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Design of an adaptive finite-time controller for synchronization of two identical/different non-autonomous chaotic flywheel governor systems

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The centrifugal flywheel governor (CFG) is a mechanical device that automatically controls the speed of an engine and avoids the damage caused by sudden change of load torque. It has been shown that this system exhibits very rich and complex dynamics such as chaos. This paper investigates the problem of robust finite-time synchronization of non-autonomous chaotic CFGs. The effects of unknown parameters, model uncertainties and external disturbances are fully taken into account. First, it is assumed that the parameters of both master and slave CFGs have the same value and a suitable adaptive finite-time controller is designed. Second, two CFGs are synchronized with the parameters of different values via a robust adaptive finite-time control approach. Finally, some numerical simulations are used to demonstrate the effectiveness and robustness of the proposed finite-time controllers.

Keywords: finite-time controller, chaos synchronization, non-autonomous centrifugal flywheel governor, chaotic system

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1. Introduction

Chaotic systems are very complex nonlinear dynamical systems that possess unpredictable and irregular behaviours. One of the main features of a chaotic system is that a tiny change in the initial conditions leads to a large difference in the long-term behaviour of the system. The other major attribute of chaotic systems is that their trajectories are always locally unbounded and globally bounded which result in strange attractors.[1] In recent years, synchronization of autonomous chaotic systems has received considerable attention among many researchers.[2–16] On the other hand, with the discovery of more and more non-autonomous chaotic systems in engineering and physics, synchronization of non-autonomous chaotic systems has become a significant research topic in nonlinear science and several control techniques for the synchronization of the non-autonomous chaotic systems have been developed, which include adaptive control,[17] linear state feedback control,[18] impulsive control,[19] sliding mode control,[20] sinusoidal state error feedback control,[21] the fuzzy observer-based method,[22] and variable substitution control.[23]

The centrifugal flywheel governor (CFG)[24] is one of the most interesting and attractive nonlinear dynamical systems. It is a mechanical device that automatically controls the speed of an engine and prevents damage caused by an abrupt change of load torque. The CFGs have found useful applications in many rotational machines such as the diesel engine, steam engine and gas turbine. For example, the ignition timing of an automotive engine has been controlled by a distributor composed of a spring and a centrifugal governor.[25] Recent research has recognized different kinds of CFG with a rich variety of nonlinear behaviour. Furthermore, it has been shown that these systems exhibit a diverse range of dynamic behaviour including regular, periodic, quasi-periodic and chaotic motions.[26–34]

Regular and chaotic dynamics of rotational machines with a centrifugal governor and subjected to external disturbances have been investigated in Ref. [26]. Zhang *et al.*[27] have studied the complex dynamical behaviour of a class of CFG systems and have proposed a parametric open-plus-closed-loop approach for controlling their chaos. Bifurcation and chaos of a CFG have been reported in Ref. [28]. Sotomayor *et al.*[29] have studied the Lyapunov stability and the Hopf bifurcation in a hexagonal centrifugal governor with a steam engine. Ge and Jhuang[30] have investigated chaos, its control and synchronization for a

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fractional order rotational mechanical system with a centrifugal governor. Nonlinear dynamics and chaos control of a rotational machine with a hexagonal centrifugal governor considering the effects of external disturbances have been studied in Ref. [31]. Anti-control and synchronization of chaos for a rotational machine system with a hexagonal centrifugal governor have been addressed in Ref. [32]. Ge and Lee[33] have proposed linear and nonlinear feedback synchronization schemes for rotational machine systems with time-delay.

However, the work mentioned above[26–33] has synchronized/stabilized the chaotic CFGs asymptotically. This means that in the previous studies the state trajectories of the slave system converge to the state trajectories of the master system with infinite settling time. In fact, in practical engineering processes, one may want to synchronize two chaotic CFGs as quickly as possible. Therefore, it is important to study the finite-time synchronization of chaotic CFGs. Moreover, the authors of Refs. [26]–[33] have studied the synchronization problem of chaotic CFGs without considering the effects of both unknown parameters and uncertainties. While, in practice, the parameters of CFGs are inevitably perturbed by external inartificial factors, such as environment temperature and mutual interference among components, and their exact values cannot be determined in advance. In addition, in real world applications, there are always some model uncertainties and external disturbances in the dynamics of the CFGs. On the other hand, since chaotic CFGs are very sensitive to any system parameter variations, the effects of unknown parameters and system uncertainties can lead to unpredictable behaviours and can even break the synchronization. As a result, finite-time synchronization of uncertain chaotic CFGs is essential both in application and research. However, to the best of our knowledge, the problem of robust synchronization of the uncertain chaotic CFG in a given finite time has not been investigated in the literature so far and has remained an open challenging problem. Therefore, the main contribution of this paper is to propose a robust adaptive controller for finite-time synchronization of non-autonomous chaotic CFGs with model uncertainties, external disturbances and fully unknown parameters. Using some adaptive laws and the finite-time control idea, a robust adaptive controller is derived to synchronize two uncertain non-autonomous CFGs in finite time, even when the parameters of the master and slave system have different values. Numerical simulations are given to illustrate the robustness and applicability of the proposed technique. The results of this paper are compared with the results of an existing work in the literature.

2. Centrifugal flywheel governor system

A mechanical CFG system[24] is depicted in Fig. 1. The motor drives the flywheel to rotate with angular velocity ω. The flywheel is joined to the axis through a gear case, so the axis rotates with angular velocity $n\omega$. Rods 1 and 2 with length l are joined to a hinge at the end of the axis. Both rods are also attached to a ball of mass m. The balls are also connected to a sleeve over the axis by rods 3 and 4. A linear spring of stiffness k is attached to the sleeve, covering the upper portion of the axis. The vapor's flux Q into the engine is adjusted by a mechanical governor on the sleeve, which is set to make the flywheel rotate at a certain angular velocity ω_0. When $\Delta\omega = \omega - \omega_0 \neq 0$, the balls will move outward or inward, and the sleeve will slide up or down.[25]

With some assumptions, the motion of the mechanical non-autonomous CFG is given by[25]

$$\begin{aligned} \ddot{\phi} &= \left(E + n^2\omega^2\right)\sin\phi\,\cos\phi - (E + g/l)\sin\phi - b\dot{\phi}, \\ \dot{\omega} &= (\alpha\cos\phi - F)/I - a\sin wt, \end{aligned} \tag{1}$$

where ϕ is the angle between the rotational axis and the rods and $n = 3$, $l = 1.5$, $a = 0.8$, $w = 1$, $E = 0.3$, $b = 0.4$, $I = 1.2$, $\alpha = 0.611$, $F = 0.3$ and $g = 9.8$ are positive constants. More details for the system (1) can be found in Ref. [25].

For the above-mentioned parameter values, the non-autonomous CFG (1) exhibits chaotic behaviour.[25] The strange attractor of the CFG with initial conditions of $\phi(0) = 0.006$, $\dot{\phi}(0) = 0.007$ and $\dot{\omega}(0) = 0.15$ is illustrated in Fig. 2.

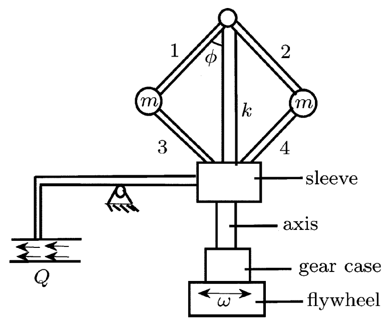

Fig. 1. Physical model of the mechanical CFG system.

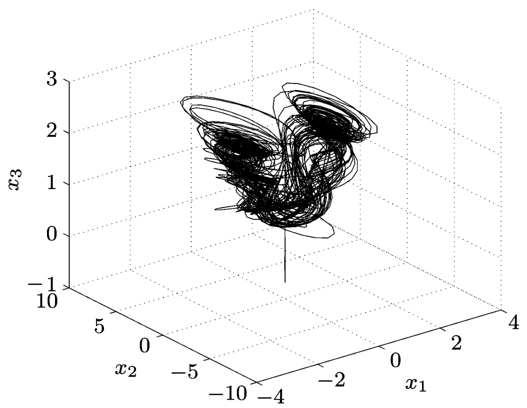

Fig. 2. Strange attractor of the chaotic non-autonomous CFG system (1).

2.1. Finite-time synchronization of two CFGs

In this section, the problem of finite-time synchronization of two identical/different CFG systems with fully unknown parameters, model uncertainties and external disturbances is solved.

2.2. Finite-time synchronization of two identical CFGs

Here, we assume that the parameters of both master and slave CFGs are identical. Defining $x_1 = \phi$, $x_2 = \dot{\phi}$ and $x_3 = \omega$, one can rewrite the non-autonomous CFG (1) with model uncertainties, external disturbances and unknown parameters as follows:

$$\begin{aligned}\dot{x}_1 &= x_2 + \Delta f_1(\boldsymbol{x}, t),\\ \dot{x}_2 &= \left(E + n^2 x_3^2\right)\sin x_1 \cos x_1 - (E + g/l)\sin x_1\\ &\quad - b x_2 + \Delta f_2(\boldsymbol{x}, t),\\ \dot{x}_3 &= (\alpha \cos x_1 - F)/I - a\sin wt + \Delta f_3(\boldsymbol{x}, t),\end{aligned} \tag{2}$$

where $\boldsymbol{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ is the state vector of the system and $\Delta f_i(\boldsymbol{x}, t)$, $i = 1, 2, 3$ represents unknown bounded time-varying model uncertainties and external disturbances of the system.

Considering the chaotic CFG (2) as the master system, the slave chaotic CFG with control inputs is defined as

$$\begin{aligned}\dot{y}_1 &= y_2 + \Delta g_1(\boldsymbol{y}, t) + u_1(t),\\ \dot{y}_2 &= \left(E + n^2 y_3^2\right)\sin y_1 \cos y_1 - (E + g/l)\sin y_1\\ &\quad - b y_2 + \Delta g_2(\boldsymbol{y}, t) + u_2(t),\\ \dot{y}_3 &= (\alpha \cos y_1 - F)/I - a\sin wt\\ &\quad + \Delta g_3(\boldsymbol{y}, t) + u_3(t),\end{aligned} \tag{3}$$

where $\boldsymbol{y} = [y_1, y_2, y_3]^{\mathrm{T}}$ is the state vector of the slave system, $\Delta g_i(\boldsymbol{y}, t)$, $i = 1, 2, 3$ represents unknown bounded time-varying model uncertainties and external disturbances of the slave system and $\boldsymbol{u}(t) = [u_1(t), u_2(t), u_3(t)]^{\mathrm{T}}$ is the vector of control inputs.

Assumption 1 In practice, system uncertainties are always bounded. In this line, we assume that

$$|\Delta f_i(\boldsymbol{x}, t)| \le a_i,\ |\Delta g_i(\boldsymbol{y}, t)| \le b_i,\ i = 1, 2, 3, \tag{4}$$

where a_i and b_i, $i = 1, 2, 3$ are positive constants.

Accordingly, we have

$$|\Delta f_i(\boldsymbol{x}, t) - \Delta g_i(\boldsymbol{y}, t)| \le d_i, \quad i = 1, 2, 3, \tag{5}$$

where d_i, $i = 1, 2, 3$ is a given positive constant.

To solve the finite-time synchronization problem, the synchronization error between the master and slave systems is defined as $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t) = [e_1, e_2, e_3]^{\mathrm{T}}$. Therefore, by subtracting Eq. (3) from Eq. (2), the error dynamics is acquired as follows:

$$\begin{aligned}\dot{e}_1 &= e_2 + \Delta f_1(\boldsymbol{x}, t) - \Delta g_1(\boldsymbol{y}, t) - u_1(t),\\ \dot{e}_2 &= E(\sin x_1 \cos x_1 - \sin y_1 \cos y_1 - \sin x_1 + \sin y_1)\\ &\quad + n^2\left(x_3^2 \sin x_1 \cos x_1 - y_3^2 \sin y_1 \cos y_1\right)\\ &\quad - (g/l)(\sin x_1 - \sin y_1) - b e_2 + \Delta f_2(\boldsymbol{x}, t)\\ &\quad - \Delta g_2(\boldsymbol{y}, t) - u_2(t),\\ \dot{e}_3 &= \alpha/I(\cos x_1 - \cos y_1) + \Delta f_3(\boldsymbol{x}, t)\\ &\quad - \Delta g_3(\boldsymbol{y}, t) - u_3(t).\end{aligned} \tag{6}$$

Assumption 2 It is assumed that the parameters n, l, a, w, E, b, I, α, F and g are fully unknown in advance. Then, defining $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3, \theta_4, \theta_5]^{\mathrm{T}} = [E, n^2, g/l, b, \alpha/I]^{\mathrm{T}}$ as the vector of the unknown parameters of the error system (6), the error system parameters are assumed to lie in a bounded range to ensure that the CFG exhibits chaos. Therefore, we have the following condition:

$$\|\boldsymbol{\theta}\| \le \Theta, \tag{7}$$

where $\|\,.\,\|$ denotes the Euclidean norm in R^n and Θ is a known positive constant.

Finite-time synchronization of two chaotic CFGs means that there is a finite time T such that the state trajectories of the slave CFG converge to the trajectories of the master CFG within the finite time T, i.e.,

$$\mathrm{Lim}_{t\to T}\|\boldsymbol{e}(t)\| = 0, \quad \|\boldsymbol{e}(t)\| \equiv 0, \quad t \ge T. \tag{8}$$

In what follows, proper control laws are introduced to guarantee the finite-time synchronization of two uncertain CFGs with identical parameters.

$$u_1(t) = e_2 + \mu\left(\Theta + \|\hat{\boldsymbol{\theta}}\|\right)\left(\frac{e_1}{\|\boldsymbol{e}\|^2}\right)$$

$$
\begin{aligned}
&\quad + (d_1+\eta_1)\,\mathrm{sgn}(e_1),\\
u_2(t) &= \hat{\theta}_1(\sin x_1\cos x_1-\sin y_1\cos y_1-\sin x_1\\
&\quad + \sin y_1)+\hat{\theta}_2(x_3^2\sin x_1\,\cos x_1\\
&\quad - y_3^2\sin y_1\,\cos y_1)-\hat{\theta}_3\,(\sin x_1-\sin y_1)\\
&\quad - \hat{\theta}_4 e_2+\mu\left(\varTheta+\|\hat{\boldsymbol{\theta}}\|\right)\left(\frac{e_2}{\|\boldsymbol{e}\|^2}\right)\\
&\quad + (d_2+\eta_2)\,\mathrm{sgn}(e_2),\\
u_3(t) &= \hat{\theta}_5\,(\cos x_1-\cos y_1)+\mu\left(\varTheta+\|\hat{\boldsymbol{\theta}}\|\right)\left(\frac{e_3}{\|\boldsymbol{e}\|^2}\right)\\
&\quad + (d_3+\eta_3)\,\mathrm{sgn}(e_3),
\end{aligned}
\tag{9}
$$

where $\hat{\boldsymbol{\theta}}=[\hat{\theta}_1,\,\hat{\theta}_2,\,\hat{\theta}_3,\,\hat{\theta}_4,\,\hat{\theta}_5]^{\mathrm{T}}$ is an estimation for unknown parameter vector $\boldsymbol{\theta}$; $\mu=\min\{\eta_i\}$, $i=1,2,3$, $\eta_i>0$ is a constant gain, if $e_i=0$, then $e_i/\|\boldsymbol{e}\|^2=0$, $i=1,\,2,\,3$ and sgn(.) is the signum function.

In order to deal with the unknown parameters of the CFGs, the following update laws are proposed:

$$
\begin{aligned}
\dot{\hat{\theta}}_1(t) &= e_2(\sin x_1\cos x_1-\sin y_1\cos y_1\\
&\quad - \sin x_1+\sin y_1),\ \hat{\theta}_1(0)=\hat{\theta}_{10},\\
\dot{\hat{\theta}}_2(t) &= e_2(x_3^2\sin x_1\,\cos x_1\\
&\quad - y_3^2\sin y_1\,\cos y_1),\ \hat{\theta}_2(0)=\hat{\theta}_{20},\\
\dot{\hat{\theta}}_3(t) &= -e_2\,(\sin x_1-\sin y_1)\,,\ \hat{\theta}_3(0)=\hat{\theta}_{30},\\
\dot{\hat{\theta}}_4(t) &= -e_2^2,\ \hat{\theta}_4(0)=\hat{\theta}_{40},\\
\dot{\hat{\theta}}_5(t) &= e_3\,(\cos x_1-\cos y_1)\,,\ \hat{\theta}_5(0)=\hat{\theta}_{50},
\end{aligned}
\tag{10}
$$

where $\hat{\theta}_{10}$, $\hat{\theta}_{20}$, $\hat{\theta}_{30}$, $\hat{\theta}_{40}$ and $\hat{\theta}_{50}$ are initial values of the update parameters $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$, $\hat{\theta}_4$ and $\hat{\theta}_5$, respectively.

Theorem 1 If the synchronization error system (6) is controlled by the control signals (9) with the update laws (10), then the state trajectories of the synchronization error system (6) will converge to zero in the finite time

$$
T_1=\frac{\sqrt{2}}{\mu}\left(\frac{1}{2}\left(\|\boldsymbol{e}(0)\|^2+\|\hat{\boldsymbol{\theta}}(0)-\boldsymbol{\theta}\|^2\right)\right)^{1/2}.
$$

Therefore, two non-autonomous chaotic CFGs (2) and (3) with identical unknown parameters, model uncertainties and external disturbances will be synchronized in the finite time T_1.

Proof See Appendix A.

Remark 1 If the error system's parameters $\boldsymbol{\theta}$ vary with time in a bounded range, the variations of the parameters will lead to some parameter uncertainties. These uncertainties can be regarded as extra model uncertainties added to the system uncertainties. It is worth noticing that the designed controller (9) and (10) can be simply extended to this case as follows:

One can rewrite the error system (6) in the following form:

$$
\begin{aligned}
\dot{\boldsymbol{e}}(t) &= \boldsymbol{f}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)+\boldsymbol{F}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)\boldsymbol{\theta}\\
&\quad + \Delta\boldsymbol{f}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)-\boldsymbol{u}(t),
\end{aligned}
\tag{11}
$$

where $\boldsymbol{f}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)=[e_2,\,0,\,0]^{\mathrm{T}}$, $\boldsymbol{F}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)=\mathrm{diag}\{\sin x_1\cos x_1-\sin y_1\cos y_1-\sin x_1+\sin y_1,\ x_3^2\sin x_1\,\cos x_1-y_3^2\sin y_1\,\cos y_1, \sin x_1-\sin y_1,\,-e_2,\ \cos x_1-\cos y_1\}$, and $\Delta\boldsymbol{f}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)=[\Delta f_1(\boldsymbol{x},t)-\Delta g_2(\boldsymbol{y},t), \Delta f_2(\boldsymbol{x},t)\text{–}\Delta g_2(\boldsymbol{y},t), \Delta f_3(\boldsymbol{x},t)-\Delta g_3(\boldsymbol{y},t)]^{\mathrm{T}}$.

In the case of time-varying parameters, we have

$$
\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}+\Delta\boldsymbol{\theta},
\tag{12}
$$

where $\bar{\boldsymbol{\theta}}$ is the unknown nominal value of $\boldsymbol{\theta}$ and $\Delta\boldsymbol{\theta}$, which is norm-bounded by a known constant, is the time-varying part of $\boldsymbol{\theta}$.

Inserting $\boldsymbol{\theta}$ from (12) into the right hand side of (11), it yields

$$
\begin{aligned}
\dot{\boldsymbol{e}}(t) &= \boldsymbol{f}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)+\boldsymbol{F}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)\left(\bar{\boldsymbol{\theta}}+\Delta\boldsymbol{\theta}\right)\\
&\quad + \Delta\boldsymbol{f}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)-\boldsymbol{u}(t)\\
&= \boldsymbol{f}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)+\boldsymbol{F}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)\bar{\boldsymbol{\theta}}\\
&\quad + \Delta\tilde{\boldsymbol{f}}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)-\boldsymbol{u}(t),
\end{aligned}
\tag{13}
$$

where

$$
\Delta\tilde{\boldsymbol{f}}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)=\boldsymbol{F}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)\Delta\boldsymbol{\theta}+\Delta\boldsymbol{f}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)
$$

is the new bounded uncertain part of the error system.

One can see that the new error system (13) is in the form of the error system (6). Therefore, the proposed finite-time controller is also valid for the case of the time-varying unknown parameters and the bounds of the uncertainties can be modified according to the bounds of $\Delta\boldsymbol{\theta}$ and $\boldsymbol{F}(\boldsymbol{x},\,\boldsymbol{y},\,\boldsymbol{e},\,t)$.

2.3. Finite-time synchronization of two different CFGs

Here, it is assumed that the unknown parameters of the CFGs are not in the same values. Consider the following master and slave CFGs with the unknown parameters of different values, model uncertainties and external disturbances.

Master system

$$
\begin{aligned}
\dot{x}_1 &= x_2+\Delta f_1(\boldsymbol{x},t),\\
\dot{x}_2 &= \left(E_{\mathrm{m}}+n_{\mathrm{m}}^2x_3^2\right)\sin x_1\,\cos x_1\\
&\quad - (E_{\mathrm{m}}+g_{\mathrm{m}}/l_{\mathrm{m}})\sin x_1-b_{\mathrm{m}}x_2+\Delta f_2(\boldsymbol{x},t),\\
\dot{x}_3 &= (\alpha_{\mathrm{m}}\cos x_1-F_{\mathrm{m}})\,/I_{\mathrm{m}}\\
&\quad - a_{\mathrm{m}}\sin w_{\mathrm{m}}t+\Delta f_3(\boldsymbol{x},t);
\end{aligned}
\tag{14}
$$

slave system:

$$\begin{aligned}\dot{y}_1 &= y_2 + \Delta g_1(\boldsymbol{y},t) + u_1(t),\\ \dot{y}_2 &= \left(E_\mathrm{s} + n_\mathrm{s}^2 y_3^2\right)\sin y_1\ \cos y_1 - (E_\mathrm{s} + g_\mathrm{s}/l_\mathrm{s})\sin y_1\\ &\quad - b_\mathrm{s} y_2 + \Delta g_2(\boldsymbol{y},t) + u_2(t),\\ \dot{y}_3 &= (\alpha_\mathrm{s}\cos y_1 - F_\mathrm{s})/I_\mathrm{s} - a_\mathrm{s}\sin w_\mathrm{s} t\\ &\quad + \Delta g_3(\boldsymbol{y},t) + u_3(t);\end{aligned} \tag{15}$$

where E_m, n_m, g_m, l_m, b_m, α_m, F_m, I_m, a_m, w_m, E_s, n_s, g_s, l_s, b_s, α_s, F_s, I_s, a_s and w_s are positive constants.

Subtracting Eq. (15) from Eq. (14), we obtain the synchronization error dynamics as follows:

$$\begin{aligned}\dot{e}_1 &= e_2 + \Delta f_1(\boldsymbol{x},t) - \Delta g_1(\boldsymbol{y},t) - u_1(t),\\ \dot{e}_2 &= \left(E_\mathrm{m} + n_\mathrm{m}^2 x_3^2\right)(\sin x_1\ \cos x_1)\\ &\quad - (E_\mathrm{m} + g_\mathrm{m}/l_\mathrm{m})\sin x_1 - b_\mathrm{m} x_2\\ &\quad - \left(E_\mathrm{s} + n_\mathrm{s}^2 y_3^2\right)(\sin y_1\ \cos y_1)\\ &\quad + (E_\mathrm{s} + g_\mathrm{s}/l_\mathrm{s})\sin y_1 - b_\mathrm{s} y_2 + \Delta f_2(\boldsymbol{x},t)\\ &\quad - \Delta g_2(\boldsymbol{y},t) - u_2(t),\\ \dot{e}_3 &= (\alpha_\mathrm{m}\cos x_1 - F_\mathrm{m})/I_\mathrm{m} - a_\mathrm{m}\sin w_\mathrm{m} t\\ &\quad - (\alpha_\mathrm{s}\cos y_1 - F_\mathrm{s})/I_\mathrm{s} - a_\mathrm{s}\sin w_\mathrm{s} t\\ &\quad + \Delta f_3(\boldsymbol{x},t) - \Delta g_3(\boldsymbol{y},t) - u_3(t).\end{aligned} \tag{16}$$

Assumption 3 It is assumed that the parameters E_m, n_m, g_m, l_m, b_m, α_m, F_m, I_m, a_m, w_m, E_s, n_s, g_s, l_s, b_s, α_s, F_s, I_s, a_s and w_s are fully unknown in advance. Furthermore, consider $\boldsymbol{\theta}_\mathrm{m} = [\theta_{\mathrm{m}1}, \theta_{\mathrm{m}2}, \theta_{\mathrm{m}3}, \theta_{\mathrm{m}4}, \theta_{\mathrm{m}5}, \theta_{\mathrm{m}6}, \theta_{\mathrm{m}7}]^\mathrm{T} = [E_\mathrm{m}, n_\mathrm{m}^2, g_\mathrm{m}/l_\mathrm{m}, b_\mathrm{m}, \alpha_\mathrm{m}/I_\mathrm{m}, F_\mathrm{m}, |a_\mathrm{m}|]^\mathrm{T}$ and $\boldsymbol{\theta}_\mathrm{s} = [\theta_{\mathrm{s}1}, \theta_{\mathrm{s}2}, \theta_{\mathrm{s}3}, \theta_{\mathrm{s}4}, \theta_{\mathrm{s}5}, \theta_{\mathrm{s}6}, \theta_{\mathrm{s}7}]^\mathrm{T} = [E_\mathrm{s}, n_\mathrm{s}^2, g_\mathrm{s}/l_\mathrm{s}, b_\mathrm{s}, \alpha_\mathrm{s}/I_\mathrm{s}, F_\mathrm{s}, |a_\mathrm{s}|]^\mathrm{T}$ to be the unknown parameter vectors of the CFG master and slave systems, respectively. Then, $\boldsymbol{\theta}_\mathrm{m}$ and $\boldsymbol{\theta}_\mathrm{s}$ are bounded by

$$\|\boldsymbol{\theta}_\mathrm{m}\| \le \varTheta_\mathrm{m} \quad \text{and} \quad \|\boldsymbol{\theta}_\mathrm{s}\| \le \varTheta_\mathrm{s}, \tag{17}$$

where $\varTheta_\mathrm{m}$ and $\varTheta_\mathrm{s}$ are given positive constants.

To guarantee the finite-time stability of the error system (16), the following control inputs are designed:

$$\begin{aligned}u_1(t) &= e_2 + \mu\left(\varTheta_\mathrm{m} + \|\hat{\boldsymbol{\theta}}_\mathrm{m}\| + \varTheta_\mathrm{s} + \|\hat{\boldsymbol{\theta}}_\mathrm{s}\|\right)\left(\frac{e_1}{\|\boldsymbol{e}\|^2}\right)\\ &\quad + (d_1 + \eta_1)\,\mathrm{sgn}(e_1),\\ u_2(t) &= \hat{\boldsymbol{\theta}}_{\mathrm{m}1}(\sin x_1\cos x_1 - \sin x_1)\\ &\quad - \hat{\boldsymbol{\theta}}_{\mathrm{s}1}(\sin y_1\cos y_1 - \sin y_1)\\ &\quad + \hat{\boldsymbol{\theta}}_{\mathrm{m}2} x_3^2\sin x_1\ \cos x_1 - \hat{\boldsymbol{\theta}}_{\mathrm{s}2} y_3^2\sin y_1\ \cos y_1\\ &\quad - \hat{\boldsymbol{\theta}}_{\mathrm{m}3}\sin x_1 + \hat{\boldsymbol{\theta}}_{\mathrm{s}3}\sin y_1 - \hat{\boldsymbol{\theta}}_{\mathrm{m}4} x_2 + \hat{\boldsymbol{\theta}}_{\mathrm{s}4} y_2\\ &\quad + \mu\left(\varTheta_\mathrm{m} + \|\hat{\boldsymbol{\theta}}_\mathrm{m}\| + \varTheta_\mathrm{s} + \|\hat{\boldsymbol{\theta}}_\mathrm{s}\|\right)\left(\frac{e_2}{\|\boldsymbol{e}\|^2}\right)\\ &\quad + (d_2 + \eta_2)\,\mathrm{sgn}(e_2),\\ u_3(t) &= \hat{\boldsymbol{\theta}}_{\mathrm{m}5}\cos x_1 - \hat{\boldsymbol{\theta}}_{\mathrm{s}5}\cos y_1 - \hat{\boldsymbol{\theta}}_{\mathrm{m}6} + \hat{\boldsymbol{\theta}}_{\mathrm{s}6} + (\hat{\boldsymbol{\theta}}_{\mathrm{m}7}\\ &\quad + \hat{\boldsymbol{\theta}}_{\mathrm{s}7})\mathrm{sgn}(e_3) + \mu\left(\varTheta_\mathrm{m} + \|\hat{\boldsymbol{\theta}}_\mathrm{m}\| + \varTheta_\mathrm{s} + \|\hat{\boldsymbol{\theta}}_\mathrm{s}\|\right)\\ &\quad \times\left(\frac{e_3}{\|\boldsymbol{e}\|^2}\right) + (d_3 + \eta_3)\,\mathrm{sgn}(e_3),\end{aligned} \tag{18}$$

where $\hat{\boldsymbol{\theta}}_\mathrm{m} = [\hat{\theta}_{\mathrm{m}1}, \hat{\theta}_{\mathrm{m}2}, \hat{\theta}_{\mathrm{m}3}, \hat{\theta}_{\mathrm{m}4}, \hat{\theta}_{\mathrm{m}5}, \hat{\theta}_{\mathrm{m}6}, \hat{\theta}_{\mathrm{m}7}]^\mathrm{T}$ and $\hat{\boldsymbol{\theta}}_\mathrm{s} = [\hat{\theta}_{\mathrm{s}1}, \hat{\theta}_{\mathrm{s}2}, \hat{\theta}_{\mathrm{s}3}, \hat{\theta}_{\mathrm{s}4}, \hat{\theta}_{\mathrm{s}5}, \hat{\theta}_{\mathrm{s}6}, \hat{\theta}_{\mathrm{s}7}]^\mathrm{T}$ are estimations for $\boldsymbol{\theta}_\mathrm{m}$ and $\boldsymbol{\theta}_\mathrm{s}$, respectively and if $e_i = 0$, then $e_i/\|\boldsymbol{e}\|^2 = 0$, $i = 1, 2, 3$.

Subsequently, the following update laws are proposed:

$$\begin{aligned}\dot{\hat{\theta}}_{\mathrm{m}1}(t) &= e_2(\sin x_1\cos x_1 - \sin x_1), \quad \hat{\theta}_{\mathrm{m}1}(0) = \hat{\theta}_{\mathrm{m}10};\\ \dot{\hat{\theta}}_{\mathrm{m}2}(t) &= e_2 x_3^2\sin x_1\ \cos x_1, \quad \hat{\theta}_{\mathrm{m}2}(0) = \hat{\theta}_{\mathrm{m}20};\\ \dot{\hat{\theta}}_{\mathrm{m}3}(t) &= -e_2\sin x_1, \quad \hat{\theta}_{\mathrm{m}3}(0) = \hat{\theta}_{\mathrm{m}30};\\ \dot{\hat{\theta}}_{\mathrm{m}4}(t) &= -e_2 x_2, \quad \hat{\theta}_{\mathrm{m}4}(0) = \hat{\theta}_{\mathrm{m}40};\\ \dot{\hat{\theta}}_{\mathrm{m}5}(t) &= e_3\cos x_1, \quad \hat{\theta}_{\mathrm{m}5}(0) = \hat{\theta}_{\mathrm{m}50};\\ \dot{\hat{\theta}}_{\mathrm{m}6}(t) &= -e_3, \quad \hat{\theta}_{\mathrm{m}6}(0) = \hat{\theta}_{\mathrm{m}60};\\ \dot{\hat{\theta}}_{\mathrm{m}7}(t) &= |e_3|, \quad \hat{\theta}_{\mathrm{m}7}(0) = \hat{\theta}_{\mathrm{m}70};\\ \dot{\hat{\theta}}_{\mathrm{s}1}(t) &= e_2(-\sin y_1\cos y_1 + \sin y_1), \quad \hat{\theta}_{\mathrm{s}1}(0) = \hat{\theta}_{\mathrm{s}10};\\ \dot{\hat{\theta}}_{\mathrm{s}2}(t) &= -e_2 y_3^2\sin y_1\ \cos y_1, \quad \hat{\theta}_{\mathrm{s}2}(0) = \hat{\theta}_{\mathrm{s}20};\\ \dot{\hat{\theta}}_{\mathrm{s}3}(t) &= e_2\sin y_1, \quad \hat{\theta}_{\mathrm{s}3}(0) = \hat{\theta}_{\mathrm{s}30};\\ \dot{\hat{\theta}}_{\mathrm{s}4}(t) &= -e_2 y_2, \quad \hat{\theta}_{\mathrm{s}4}(0) = \hat{\theta}_{\mathrm{s}40};\\ \dot{\hat{\theta}}_{\mathrm{s}5}(t) &= -e_3\cos y_1, \quad \hat{\theta}_{\mathrm{s}5}(0) = \hat{\theta}_{\mathrm{s}50};\\ \dot{\hat{\theta}}_{\mathrm{s}6}(t) &= e_3, \quad \hat{\theta}_{\mathrm{s}6}(0) = \hat{\theta}_{\mathrm{s}60};\\ \dot{\hat{\theta}}_{\mathrm{s}7}(t) &= |e_3|, \quad \hat{\theta}_{\mathrm{s}7}(0) = \hat{\theta}_{\mathrm{s}70};\end{aligned} \tag{19}$$

where $\hat{\theta}_{\mathrm{m}10}$, $\hat{\theta}_{\mathrm{m}20}$, $\hat{\theta}_{\mathrm{m}30}$, $\hat{\theta}_{\mathrm{m}40}$, $\hat{\theta}_{\mathrm{m}50}$, $\hat{\theta}_{\mathrm{m}60}$, $\hat{\theta}_{\mathrm{m}70}$, $\hat{\theta}_{\mathrm{s}10}$, $\hat{\theta}_{\mathrm{s}20}$, $\hat{\theta}_{\mathrm{s}30}$, $\hat{\theta}_{\mathrm{s}40}$, $\hat{\theta}_{\mathrm{s}50}$, $\hat{\theta}_{\mathrm{s}60}$ and $\hat{\theta}_{\mathrm{s}70}$ are the initial values of the update parameters $\hat{\theta}_{\mathrm{m}1}$, $\hat{\theta}_{\mathrm{m}2}$, $\hat{\theta}_{\mathrm{m}3}$, $\hat{\theta}_{\mathrm{m}4}$, $\hat{\theta}_{\mathrm{m}5}$, $\hat{\theta}_{\mathrm{m}6}$, $\hat{\theta}_{\mathrm{m}7}$, $\hat{\theta}_{\mathrm{s}1}$, $\hat{\theta}_{\mathrm{s}2}$, $\hat{\theta}_{\mathrm{s}3}$, $\hat{\theta}_{\mathrm{s}4}$, $\hat{\theta}_{\mathrm{s}5}$, $\hat{\theta}_{\mathrm{s}6}$ and $\hat{\theta}_{\mathrm{s}7}$, respectively.

Theorem 2 Suppose that the control inputs (18) with the update laws (19) are used to control the synchronization error system (16). Consequently, the state trajectories of the error system (16) converge to zero in the finite time

$$T_2 = \frac{\sqrt{2}}{\mu}\left(\frac{1}{2}\Big(\|\boldsymbol{e}(0)\|^2 + \|\hat{\boldsymbol{\theta}}_\mathrm{m}(0) - \boldsymbol{\theta}_\mathrm{m}\|^2 + \|\hat{\boldsymbol{\theta}}_\mathrm{s}(0) - \boldsymbol{\theta}_\mathrm{s}\|^2\Big)\right)^{1/2}.$$

This means that two chaotic non-autonomous CFGs (15) and (16) with fully unknown parameters of different values, model uncertainties and external disturbances are finite-time synchronized with the convergence time T_2.

Proof See Appendix B.

3. Illustrative examples

In this section, some numerical simulations are presented to validate the robustness and feasibility of the proposed finite-time controllers in the synchronization of two identical/different uncertain chaotic non-autonomous CFGs with unknown parameters. The simulations are performed using MATLAB software. Two illustrative examples are presented. In the examples, the following uncertainties and disturbances are applied:

$$\begin{aligned}\Delta f_1(\boldsymbol{x},t) &= -0.3\sin(x_1) - 0.2\cos(5t),\\ \Delta f_2(\boldsymbol{x},t) &= -0.2\cos(3x_2) - 0.25\sin(4t),\\ \Delta f_3(\boldsymbol{x},t) &= -0.2\cos(2x_3) + 0.25\sin(4t),\\ \Delta g_1(\boldsymbol{y},t) &= 0.25\cos(2y_1) - 0.3\sin(3t),\\ \Delta g_2(\boldsymbol{y},t) &= -0.2\sin(5y_2) + 0.25\cos(2t),\\ \Delta g_3(\boldsymbol{y},t) &= -0.35\cos(6y_3) + 0.25\sin(3t). \quad (20)\end{aligned}$$

In all simulations, the values of the constant gains η_1, η_2 and η_3 are chosen to be 1 and the initial values of all update parameters are set to be 0.5. Note that the control inputs are activated at $t = 5$ s.

3.1. Example 1

In this example, the efficient performance of the proposed finite-time controller (9) and (10) in synchronization of two identical CFGs with unknown parameters is illustrated. The parameters $n = 3$, $l = 1.5$, $a = 0.8$, $w = 1$, $E = 0.3$, $b = 0.4$, $I = 1.2$, $\alpha = 0.611$, $F = 0.3$ and $g = 9.8$ are selected for both master and slave systems. Θ is chosen to be equal to 10. The initial values of the master and slave systems are chosen as $x_1(0) = 0.3$, $x_2(0) = 0.2$, $x_3(0) = 0.1$ and $y_1(0) = 0.01$, $y_2(0) = 0.02$, $y_3(0) = 0.03$, respectively. According to Theorem 1 and Eqs. (9) and (10), the proper control laws are applied.

The state trajectories of the synchronization error system (6) are depicted in Fig. 3. It is clear that the synchronization errors converge to zero quickly. This means that by applying the proposed robust adaptive controller, the state trajectories of the uncertain CFG slave system (3) approach the state trajectories of the uncertain CFG master system (2) in a finite time. The time history of the update parameter vector $\hat{\boldsymbol{\theta}}$ is revealed in Fig. 4. One can see that all the update parameters converge to fixed values.

For comparison, the adaptive controller proposed in Ref. [34] is used to synchronize two identical CFGs with unknown parameters. Figures 5 and 6 show the state trajectories of the error system (6) with and without the uncertainties (20), respectively. It can be seen that in the absence of the uncertainties the synchronization errors converge to zero slowly. However, when the uncertainties exist in the systems' dynamics, the synchronization errors have permanent oscillations and the synchronization purpose is not

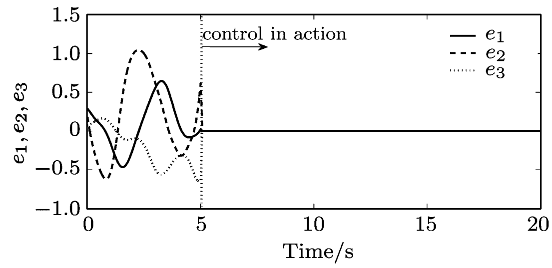

Fig. 3. State trajectories of the synchronization error system (6) with the controller of Eqs. (9) and (10).

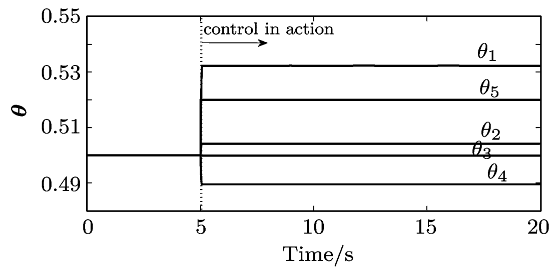

Fig. 4. Time response of the update parameter vector $\hat{\boldsymbol{\theta}}$ obtained by Eq. (10).

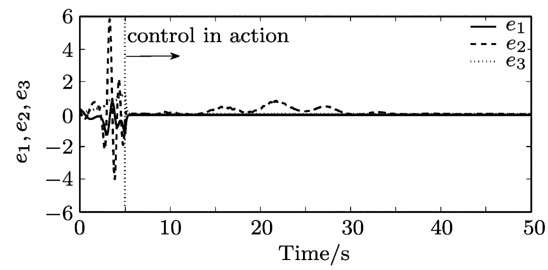

Fig. 5. State trajectories of the synchronization error system (6) with the adaptive controller in Ref. [34] and without uncertainties.

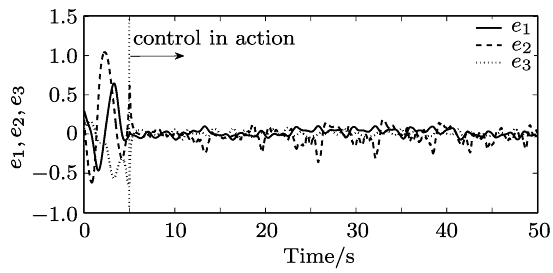

Fig. 6. State trajectories of the synchronization error system (6) with the adaptive controller in Ref. [34] and the uncertainties (20).

completely achieved. In other words, the controller in Ref. [34] is not robust against the system uncertainties. On the other hand, our controller (9) and (10) has both robustness against the uncertainties and fast convergence properties.

3.2. Example 2

In this example, it is assumed that the unknown parameters of the master and slave systems do not have the same values. The parameters $n_{\mathrm{m}} = 4$, $l_{\mathrm{m}} = 0.5$, $a_{\mathrm{m}} = 0.4$, $w_{\mathrm{m}} = 2$, $E_{\mathrm{m}} = 0.5$, $b_{\mathrm{m}} = 0.3$, $I_{\mathrm{m}} = 0.5$, $\alpha_{\mathrm{m}} = 0.432$, $F_{\mathrm{m}} = 0.35$ and $g_{\mathrm{m}} = 10$ are selected for the master system and the parameters $n_{\mathrm{s}} = 3$, $l_{\mathrm{s}} = 1.5$, $a_{\mathrm{s}} = 0.8$, $w_{\mathrm{s}} = 1$, $E_{\mathrm{s}} = 0.3$, $b_{\mathrm{s}} = 0.4$, $I_{\mathrm{s}} = 1.2$, $\alpha_{\mathrm{s}} = 0.611$, $F_{\mathrm{s}} = 0.3$ and $g_{\mathrm{s}} = 9.8$ are chosen for the slave system. Θ_{m} and Θ_{s} are set to be 26 and 10, respectively. The initial values of the master and slave systems are selected as $x_1(0) = 0.2$, $x_2(0) = 0.1$, $x_3(0) = 0.3$ and $y_1(0) = 0.03$, $y_2(0) = 0.01$, $y_3(0) = 0.02$, respectively. Using Eqs. (18) and (19) and Theorem 2, suitable control laws are employed.

Figure 7 displays the state trajectories of the synchronization error system (16). Obviously, the synchronization errors reach zero rapidly. This indicates that when the proposed robust adaptive controller (18) and (19) is turned on, the state trajectories of the uncertain CGF slave system (15) get to the state trajectories of the uncertain CGF master system (14) as quickly as possible. The time histories of the update parameter vectors $\hat{\boldsymbol{\theta}}_{\mathrm{m}}$ and $\hat{\boldsymbol{\theta}}_{\mathrm{s}}$ are depicted in Figs. 8 and 9, respectively. It is seen that all the update parameters are bounded.

To compare the performance of the proposed technique, the adaptive synchronization strategy introduced in Ref. [34] is used to synchronize two CFGs with different parameters. System synchronization errors with and without the uncertainties (20) appear in Figs. 10 and 11, respectively. One can see that the proposed adaptive controller in Ref. [34] can asymptotically synchronize two different CFGs without the uncertainties. However, when the uncertainties (2) are present in the dynamics of the systems, there are nonzero steady state oscillations in the error trajectories. This means that the proposed adaptive controller in Ref. [34] is not robust to system uncertainties. However, the method proposed in our paper can robustly and quickly synchronize two different CFGs in the presence of model uncertainties and external disturbances in both the master and slave systems.

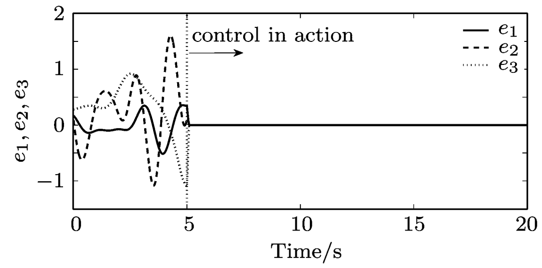

Fig. 7. State trajectories of the synchronization error system (16) with the controller of Eqs. (18) and (19).

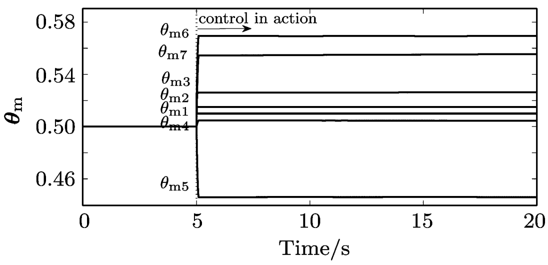

Fig. 8. Time response of the update parameter vector $\hat{\boldsymbol{\theta}}_{\mathrm{m}}$ obtained by Eq. (19).

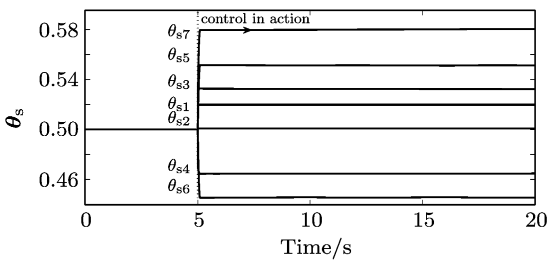

Fig. 9. Time response of the update parameter vector $\hat{\boldsymbol{\theta}}_{\mathrm{s}}$ obtained by Eq. (19).

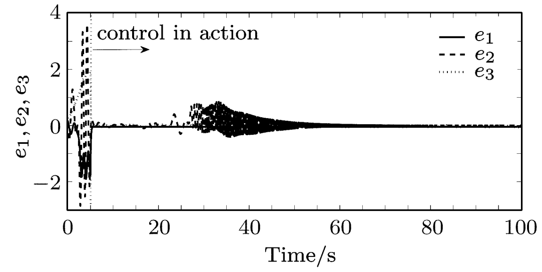

Fig. 10. State trajectories of the synchronization error system (16) with the adaptive controller in Ref. [34] and without uncertainties.

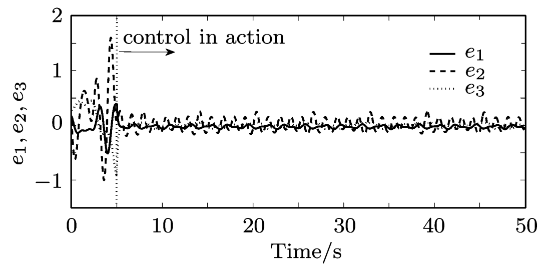

Fig. 11. State trajectories of the synchronization error system (16) with the adaptive controller in Ref. [34] and the uncertainties (20).

4. Conclusions

This paper is dedicated to solving the problem of robust finite-time synchronization of non-autonomous centrifugal flywheel governors (CFGs). It is assumed that the parameters of both master and slave CFGs are completely unknown with identical/different values. Besides, unknown model uncertainties and external disturbances are added to the systems' dynamics. Based on the finite-time control theory and some update parameters, proper finite-time robust adaptive controllers are derived. Numerical simulations reveal that the proposed controllers can synchronize two identical/different chaotic non-autonomous CFGs, even when the parameters of the systems are fully unknown and some uncertainties are present in the dynamics of both master and slave systems. The efficiency and benefits of our method are highlighted in comparison with an adaptive method proposed in the literature. Note that the proposed finite-time controllers can be extended to the synchronization of complex multi-scroll chaotic systems,[35–40] which remains the future work of the author.

Appendix A

Before proving Theorem 1, the following lemmas, which are needed to prove the finite-time stability of the error system, are presented.

Lemma 1[41] Suppose that a continuous, positive-definite function $V(t)$ satisfies the following differential inequality:

$$\dot{V}(t) \leq -pV^{\xi}(t) \quad \forall \quad t \geq t_0,\ V(t_0) \geq 0, \tag{A1}$$

where $p > 0$, $0 < \xi < 1$ are two constants. Then, $V(t)$ satisfies the following inequality:

$$V^{1-\xi}(t) \leq V^{1-\xi}(t_0) - p(1-\xi)(t-t_0), \quad t_0 \leq t \leq t_1, \tag{A2}$$

and $V(t) \equiv 0$, $\forall\, t \geq t_1$ with t_1 given by

$$t_1 = t_0 + \frac{V^{1-\xi}(t_0)}{p(1-\xi)}. \tag{A3}$$

Lemma 2 For $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$, the following inequality holds:

$$|\alpha_1| + |\alpha_2| + |\alpha_n| \geq \sqrt{\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2}. \tag{A4}$$

Proof of Theorem 1 Choose a positive definite function in the form of

$$V_1(t) = \frac{1}{2}\|\boldsymbol{e}\|^2 + \frac{1}{2}\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2. \tag{A5}$$

Taking the time derivative of $V(t)$, one has

$$\dot{V}_1(t) = \sum_{i=1}^{3} e_i \dot{e}_i + \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)^{\mathrm{T}} \dot{\hat{\boldsymbol{\theta}}}. \tag{A6}$$

Inserting $\dot{e}_i$ from Eq. (6) into the above equation, we obtain

$$\begin{aligned}\dot{V}_1(t) = {} & e_1 \left(e_2 + \Delta f_1(\boldsymbol{x}, t) - \Delta g_1(\boldsymbol{y}, t) - u_1(t)\right) + e_2 \left(E \left(\sin x_1 \cos x_1 - \sin y_1 \cos y_1 - \sin x_1 + \sin y_1\right)\right. \\ & + n^2 \left(x_3^2 \sin x_1 \cos x_1 - y_3^2 \sin y_1 \cos y_1\right) - (g/l)\left(\sin x_1 - \sin y_1\right) - b e_2 + \Delta f_2(\boldsymbol{x}, t) - \Delta g_2(\boldsymbol{y}, t) - u_2(t)\big) \\ & + e_3 \left(\alpha/I \left(\cos x_1 - \cos y_1\right) + \Delta f_3(\boldsymbol{x}, t) - \Delta g_3(\boldsymbol{y}, t) - u_3(t)\right) + (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathrm{T}} \dot{\hat{\boldsymbol{\theta}}}.\end{aligned} \tag{A7}$$

It is obvious that

$$\begin{aligned}\dot{V}_1(t) \leq {} & e_1 \left(e_2 - u_1(t)\right) + |e_1| \left|\Delta f_1(\boldsymbol{x}, t) - \Delta g_1(\boldsymbol{y}, t)\right| + e_2 \left(E \left(\sin x_1 \cos x_1 - \sin y_1 \cos y_1 - \sin x_1 + \sin y_1\right)\right. \\ & + n^2 \left(x_3^2 \sin x_1 \cos x_1 - y_3^2 \sin y_1 \cos y_1\right) - (g/l)\left(\sin x_1 - \sin y_1\right) - b e_2 - u_2(t)\big) + |e_2| \left|\Delta f_2(\boldsymbol{x}, t)\right. \\ & - \Delta g_2(\boldsymbol{y}, t)\big| + e_3 \left(\alpha/I \left(\cos x_1 - \cos y_1\right) - u_3(t)\right) + |e_3| \left|\Delta f_3(\boldsymbol{x}, t) - \Delta g_3(\boldsymbol{y}, t)\right| + (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathrm{T}} \dot{\hat{\boldsymbol{\theta}}}.\end{aligned} \tag{A8}$$

Using $\boldsymbol{\theta}^{\mathrm{T}} \dot{\hat{\boldsymbol{\theta}}} = E e_2 \left(\sin x_1 \cos x_1 - \sin y_1 \cos y_1 - \sin x_1 + \sin y_1\right) + n^2 e_2 \left(x_3^2 \sin x_1 \cos x_1 - y_3^2 \sin y_1 \cos y_1\right) (g/l) e_2 \left(\sin x_1 - \sin y_1\right) - b e_2^2 + (\alpha/I)\, e_3 \left(\cos x_1 - \cos y_1\right)$ and Assumption 1, we have

$$\dot{V}_1(t) \leq e_1 \left(e_2 - u_1(t)\right) + d_1 |e_1| - e_2 u_2(t) + d_2 |e_2| - e_3 u_3(t) + d_3 |e_3| + \hat{\boldsymbol{\theta}}^{\mathrm{T}} \dot{\hat{\boldsymbol{\theta}}}. \tag{A9}$$

Using the control inputs $u_1(t)$, $u_2(t)$ and $u_3(t)$ from Eq. (9), one has

$$\dot{V}_1(t) \leq e_1 \left(e_2 - \left[e_2 + \mu \left(\varTheta + \|\hat{\boldsymbol{\theta}}\|\right) \left(\frac{e_1}{\|\boldsymbol{e}\|^2}\right) + (d_1 + \eta_1)\operatorname{sgn}(e_1)\right]\right) + d_1 |e_1|$$

$$
\begin{aligned}
&- e_2\bigg[\hat{\theta}_1\left(\sin x_1\cos x_1-\sin y_1\cos y_1-\sin x_1+\sin y_1\right)+\hat{\theta}_2\left(x_3^2\sin x_1\,\cos x_1-y_3^2\sin y_1\,\cos y_1\right)\\
&- \hat{\theta}_3\left(\sin x_1-\sin y_1\right)-\hat{\theta}_4 e_2+\mu\left(\Theta+\|\hat{\boldsymbol{\theta}}\|\right)\left(\frac{e_2}{\|\boldsymbol{e}\|^2}\right)+\left(d_2+\eta_2\right)\operatorname{sgn}(e_2)\bigg]+d_2\left|e_2\right|\\
&- e_3\left[\hat{\theta}_5\left(\cos x_1-\cos y_1\right)+\mu\left(\Theta+\|\hat{\boldsymbol{\theta}}\|\right)\left(\frac{e_3}{\|\boldsymbol{e}\|^2}\right)+\left(d_3+\eta_3\right)\operatorname{sgn}(e_3)\right]+d_3\left|e_3\right|+\hat{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}. \qquad \text{(A10)}
\end{aligned}
$$

Based on $\hat{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}} = \hat{\theta}_1 e_2\left(\sin x_1\,\cos x_1-\sin y_1\,\cos y_1-\sin x_1+\sin y_1\right)+\hat{\theta}_2 e_2\left(x_3^2\sin x_1\,\cos x_1-y_3^2\sin y_1\,\cos y_1\right)$ $+\hat{\theta}_3 e_2\left(\sin x_1-\sin y_1\right)-\hat{\theta}_4 e_2^2+\hat{\theta}_5 e_3\left(\cos x_1-\cos y_1\right), e_1^2/\|\boldsymbol{e}\|^2+e_2^2\|\boldsymbol{e}\|^2+e_3^2\|\boldsymbol{e}\|^2=1$ and $e_i\operatorname{sgn}(e_i)=|e_i|$, one has

$$
\dot{V}_1(t)\le-\eta_1\left|e_1\right|-\eta_2\left|e_2\right|-\eta_3\left|e_3\right|-\mu\left(\Theta+\|\hat{\boldsymbol{\theta}}\|\right). \qquad \text{(A11)}
$$

From Lemma 2, we can obtain

$$
\dot{V}_1(t)\le-\mu\|\boldsymbol{e}\|-\mu\left(\Theta+\|\hat{\boldsymbol{\theta}}\|\right). \qquad \text{(A12)}
$$

On the basis of Assumption 2 and $\|\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\|\le\|\hat{\boldsymbol{\theta}}\|+\|\boldsymbol{\theta}\|\le\|\hat{\boldsymbol{\theta}}\|+\Theta$, we have

$$
\dot{V}_1(t)\le-\mu\left(\|\boldsymbol{e}\|+\|\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\|\right). \qquad \text{(A13)}
$$

Again according to Lemma 2, one has

$$
\dot{V}_1(t)\le-\sqrt{2}\mu\left(\frac{1}{2}\left(\|\boldsymbol{e}\|^2+\|\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\|^2\right)\right)^{1/2}=-\sqrt{2}\mu V_1^{1/2}(t). \qquad \text{(A14)}
$$

Therefore, from Lemma 1, the error system trajectories $\boldsymbol{e}(t)$ of Eq. (6) will converge to zero, in the finite time

$$
T_1=\frac{\sqrt{2}}{\mu}\left(\frac{1}{2}\left(\|\boldsymbol{e}(0)\|^2+\|\hat{\boldsymbol{\theta}}(0)-\boldsymbol{\theta}\|^2\right)\right)^{1/2}.
$$

Hence, the chaotic non-autonomous CFGs (2) and (3) with identical unknown parameters, model uncertainties and external disturbances will be finite-time synchronized and the proof is achieved completely.

Appendix B

Proof of Theorem 2 Selecting a positive definite function in the form of

$$
V_2(t)=\frac{1}{2}\|\boldsymbol{e}\|^2+\frac{1}{2}\|\hat{\boldsymbol{\theta}}_{\mathrm{m}}-\boldsymbol{\theta}_{\mathrm{m}}\|^2+\frac{1}{2}\|\hat{\boldsymbol{\theta}}_{\mathrm{s}}-\boldsymbol{\theta}_{\mathrm{s}}\|^2
$$

and taking its derivative with respect to time, one has

$$
\dot{V}_2(t)=\sum_{i=1}^{3}e_i\dot{e}_i+\left(\hat{\boldsymbol{\theta}}_{\mathrm{m}}-\boldsymbol{\theta}_{\mathrm{m}}\right)^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{m}}+\left(\hat{\boldsymbol{\theta}}_{\mathrm{s}}-\boldsymbol{\theta}_{\mathrm{s}}\right)^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{s}}. \qquad \text{(B1)}
$$

Replacing for $\dot{e}_i$ from Eq. (16), we have

$$
\begin{aligned}
\dot{V}_2(t)=\;& e_1\left(e_2+\Delta f_1(\boldsymbol{x},t)-\Delta g_1(\boldsymbol{y},t)-u_1(t)\right)+e_2\left(\left(E_{\mathrm{m}}+n_{\mathrm{m}}^2x_3^2\right)\left(\sin x_1\,\cos x_1\right)-\left(E_{\mathrm{m}}+g_{\mathrm{m}}/l_{\mathrm{m}}\right)\sin x_1\right.\\
&-b_{\mathrm{m}}x_2-\left(E_{\mathrm{s}}+n_{\mathrm{s}}^2y_3^2\right)\left(\sin y_1\,\cos y_1\right)+\left(E_{\mathrm{s}}+g_{\mathrm{s}}/l_{\mathrm{s}}\right)\sin y_1-b_{\mathrm{s}}y_2+\Delta f_2(\boldsymbol{x},t)-\Delta g_2(\boldsymbol{y},t)-\left.u_2(t)\right)\\
&+e_3((\alpha_{\mathrm{m}}\cos x_1-F_{\mathrm{m}})/I_{\mathrm{m}}-a_{\mathrm{m}}\sin w_{\mathrm{m}}t-(\alpha_{\mathrm{s}}\cos y_1-F_{\mathrm{s}})/I_{\mathrm{s}}-a_{\mathrm{s}}\sin w_{\mathrm{s}}t+\Delta f_3(\boldsymbol{x},t)-\Delta g_3(\boldsymbol{y},t)\\
&-u_3(t))+(\hat{\boldsymbol{\theta}}_{\mathrm{m}}-\boldsymbol{\theta}_{\mathrm{m}})^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{m}}+(\hat{\boldsymbol{\theta}}_{\mathrm{s}}-\boldsymbol{\theta}_{\mathrm{s}})^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{s}}. \qquad \text{(B2)}
\end{aligned}
$$

It is clear that

$$
\dot{V}_2(t)\le e_1\left(e_2-u_1(t)\right)+|e_1|\left|\Delta f_1(\boldsymbol{x},t)-\Delta g_1(\boldsymbol{y},t)\right|+e_2\left(\left(E_{\mathrm{m}}+n_{\mathrm{m}}^2x_3^2\right)\left(\sin x_1\,\cos x_1\right)-\left(E_{\mathrm{m}}+g_{\mathrm{m}}/l_{\mathrm{m}}\right)\sin x_1\right.
$$

$$-b_{\mathrm{m}}x_2-\left(E_{\mathrm{s}}+n_{\mathrm{s}}^2y_3^2\right)(\sin y_1\,\cos y_1)+(E_{\mathrm{s}}+g_{\mathrm{s}}/l_{\mathrm{s}})\sin y_1-b_{\mathrm{s}}y_2-u_2(t)\big)+|e_2|\,|\Delta f_2(\boldsymbol{x},t)-\Delta g_2(\boldsymbol{y},t)|$$
$$+e_3\left((\alpha_{\mathrm{m}}\cos x_1-F_{\mathrm{m}})/I_{\mathrm{m}}-(\alpha_{\mathrm{s}}\cos y_1-F_{\mathrm{s}})/I_{\mathrm{s}}-u_3(t)\right)+|e_3|\left(|a_{\mathrm{m}}|+|a_{\mathrm{s}}|\right)+|e_3|\,|\Delta f_3(\boldsymbol{x},t)$$
$$-\Delta g_3(\boldsymbol{y},t)|+(\hat{\boldsymbol{\theta}}_{\mathrm{m}}-\boldsymbol{\theta}_{\mathrm{m}})^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{m}}+(\hat{\boldsymbol{\theta}}_{\mathrm{s}}-\boldsymbol{\theta}_{\mathrm{s}})^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{s}}. \tag{B3}$$

Using $\boldsymbol{\theta}_{\mathrm{m}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{m}}=E_{\mathrm{m}}e_2\left(\sin x_1\,\cos x_1-\sin x_1\right)+n_{\mathrm{m}}^2e_2\left(x_3^2\sin x_1\,\cos x_1\right)-(g_{\mathrm{m}}/l_{\mathrm{m}})e_2\sin x_1-b_{\mathrm{m}}e_2x_2+(\alpha_{\mathrm{m}}/I_{\mathrm{m}})\,e_3\cos x_1-(F_{\mathrm{m}}/I_{\mathrm{m}})\,e_3+|a_{\mathrm{m}}|\,|e_3|$, $\boldsymbol{\theta}_{\mathrm{s}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{s}}=E_{\mathrm{s}}e_2\left(-\sin y_1\,\cos y_1+\sin y_1\right)-n_{\mathrm{s}}^2e_2\left(y_3^2\sin y_1\,\cos y_1\right)+(g_{\mathrm{s}}/l_{\mathrm{s}})e_2\sin y_1+b_{\mathrm{s}}e_2y_2-(\alpha_{\mathrm{s}}/I_{\mathrm{s}})\,e_3\cos y_1-(F_{\mathrm{s}}/I_{\mathrm{s}})\,e_3+|a_{\mathrm{s}}|\,|e_3|$ and inequality (5), one obtains

$$\dot{V}_2(t)\le e_1\left(e_2-u_1(t)\right)+d_1\,|e_1|-e_2u_2(t)+d_2\,|e_2|-e_3u_3(t)+d_3\,|e_3|+\hat{\boldsymbol{\theta}}_{\mathrm{m}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{m}}+\hat{\boldsymbol{\theta}}_{\mathrm{s}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{s}}. \tag{B4}$$

Using the control laws $u_1(t)$, $u_2(t)$ and $u_3(t)$ in Eq. (18), we have

$$\begin{aligned}\dot{V}_2(t)\le{}& e_1\left(e_2-\left[e_2+\mu\left(\varTheta_{\mathrm{m}}+\|\hat{\boldsymbol{\theta}}_{\mathrm{m}}\|+\varTheta_{\mathrm{s}}+\|\hat{\boldsymbol{\theta}}_{\mathrm{s}}\|\right)\left(\frac{e_1}{\|\boldsymbol{e}\|^2}\right)+(d_1+\eta_1)\,\mathrm{sgn}(e_1)\right]\right)+d_1\,|e_1|\\ &-e_2\Bigg[\hat{\theta}_{\mathrm{m}1}\left(\sin x_1\cos x_1-\sin x_1\right)-\hat{\theta}_{\mathrm{s}1}\left(\sin y_1\cos y_1-\sin y_1\right)+\hat{\theta}_{\mathrm{m}2}x_3^2\sin x_1\,\cos x_1-\hat{\theta}_{\mathrm{s}2}y_3^2\sin y_1\,\cos y_1\\ &-\hat{\theta}_{\mathrm{m}3}\sin x_1+\hat{\theta}_{\mathrm{s}3}\sin y_1-\hat{\theta}_{\mathrm{m}4}x_2+\hat{\theta}_{\mathrm{s}4}y_2+\mu\left(\varTheta_{\mathrm{m}}+\|\hat{\boldsymbol{\theta}}_{\mathrm{m}}\|+\varTheta_{\mathrm{s}}+\|\hat{\boldsymbol{\theta}}_{\mathrm{s}}\|\right)\left(\frac{e_2}{\|\boldsymbol{e}\|^2}\right)+(d_2+\eta_2)\,\mathrm{sgn}(e_2)\Bigg]\\ &+d_2\,|e_2|-e_3\Bigg[\hat{\theta}_{\mathrm{m}5}\cos x_1-\hat{\theta}_{\mathrm{s}5}\cos y_1-\hat{\theta}_{\mathrm{m}6}+\hat{\theta}_{\mathrm{s}6}+\left(\hat{\theta}_{\mathrm{m}7}+\hat{\theta}_{\mathrm{s}7}\right)\mathrm{sgn}(e_3)\\ &+\mu\left(\varTheta_{\mathrm{m}}+\|\hat{\boldsymbol{\theta}}_{\mathrm{m}}\|+\varTheta_{\mathrm{s}}+\|\hat{\theta}_{\mathrm{s}}\|\right)\left(\frac{e_3}{\|\boldsymbol{e}\|^2}\right)\Bigg]+(d_3+\eta_3)\,\mathrm{sgn}(e_3)+d_3\,|e_3|+\hat{\boldsymbol{\theta}}_{\mathrm{m}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{m}}+\hat{\boldsymbol{\theta}}_{\mathrm{s}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{s}}.\end{aligned} \tag{B5}$$

By means of $\hat{\boldsymbol{\theta}}_{\mathrm{m}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{m}}=\hat{\theta}_{\mathrm{m}1}e_2\left(\sin x_1\,\cos x_1-\sin x_1\right)+\hat{\theta}_{\mathrm{m}2}e_2\left(x_3^2\sin x_1\,\cos x_1\right)-\hat{\theta}_{\mathrm{m}3}\sin x_1-\hat{\theta}_{\mathrm{m}4}e_2x_2+\hat{\theta}_{\mathrm{m}5}e_3\cos x_1-\hat{\theta}_{\mathrm{m}6}e_3+\hat{\theta}_{\mathrm{m}7}\,|e_3|$, $\hat{\boldsymbol{\theta}}_{\mathrm{s}}^{\mathrm{T}}\dot{\hat{\boldsymbol{\theta}}}_{\mathrm{s}}=\hat{\theta}_{\mathrm{s}1}e_2\left(-\sin y_1\,\cos y_1+\sin y_1\right)-\hat{\theta}_{\mathrm{s}2}e_2\left(y_3^2\sin y_1\,\cos y_1\right)+\hat{\theta}_{\mathrm{s}3}e_2\sin y_1+\hat{\theta}_{\mathrm{s}4}e_2y_2-\hat{\theta}_{\mathrm{s}5}e_3\cos y_1-\hat{\theta}_{\mathrm{s}6}e_3+\hat{\theta}_{\mathrm{s}7}\,|e_3|$, $e_1^2/\|\boldsymbol{e}\|^2+e_2^2/\|\boldsymbol{e}\|^2+e_3^2/\|\boldsymbol{e}\|^2=1$ and $e_i\mathrm{sgn}(e_i)=|e_i|$, we have

$$\dot{V}_2(t)\le-\eta_1\,|e_1|-\eta_2\,|e_2|-\eta_3\,|e_3|-\mu\left(\varTheta_{\mathrm{m}}+\|\hat{\boldsymbol{\theta}}_{\mathrm{m}}\|+\varTheta_{\mathrm{s}}+\|\hat{\boldsymbol{\theta}}_{\mathrm{s}}\|\right). \tag{B6}$$

Using the fact in Lemma 2 yields

$$\dot{V}_2(t)\le-\mu\|\boldsymbol{e}\|-\mu\left(\varTheta_{\mathrm{m}}+\|\hat{\boldsymbol{\theta}}_{\mathrm{m}}\|+\varTheta_{\mathrm{s}}+\|\hat{\boldsymbol{\theta}}_{\mathrm{s}}\|\right). \tag{B7}$$

By Assumption 2 and inequality $\|\hat{\boldsymbol{\theta}}_i-\boldsymbol{\theta}_i\|\le\|\hat{\boldsymbol{\theta}}_i\|+\|\boldsymbol{\theta}_i\|\le\|\hat{\boldsymbol{\theta}}_i\|+\varTheta_i$, $i=\mathrm{m},\,\mathrm{s}$, one obtains

$$\dot{V}_2(t)\le-\mu\left(\|\boldsymbol{e}\|+\|\hat{\boldsymbol{\theta}}_{\mathrm{m}}-\theta_{\mathrm{m}}\|+\|\hat{\boldsymbol{\theta}}_{\mathrm{s}}-\boldsymbol{\theta}_{\mathrm{s}}\|\right). \tag{B8}$$

Using Lemma 2, we have

$$\dot{V}_2(t)\le-\sqrt{2}\mu\left(\frac{1}{2}\left(\|\boldsymbol{e}\|^2+\|\hat{\boldsymbol{\theta}}_{\mathrm{m}}-\boldsymbol{\theta}_{\mathrm{m}}\|^2+\|\hat{\boldsymbol{\theta}}_{\mathrm{s}}-\boldsymbol{\theta}_{\mathrm{s}}\|^2\right)\right)^{1/2}=-\sqrt{2}\mu V_2^{1/2}(t). \tag{B9}$$

As a result, according to Lemma 1, the error trajectories $\boldsymbol{e}(t)$ of Eq. (16) will converge to zero, in the finite time

$$T_2=\frac{\sqrt{2}}{\mu}\left(\frac{1}{2}\left(\|\boldsymbol{e}(0)\|^2+\|\hat{\boldsymbol{\theta}}_{\mathrm{m}}(0)-\boldsymbol{\theta}_{\mathrm{m}}\|^2+\|\hat{\boldsymbol{\theta}}_{\mathrm{s}}(0)-\boldsymbol{\theta}_{\mathrm{s}}\|^2\right)\right)^{1/2}.$$

Therefore, the chaotic non-autonomous CFGs (14) and (15) with unknown parameters of different values, model uncertainties and external disturbances will be finite-time synchronized and the proof is complete.

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