Research in Astronomy and Astrophysics

RESEARCH PAPERS

Recursive harmonic analysis for computing Hansen coefficients

Research in Astron. Astrophys. **2010** Vol. **10** No. **12**, 1298–1306
http://www.raa-journal.org *http://www.iop.org/journals/raa*

Research in Astronomy and Astrophysics

Recursive harmonic analysis for computing Hansen coefficients

Mohamed Adel Sharaf[1] and Hadia Hassan Selim[2]

[1] Astronomy Department, Faculty of Science, King Abdul Aziz University, Jeddah, Saudi Arabia; *Sharaf_adel@hotmail.com*

[2] Astronomy Department, National Research Institute of Astronomy and Geophysics, Helwan, Egypt; *hassanselim@hotmail.com*

Received 2010 March 17; accepted 2010 June 18

Abstract We report on a simple pure numerical method developed for computing Hansen coefficients by using a recursive harmonic analysis technique. The precision criteria of the computations are very satisfactory and provide materials for computing Hansen's and Hansen's like expansions, and also to check the accuracy of some existing algorithms.

Key words: celestial mechanics — methods: numerical

1 INTRODUCTION

Hansen coefficients (Cefola 1977) are an important class of functions which are frequently used in many branches of Celestial Mechanics, such as planetary theory (Newcomb 1895) and artificial satellite motion (Allan1967; Hughes1977). Moreover, there are extensive forms of Hansen like expansions (Klioner et al. 1998; Sharaf 1985, 1986) which play important roles in the expansion theories of elliptic motion.

Giacaglia (1976) noted that Hansen's coefficients appear in satellite theory in the expression of the perturbing function due to the primary and due to the presence of a third body and they are usually called Eccentricity Functions. He derived the recurrence relation for these functions and their derivatives, as they appear in the evaluation of geopotential and third body perturbations of an artificial satellite. Also Giacaglia (1987) proved Hansen's coefficients, for Fourier series in terms of the mean anomaly, correspond to a rotation of the orbital plane proportional to the eccentricity of the orbit. They are given in terms of Bessel functions and generalized associated Legendre functions which arise through the transformation of spherical harmonics under rotation. Hughes (1981) computed tables of analytical expressions for the Hansen coefficients $\mathrm{X}_o^{n,\pm m}(e)$ and $\mathrm{X}_o^{-(n+1),\pm m}(e)$ when $1 \leq n \leq 30$ and $0 \leq m \leq n$. Branham (1990) derived a recursive calculation method for Hansen coefficients which is used in expansions of elliptic motion by three methods: Tisserand's method, the von Zeipel-Andoyer method with explicit representation of the polynomials required to compute the Hansen coefficients and the von Zeipel-Andoyer method with the value of the polynomials calculated recursively. Vakhidov (2000) studied in detail efficient approximations of Hansen coefficients using polynomials in terms of the eccentricity. He & Zhang (1990) used Hansen coefficients to compute general perturbations of the asteroids in the Flora group due to Jupiter. Breiter et al. (2004) showed that most of the theory of Hansen coefficients remains valid for $\mathrm{X}_k^{\gamma j}$, when γ is a real number; also, the generalized coefficients can be applied in a variety of perturbed problems that involve some drag effects. Sadov (2008) deals analytically with the properties of Hansen's coefficients in the theory of elliptic motion considered as functions of the parameter $\eta = \sqrt{1-e^2}$, where e is the eccentricity.

In the present paper, we develop a simple pure numerical method for computing Hansen coefficients by using a recursive harmonic analysis technique. The precision criteria of the computations are very satisfactory. The importance of the method is that it not only provides materials for computing Hansen's and also Hansen's like expansions, but also it can be used, due to its simplicity and accuracy, to check the accuracy of the different algorithms which already exist.

2 BASIC FORMULATIONS

2.1 Properties of Least Squares

Let y be represented by the general linear expression of the form $\sum_{i=1}^{L} c_i\phi(x)$ where the ϕ expressions are linear independent functions of x. Let $\boldsymbol{c}$ be the vector of the exact values of the c coefficients and $\widehat{\boldsymbol{c}}$ be the least squares estimators of $\boldsymbol{c}$ obtained from the solution of the normal equations $\boldsymbol{G}\widehat{\boldsymbol{c}} = \boldsymbol{b}$. The coefficient matrix $\boldsymbol{G}(L \times L)$ is symmetric positive definite, that is, all its eigenvalues $V_i; i = 1, 2, .., L$ are positive. Let $EC(z)$ denote the expectation of z and σ^2 the variance of the fit, defined as

$$\sigma^2 = q_n/(N-L), \tag{1}$$

where

$$q_n = (\boldsymbol{y} - \Phi^T\widehat{\boldsymbol{c}})^T(\boldsymbol{y} - \Phi^T\widehat{\boldsymbol{c}}). \tag{2}$$

N is the number of observations, $\boldsymbol{y}$ is a vector with elements y_k and $\Phi(L \times N)$ has elements $\varphi_{ik} = \varphi_i(x_k)$. The transpose of a vector or a matrix is indicated by the superscript 'T'.

According to the least squares criterion, it could be shown that (Sharaf et al. 2000)

- The estimators $\widehat{\boldsymbol{c}}$ given by the least squares method give the minimum of q_n.
- The estimators $\widehat{\boldsymbol{c}}$ of the coefficients $\boldsymbol{c}$, obtained by the method of least squares, are unbiased, i.e. $EC(\widehat{\boldsymbol{c}}) = \boldsymbol{c}$.
- The variance-covariance matrix Var $(\widehat{\boldsymbol{c}})$ of the unbiased estimators $\widehat{\boldsymbol{c}}$ is given by

$$\mathrm{Var}(\widehat{\boldsymbol{c}}) = \sigma^2\boldsymbol{G}^{-1}, \tag{3}$$

  where $\boldsymbol{G}^{-1}$ is the inverse of $\boldsymbol{G}$.
- The average squared distance between $\boldsymbol{c}$ and $\widehat{\boldsymbol{c}}$ is

$$EC(D^2) = \sigma^2 \sum_{1}^{L} \frac{1}{V_i}. \tag{4}$$

2.2 Harmonic Analysis of a Periodic Function

Let it be required to find a sum

$$a_0 + \sum_{j=1}^{s} a_j \cos jx + \sum_{j=1}^{s} b_j \sin jx, \tag{5}$$

which furnishes the best possible representation of a function $u(x)$, when we are given that $u(x)$ assumes the values $u_0, u_1, ..., u_{i-1}$ when x takes $x_0, x_1, ..., x_{i-1}$ respectively, and m is some number greater than $2s$. The problem is to determine the $(2s+1)$ constants, a_0, a_j and $b_j, j = 1, 2, ..., s$ so as to make the expression (5) assume, as nearly as possible, the l values $u_0, u_1, ..., u_{i-1}$ when

x takes the values $x_0, x_1, ..., x_{i-1}$. To do so, we shall make use of the method of least squares and we get

$$\frac{1}{2}a_0\eta_{0i} + \sum_{j=1}^{s} a_j\eta_{ij} + \sum_{j=1}^{s} b_j\beta_{ij} = d_i, \quad i = 0, 1, ..., s;$$
$$\frac{1}{2}a_0\beta_{0q} + \sum_{j=1}^{s} a_j\beta_{qj} + \sum_{j=1}^{s} b_j\gamma_{qj} = c_q, \quad q = 1, 2, ..., s; \tag{6}$$

where

$$\begin{aligned}
\eta_{ij} &= \eta_{ji} = \sum_{k=0}^{i-1} \cos ix_k, \quad i = 0, 1, ..., s, \quad j = 0, 1, ...s; \\
\beta_{qj} &= \sum_{k=0}^{i-1} \cos jx_k \sin qx_k, \quad j = 0, 1, ..., s, \quad q = 1, 2, ..., s; \\
\gamma_{qj} &= \gamma_{jq} = \sum_{k=0}^{i-1} \sin qx_k \sin jx_k, \quad q = 1, 2, ..., s, \quad j = 1, 2, ..., s; \\
d_i &= \sum_{k=0}^{i-1} u_k \cos ix_k, \quad i = 0, 1, ..., s; \\
c_q &= \sum_{k=0}^{i-1} u_k \sin qx_k, \quad q = 1, 2, ..., s.
\end{aligned} \tag{7}$$

The system of Equations (7) is the normal equation. These equations represent a set of linear equations in $(2s + 1)$ unknown a and b coefficients and could be solved by any of the methods adopted for linear systems. However, the coefficient matrix of this set could be reduced to a diagonal one by a certain choice of the arguments x_k and, in this case, the a and b values are exactly determined and the problem is known as harmonic analysis.

In the method of harmonic analysis, the arguments x_k take the special values:

$$0,\ \frac{2\pi}{l},\ 2\frac{2\pi}{l},\ 3\frac{2\pi}{l},\ ...,\ (l-1)\frac{2\pi}{l}\,. \tag{8}$$

For these values, the set of η, β and γ values of Equations (7) becomes

$$\begin{aligned}
&\text{for} \quad i = j \neq 0 :\ \eta_{ij} = \gamma_{ji} = \frac{1}{2}l; \quad \beta_{ij} = 0; \\
&\text{for} \quad i \neq j :\ \eta_{ij} = \gamma_{ij} = \beta_{ij} = 0\,.
\end{aligned}$$

Consequently, the a and b coefficients could then be exactly computed from

$$a_j = \frac{\mu}{l}\sum_{k=0}^{i-1} u_k \cos j\,\frac{2\pi}{l}k; \quad j = 0, 1, .., s,$$
$$b_q = \frac{2}{l}\sum_{k=0}^{i-1} u_k \sin q\,\frac{2\pi}{l}k; \quad q = 1, 2, ..., s, \tag{9}$$

where $\mu = 1$ if $j = 0$; $\mu = 2$ if $j > 0$.

2.3 Hansen Coefficients

Consider elliptic motion expansions of $(r/a)^n \cos mv$ and $(r/a)^n \sin mv$ in terms of the mean anomaly M that is

$$\left(\frac{r}{a}\right)^n \cos mv = \sum_{k=0} A_k^{n,m} \cos kM,$$
$$\left(\frac{r}{a}\right)^n \sin mv = \sum_{k=1} B_k^{n,m} \sin kM, \tag{10}$$

where a is the semimajor axis, r the radial distance, n is a positive or negative integer, while m is a positive integer and v the true anomaly in elliptic motion. The set of A and B coefficients is called Hansen's coefficients, which are functions of the eccentricity e.

The relations between the eccentric anomaly E and the anomalies M and v are given for elliptic motion as follows:

- The relation between E and M is the well-known Kepler's equation
$$M = E - e \sin E. \tag{11}$$
- The fundamental relations between v and E in an elliptic orbit are
$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}. \tag{12}$$
These equations are the most useful relations between $v(E)$ and $E(v)$, since $\frac{v}{2}$ and $\frac{E}{2}$ are always in the same quadrant. There is a possibility of numerical trouble when Equation (12) is used with angles that are near $\pm\frac{\pi}{2}$, as the two tangents become infinite. In order to avoid this difficulty, Broucke & Cefola (1973) established the formula
$$\tan \frac{1}{2}(v - E) = \frac{\beta \sin E}{1 - \beta \cos E}, \tag{13}$$
where
$$\beta = \frac{1 - \sqrt{1-e^2}}{e} = \frac{e}{1 + \sqrt{1-e^2}}. \tag{14}$$
Equation (13) is free of numerical trouble, no matter what the values of the angles are. Moreover, it can be easily used because the angle $(v - E)/2$ is always less than $\frac{\pi}{2}$ for all elliptic orbits.
- Finally the relation between r and E is
$$\left(\frac{r}{a}\right) = 1 - e \cos E. \tag{15}$$

3 COMPUTATIONAL DEVELOPMENTS

3.1 Practical Computation of the a and b Coefficients

The a and b coefficients of Equation (9) could be computed efficiently (Ralston & Rabinowitz 1978) from

$$a_j = \frac{\mu}{l}\left(u_o + F_{1,j} \cos \frac{2\pi}{l} j - F_{2,j}\right); \quad j = 0, 1, ..., s, \tag{16}$$

$$b_q = \frac{2}{l} F_{1,q} \sin \frac{2\pi}{l} q; \quad q = 1, 2, ..., s, \tag{17}$$

where, for any j, the F values are computed recursively from

$$F_{k,j} = u_k + 2 \cos x_j F_{k+1,j} - F_{k+2,j} \tag{18}$$

by using the initial conditions $F_{i,j} = F_{i+1,j} = 0$, starting with $k = l - 1$ to successively compute $F_{i-1,j}, F_{i-2,j}, ..., F_{1,j}$.

3.2 Error Estimates

- The variance of the fit (Eq. (1)) is given by

$$\sigma^2 = \frac{\delta^2}{l - L}, \tag{19}$$

where the sum of squares of the residuals δ^2 is given as (Ralston & Rabinowitz 1978)

$$\delta^2 = \sum_{i=0}^{i-1} u_i^2 - \frac{1}{2}\left[2a_0^2 + \sum_{j=1}^{s}(a_j^2 + b_j^2)\right]. \tag{20}$$

Clearly both σ and δ depend on the number s of the a and b coefficients. If the precision is measured by probable error (PE), then

$$\mathrm{PE} = 0.6745\sigma\,. \tag{21}$$

- Since the coefficient matrix $\boldsymbol{G}$ of the harmonic analysis is diagonal with elements of the same value $l/2$, then according to Equation (3), the standard error of each of the a and b coefficients is

$$\sigma_{\mathrm{coeff}} = \sigma\sqrt{\frac{2}{l}}\,. \tag{22}$$

The corresponding probable error for each coefficient is

$$\mathrm{PE}_{\mathrm{coeff}} = 0.6745\sigma_{\mathrm{coeff}}. \tag{23}$$

- The average squared distance between the exact and the least squares values (Eq. (9)) is given according to Equation (4) as

$$Q = \mathrm{CE}(D^2) = \frac{2s}{l}\sigma^2. \tag{24}$$

3.3 Choosing the Number of Coefficients

In practice, since we do not know s, we would evaluate the a and b coefficients for $s = 1, 2$, then compute δ^2 (Eq. (19)), and continue as long as δ^2 decreases significantly (within a given tolerance (Tol)) with increasing s.

3.4 The Special Values

The special values of the left hand side of Equation (10) are computed as follows:

1. $M_i = \frac{2\pi i}{l}; i = 0, 1, ..., l - 1$.
2. For each M solve Kepler's equation (Eq. (11)) by the Newton-Raphson iterative method (or any other method). Let E_0 be an initial approximation of E; define for $k = 0, 1, 2, ...$

$$E_{k+1} = E_k - \frac{E_k - e\sin E_k - M}{1 - e\sin E_k}\,.$$

Each E_{k+1} should approximate E more closely than E_k. For the initial approximation E_0, use the value (Battin1999)

$$E_0 = M + \frac{e\sin M}{1 - \sin(M + e) + \sin M}\,.$$

The above procedure is terminated if the following conditions are satisfied $\varepsilon_2 \le \varepsilon_1$ and $\mid H(E_{i+1)} \mid \le 100\varepsilon_1, \varepsilon_2 = \mid \frac{E_{i+1} - E_i}{E_{i+1}} \mid$ if $\mid E_{i+1} \mid > 1; \varepsilon_2 = \mid E_{i+1} - E_i \mid$ if $\mid E_{i+1} \mid < 1$, where ε_1 is a given tolerance and $H(E) = M - E - e\sin E$.

3. For each E compute v using Equation (14) and $(\frac{r}{a})^n$ from Equation (10).
4. For each v compute $\cos(mv)$.
5. Finally, find the product of the values of $(\frac{r}{a})^n$ (of step 3) and $\cos(mv)$ (of step 4).

3.5 Numerical Results

The above computational developments are applied for calculating Hansen's coefficients of Equation (10) with input constants $l = 100, \mathrm{Tol} = 10^{-6}$ and $\varepsilon_1 = 10^{-8}$. The numerical results are listed in Tables 1 to 6 for different values of n, m and different eccentricities of some members of the solar system. In these tables, $\delta_A^2(\delta_B^2)$ represents the sum of the squares of the residuals of Equation (19) for A (B) coefficients, $\sigma_{\mathrm{coeff}.A}(\sigma_{\mathrm{coeff}.B})$ represents the common standard error of Equation (21) for A (B) coefficients, and finally $Q_A(Q_B)$ represents the average squared distance between the exact and least squares values of Equation (23) for A (B) coefficients.

Table 1 Hansen Coefficients for the Planet Earth: $e = 0.016708617, n = -3, m = 6$

k	A_k	B_k
0	-2.80505×10^{-16}	
1	-1.69927×10^{-10}	-1.69927×10^{-10}
2	1.31491×10^{-7}	1.31491×10^{-7}
3	-0.0000259261	-0.0000259261
4	0.00209013	0.00209013
5	-0.0749101	-0.0749101
6	0.99039	0.99039
7	0.124591	0.124591
8	0.00917108	0.00917108
9	0.000516607	0.000516607
10	0.0000246565	0.0000246565
11	1.05004×10^{-6}	1.05004×10^{-6}
$d_A^2 = 8.52652 \times 10^{-14}$		$d_B^2 = 6.39488 \times 10^{-14}$
$s_{\infty ee.A} = 4.37729 \times 10^{-9}$		$s_{\infty ee.B} = 3.79085 \times 10^{-9}$
$Q_A = 2.10768 \times 10^{-16}$		$Q_B = 1.8076 \times 10^{-16}$

Table 2 Hansen Coefficients for the Minor Planet Pluto: $e = 0.249050, n = -3, m = 6$

k	A_k	B_k
0	0.0508079	
1	-0.325005	-0.319177
2	0.969155	0.969203
3	-1.34716	-1.34716
4	0.536896	0.536896
5	0.248699	0.248699
6	0.0765758	0.0765758
7	0.0211903	0.0211903
8	0.0056458	0.0056458
9	0.00148234	0.00148234
10	0.000386931	0.000386931
11	0.000100722	0.000100722
12	0.0000261571	0.0000261571
13	6.76933×10^{-6}	6.76933×10^{-6}
$d_A^2 = 1.62174 \times 10^{-10}$		$d_B^2 = 1.6226 \times 10^{-10}$
$s_{\infty ee.A} = 1.93084 \times 10^{-7}$		$s_{\infty ee.B} = 1.93135 \times 10^{-7}$
$Q_A = 4.84659 \times 10^{-13}$		$Q_B = 4.84914 \times 10^{-13}$

Table 3 Hansen Coefficients for the Asteroid Ceres: $e = 0.078, n = 8, m = 2$

k	A_k	B_k
0	0.0854431	
1	−0.492936	−0.479094
2	1.08609	1.08564
3	−0.157994	−0.157993
4	0.00140598	0.00140603
5	0.000192711	0.000192714
6	0.0000113508	0.0000113509
7	6.01265×10^{-7}	6.01269×10^{-7}
$d_A^2 = 4.26326 \times 10^{-14}$ $s_{\infty ee.A} = 3.02792 \times 10^{-9}$ $Q_A = 6.41781 \times 10^{-17}$		$d_B^2 = 4.26326 \times 10^{-14}$ $s_{\infty ee.B} = 3.02792 \times 10^{-9}$ $Q_B = 6.41781 \times 10^{-17}$

Table 4 Hansen Coefficients for the Asteroid Sekhmet: $e = 0.296, n = -1, m = 5$

k	A_k	B_k
0	−0.0000795273	
1	0.00983893	0.00983416
2	−0.114213	−0.114214
3	0.431088	0.431088
4	−0.482649	−0.482649
5	−0.260258	−0.260258
6	0.191795	0.191795
7	0.410314	0.410314
8	0.411965	0.411965
9	0.318733	0.318733
10	0.213837	0.213837
11	0.130854	0.130854
12	0.0750457	0.0750457
13	0.0410069	0.0410069
14	0.021582	0.021582
15	0.0110231	0.0110231
16	0.00549385	0.00549385
17	0.00268279	0.00268279
18	0.00128767	0.00128767
19	0.000608982	0.000608982
20	0.000284349	0.000284349
21	0.000131294	0.000131294
22	0.0000600289	0.0000600289
23	0.0000272069	0.0000272069
24	0.0000122351	0.0000122351
25	5.46368×10^{-6}	5.46368×10^{-6}
$d_A^2 = 3.64729 \times 10^{-10}$ $s_{\infty ee.A} = 3.11867 \times 10^{-7}$ $Q_A = 2.43152 \times 10^{-12}$		$d_B^2 = 3.64665 \times 10^{-10}$ $s_{\infty ee.B} = 3.1184 \times 10^{-7}$ $Q_B = 2.4311 \times 10^{-12}$

Table 5 Hansen Coefficients for the Comet Wild2: $e = 0.541, n = 3, m = 2$

k	A_k	B_k
0	−0.187235	
1	0.954443	0.943797
2	−1.75935	−1.75982
3	1.04451	1.04448
4	0.271626	0.271625
5	−0.101094	−0.101092

Table 5 —*Continued*

k	A_k	B_k
6	-0.158792	-0.158791
7	-0.112669	-0.112669
8	-0.05443	-0.0544298
9	-0.0112087	-0.0112086
10	0.0142725	0.0142726
11	0.026263	0.026263
12	0.0296931	0.0296931
13	0.0283862	0.0283862
14	0.0248699	0.0248699
15	0.0206484	0.0206484
16	0.0165297	0.0165297
17	0.0128893	0.0128893
18	0.00985376	0.00985376
19	0.00741822	0.00741822
20	0.005551671	0.005551671
21	0.00406198	0.00406198
22	0.00296636	0.00296636
23	0.00215139	0.00215139
24	0.00155123	0.00155123
25	0.00111291	0.00111291
26	0.000795003	0.000795003
27	0.000565771	0.000565771
28	0.000401309	0.000401309
29	0.000283824	0.000283824
30	0.000200214	0.000200214
31	0.000140908	0.000140908
32	0.0000989644	0.0000989644
33	0.0000693761	0.0000693761
34	0.000048552	0.000048552
35	0.0000339265	0.0000339265
36	0.0000236736	0.0000236736
37	0.0000164982	0.0000164982
38	0.0000114842	0.0000114842
39	7.9854×10^{-6}	7.9854×10^{-6}
$d_A^2 = 1.18559 \times 10^{-8}$ $s_{\infty ee.A} = 4.05228 \times 10^{-7}$ $Q_A = 6.40418 \times 10^{-12}$		$d_B^2 = 1.18562 \times 10^{-8}$ $s_{\infty ee.B} = 4.05232 \times 10^{-7}$ $Q_B = 6.4043 \times 10^{-12}$

Table 6 Hansen Coefficients for the Comet Lexell: $e = 0.786, n = 8, m = 4$

k	A_k	B_k
0	28.4068	
1	-47.0631	-25.693
2	23.9162	21.1464
3	-4.70405	-4.84203
4	-0.605464	-0.619241
5	-0.0262285	-0.0283248
6	0.0293643	0.0289445
7	0.0217703	0.0216686
8	0.0121887	0.0121605
9	0.00637396	0.00636531
10	0.00325957	0.00325672
11	0.00164622	0.00164524
12	0.000817397	0.000817052
13	0.000392683	0.000392564
14	0.000176143	0.000176105
15	0.0000672271	0.0000672181
16	0.0000140575	0.0000140581

Table 6 —*Continued*

k	A_k	B_k
17	-0.0000103329	-0.0000103297
18	-0.0000200483	-0.000020045
19	-0.0000224845	-0.0000224818
20	-0.0000215081	-0.0000215061
21	-0.0000191168	-0.0000191154
22	-0.0000163167	-0.0000163158
23	-0.0000135893	-0.0000135887
24	-0.0000111418	-0.0000111414
25	-9.0409×10^{-6}	-9.04071×10^{-6}
$d_A^2 = 7.42148 \times 10^{-9}$		$d_B^2 = 7.33417 \times 10^{-9}$
$s_{\infty ee.A} = 1.40679 \times 10^{-6}$		$s_{\infty ee.B} = 1.39849 \times 10^{-6}$
$Q_A = 4.94765 \times 10^{-11}$		$Q_B = 4.88944 \times 10^{-11}$

4 CONCLUSIONS

In concluding the present paper, a pure numerical method is developed for computing Hansen coefficients by using a recursive harmonic analysis technique. The precision criteria, which are: the variance of the fit, the standard errors of the coefficients and the average squared distance between the exact and least squares values, are all very satisfactory. The method not only provides materials for computing Hansen's and also Hansen's like expansions, but also can be used to check the accuracy of the different algorithms that already exist.

References

Allan, R. R. 1967, Planet. Space Sci., 15, 53

Battin, R. H. 1999, Revised Edition, AIAA, Education Series, Reston, Virginia

Branham, R. L., Jr. 1990, Celestial Mechanics and Dynamical Astronomy, 49, 209

Breiter, S., Métris, G., & Vokrouhlický, D. 2004, Celestial Mechanics and Dynamical Astronomy, 88, 153

Broucke, R., & Cefola, P. 1973, Celestial Mechanics, 7, 388

Cefola, P. J. 1977, Charles Stark Draper Laboratory Report

Giacaglia, G. E. O. 1976, Celestial Mechanics, 14, 515

Giacaglia, G. E. O. 1987, Publ. Astron. Soc. Japan., 39, 171

Hughes, S. 1977, Planetary Space Sci., 25, 809

Hughes, S. 1981, Celestial Mechanics, 25, 101

Klioner, S. A., Vakhidov, A. A., & Vasiliev, N. N. 1997, Celestial Mechanics and Dynamical Astronomy, 68, 257

He, M.-F., & Zhang, J. 1990, Chinese Astronomy and Astrophysics, 14, 306

Newcomb, S. 1895, (United States. Nautical Almanac Office. Astronomical paper; v.5, pt.1 (1895)), (Washington: U. S. Nautical Almanac Office, 1895), 1

Ralston, R., & Rabinowitz, P.,1978, A First Course in Numerical Analysis, McGraw-Hill Kogakusha, Ltd. Tokyo, Japan

Sadov, S. Y. 2008, Celestial Mechanics and Dynamical Astronomy, 100, 287

Sharaf, M. A. 1985, Ap&SS116, 251

Sharaf, M. A. 1986, Ap&SS, 125, 259

Sharaf, M. A., Bassuny, A. A., & Korany, B. A. 2000, Astrophy. Letter & Communications, 40, 39

Vakhidov, A. A. 2000, Computer Physics Communications, 124, 40