\Delta\theta}$ is the number of asymptotic particles with momenta within $[\theta - \Delta\theta/2, \theta + \Delta\theta/2]$, but also it is an *extensive* quantity. That is, $Q_{\theta,\Delta\theta}$ is extensive for all $L \gg 1/\Delta\theta$ (it is like a Hamiltonian with an interaction range $1/\Delta\theta$). GGEs in infinite volumes may then be described using the 'continuous basis' $Q_\theta = \lim_{\Delta\theta\to 0} Q_{\theta,\Delta\theta}$ [25, 26]

$$\langle A \rangle = \frac{\mathrm{Tr}\left(\mathrm{e}^{-\int \mathrm{d}\theta \beta^\theta Q_\theta} A\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\int \mathrm{d}\theta \beta^\theta Q_\theta}\right)}. \tag{4}$$

Thus, the manifold of ergodic states is a manifold of functions $\theta \mapsto \beta^\theta$. Such states are completely characterised by the average phase-space density of asymptotic states (density per unit momentum and length)

$$\rho_{\mathrm{p}}(\theta) = \lim_{L \gg 1/\Delta\theta \to \infty} \frac{1}{L} \langle Q_{\theta,\Delta\theta} \rangle_L. \tag{5}$$

In specific applications, the density of asymptotic states takes special meanings: in the Bethe ansatz formulation, this is the density of Bethe roots [22, 23], while in the classical Toda chain, this is related to the spectral density of the Lax matrix [27–29].

In order to obtain the hydrodynamic equations, one needs to further consider the continuity equations associated with the conserved quantities Q_θ. Namely, writing

$$Q_{\theta,\Delta\theta} = \int_0^L \mathrm{d}x\, q_{\theta,\Delta\theta}(x), \tag{6}$$

we find

$$\partial_t q_{\theta,\Delta\theta} + \partial_x j_{\theta,\Delta\theta} = 0. \tag{7}$$

This equation illustrates what is the second ingredient needed in the development of GHD, i.e. the average currents $\langle j_{\theta,\Delta\theta}\rangle$, in infinite volumes. Although explicit expressions for currents $j_{\theta,\Delta\theta}$ are still unknown, the averages could be evaluated. In particular, their expression, proposed in [30, 31], reads as

$$\lim_{L\gg 1/\Delta\theta\to\infty} \langle j_{\theta,\Delta\theta}\rangle = v^{\mathrm{eff}}(\theta)\rho_{\mathrm{p}}(\theta). \tag{8}$$

The effective velocity $v^{\mathrm{eff}}(\theta)$ encodes the modification of the group velocity $v(\theta)$ of the asymptotic particle θ due to the interaction. It represents the velocity effectively emerging on large scales as a test 'quasi-particle', where intuitively a tracer of a given asymptotic momentum θ, travels through the gas and interacts with it. It is a function of the state and, in fact, a nonlinear functional of $\rho_{\mathrm{p}}(\theta)$. Specifically, it solves the following linear integral equation

$$v^{\mathrm{eff}}(\theta) = v(\theta) + \int \mathrm{d}\theta'\, \rho_{\mathrm{p}}(\theta')\tilde{\varphi}(\theta,\theta')(v^{\mathrm{eff}}(\theta') - v^{\mathrm{eff}}(\theta)). \tag{9}$$

This equation encodes the interaction specific to the model, within the differential scattering phase $\mathrm{i}\tilde{\varphi}(\theta,\theta') = \mathrm{d}S(\theta,\theta')/\mathrm{d}p(\theta)$, where $S(\theta,\theta')$ is the two-body scattering amplitude, and $p(\theta)$ is the momentum of the asymptotic particle θ. Note that this type of effective velocity first appeared in studies of classical systems such as soliton gases [32–34] and one-dimensional hard rods [1, 35], and has a simple interpretation in terms of accumulated (Wigner) time delays due to scattering phase shifts.

The most basic GHD equation is then obtained by considering a space-time dependent ρ_{p}, which represents an ergodic state in every cell of space-time. Plugging (5) and (8) into (7), we then have

$$\partial_t \rho_{\mathrm{p}} + \partial_x (v^{\mathrm{eff}}\rho_{\mathrm{p}}) = 0. \tag{10}$$

This equation, together with some appropriate initial conditions, gives a complete characterisation of the space-time dependent ergodic state.

Making the connection between integrability and hydrodynamics has turned out to be extremely fruitful. In essence, this is because it combined the exact descriptions of the former with the simplicity and versatility of the latter. This led to a vast number of surprising new developments in integrability, fluid theory, and statistical mechanics. In this Special Issue, we aim at giving a comprehensive and self-contained account of these developments: the nine contributions presented here discuss all of the main different research directions that stemmed out of GHD. Specifically

J. Stat. Mech. (2022) 014001

- Alba *et al* [36] introduce the GHD formalism for quantum many-body systems and show how it can be applied to describe the asymptotic dynamics of entanglement and correlations after inhomogeneous quenches in quantum integrable models.
- De Nardis *et al* [37] discuss the applications of GHD to the calculation of dynamical correlations and the related transport coefficients in both quantum and classical integrable systems.
- Bulchandani *et al* [38] review several recent advances, triggered by the introduction of GHD, in the study of anomalous transport in spin chains.
- Bouchoule and Dubail [39] discuss theoretical and experimental developments occurring in the research on 1D Bose gas since the inception of GHD.
- Bastianello *et al* [40] give an overview of the applications of GHD to the study of *nearly* integrable models, i.e. systems in which integrability is weakly broken.
- Cubero *et al* [41] and Borsi *et al* [42] are instead concerned with the microscopic foundation of GHD, especially with the proof of the 'current formula' (8). In particular, Cubero, Yoshimura, and Spohn present two complementary approaches, one based on a form factor expansion and the other on a collision rate ansatz, while Borsi, Pristyák, and Pozsgay discuss three different proofs of (8) via Bethe ansatz techniques.
- Finally, El [43] and Buča *et al* [44] discuss two examples of classical systems where the emergence of GHD can be proven with a higher level of mathematical rigor. Specifically, El presents a comprehensive discussion of soliton gases in integrable dispersive hydrodynamic systems, while Buča, Klobas, and Prosen review recent results on the out-of-equilibrium dynamics of the classical reversible cellular automaton 'rule 54'.

This Special Issue provides a comprehensive portrait of the current research directions related to GHD, but we certainly do not expect it to remain up to date for long. Indeed, several interesting new directions are currently being developed. Arguably, the key missing link in our current comprehension of integrable systems out-of-equilibrium is how and when the hydrodynamic description emerges from the full (quantum or classical) many-body dynamics. Moreover, we also expect further developments in the understanding of the general structure of hydrodynamics (in particular higher-order gradient corrections needed to capture low-frequency transport) and its universality. Finally, the role of the inevitable integrability-breaking perturbations present in any experiment is still poorly understood.

References

[1] Spohn H 2012 *Large Scale Dynamics of Interacting Particles* (Berlin: Springer)
[2] Spitzer F 1970 *Adv. Math.* **5** 246
[3] Liggett T M 2012 *Interacting Particle Systems* vol 276 (Berlin: Springer)
[4] Deutsch J M 1991 *Phys. Rev.* A **43** 2046
[5] Srednicki M 1994 *Phys. Rev.* E **50** 888
[6] Rigol M, Dunjko V and Olshanii M 2008 *Nature* **452** 854

[7] D'Alessio L, Kafri Y, Polkovnikov A and Rigol M 2016 *Adv. Phys.* **65** 239
[8] Robinson D W and Ruelle D 1967 *Ann. Inst. Henri Poincaré Phys. Théor.* **6** 299
[9] Robinson D W 1968 *Commun. Math. Phys.* **7** 337
[10] Jaynes E T 1957 *Phys. Rev.* **106** 620
[11] Jaynes E T 1957 *Phys. Rev.* **108** 171
[12] Rigol M, Dunjko V, Yurovsky V and Olshanii M 2007 *Phys. Rev. Lett.* **98** 050405
[13] Gogolin C and Eisert J 2016 *Rep. Prog. Phys.* **79** 056001
[14] Eisert J, Friesdorf M and Gogolin C 2015 *Nat. Phys.* **11** 124
[15] Polkovnikov A, Sengupta K, Silva A and Vengalattore M 2011 *Rev. Mod. Phys.* **83** 863
[16] Essler F H L and Fagotti M 2016 *J. Stat. Mech.* 064002
[17] Doyon B 2017 *Commun. Math. Phys.* **351** 155
[18] Calabrese P, Essler F H L and Mussardo G 2016 *J. Stat. Mech.* 064001
[19] Kinoshita T, Wenger T and Weiss D S 2006 *Nature* **440** 900
[20] Caux J-S and Essler F H L 2013 *Phys. Rev. Lett.* **110** 257203
[21] Cassidy A C, Clark C W and Rigol M 2011 *Phys. Rev. Lett.* **106** 140405
[22] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (*Cambridge Monographs on Mathematical Physics*) (Cambridge: Cambridge University Press)
[23] Takahashi M 1999 *Thermodynamics of One-Dimensional Solvable Models* (Cambridge: Cambridge University Press)
[24] Zamolodchikov A B 1990 *Nucl. Phys.* B **342** 695
[25] Ilievski E, Quinn E and Caux J-S 2017 *Phys. Rev.* B **95** 115128
[26] Pozsgay B, Vernier E and Werner M A 2017 *J. Stat. Mech.* 093103
[27] Flaschka H 1974 *Phys. Rev.* B **9** 1924
[28] Doyon B 2019 *J. Math. Phys.* **60** 073302
[29] Spohn H 2021 Hydrodynamic equations for the Toda lattice (arXiv:2101.06528)
[30] Castro-Alvaredo O A, Doyon B and Yoshimura T 2016 *Phys. Rev.* X **6** 041065
[31] Bertini B, Collura M, De Nardis J and Fagotti M 2016 *Phys. Rev. Lett.* **117** 207201
[32] Zakharov V 1971 *Sov. Phys - JETP* **33** 538
[33] El G A 2003 *Phys. Lett.* A **311** 374
[34] El G A and Kamchatnov A M 2005 *Phys. Rev. Lett.* **95** 204101
[35] Boldrighini C, Dobrushin R L and Sukhov Y M 1983 *J. Stat. Phys.* **31** 577
[36] Alba V, Bertini B, Fagotti M, Piroli L and Ruggiero P 2021 *J. Stat. Mech.* 114004
[37] De Nardis J, Doyon B, Medenjak M and Panfil M 2022 *J. Stat. Mech.* 014002
[38] Bulchandani V B, Gopalakrishnan S and Ilievski E 2021 *J. Stat. Mech.* 084001
[39] Bouchoule I and Dubail J 2022 *J. Stat. Mech.* 014003
[40] Bastianello A, De Luca A and Vasseur R 2021 *J. Stat. Mech.* 114003
[41] Cubero A C, Yoshimura T and Spohn H 2021 *J. Stat. Mech.* 114002
[42] Borsi M, Pozsgay B and Pristyák L 2021 *J. Stat. Mech.* 094001
[43] El G A 2021 *J. Stat. Mech.* 114001
[44] Buča B, Klobas K and Prosen T 2021 *J. Stat. Mech.* 074001

J. Stat. Mech. (2022) 014001