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Letter

Quantum nonlocality without entanglement in a 2*n*-partite system

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Abstract

In recent years, researchers focused their attention on the construction of nonlocal product states in multipartite quantum systems. This paper proposes a novel partitioning method for multipartite quantum systems, aiming to improve the operation efficiency. Firstly, we divide $2n$ subsystems into n parts two by two and implement orthogonality-preserving local measurement on the partitioned composite systems. Subsequently, based on the partitioning mode, nonlocal orthogonal product states in $(\mathbb{C}^3)^{\otimes 6}$ and $(\mathbb{C}^4)^{\otimes 6}$ are given. Finally, we construct nonlocal orthogonal product states in $(\mathbb{C}^d)^{\otimes 2n}$ and discuss the cases where d is odd and even. Our results demonstrate the phenomenon of nonlocality without entanglement in a $2n$-partite system.

Keywords: Hilbert space, quantum nonlocal, multipartite systems, product basis

1. Introduction

The local discrimination of quantum states is a fundamental problem in quantum information theory. When a set of orthogonal states cannot be distinguished through local operations and classical communication (LOCC), it is referred to as locally indistinguishable [1, 2]. Entangled states are nonlocal because they violate Bell inequalities [3–5]. Locally indistinguishable sets also exhibit quantum nonlocality, which is different from Bell nonlocality. Consequently, locally indistinguishable sets have found useful application in quantum cryptography primitives such as quantum secret sharing [6] and quantum information masking [7, 8].

In 1999, Bennett *et al* [9] firstly presented an initial result by demonstrating a LOCC indistinguishable orthogonal product basis in $\mathbb{C}^3 \otimes \mathbb{C}^3$. Under their influence many orthogonal product sets with quantum nonlocality were provided in bipartite [10, 11] and multipartite systems [12–26]. Zhang *et al* [27] found a set of orthogonal product states which cannot be locally distinguished in $\mathbb{C}^m \otimes \mathbb{C}^n$, where $3 \leqslant m \leqslant n$. Afterwards, significant advancements were achieved in multipartite quantum systems. For instance, Zhang *et al* [28] provided an important method to verify the local indistinguishability of orthogonal product states and showed that no matter which local party goes first to be operated, only trivial measurements can be performed. Subsequently, Zhen *et al* [29] successfully proposed the current minimum number of locally indistinguishable quantum states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$, for $3 \leqslant d_1 \leqslant d_2 \cdots \leqslant d_n$ and $n \geqslant 3$. In addition, Cao *et al* [30] presented the locally stable sets and proved that the number of quantum states constructed by them reaches the lower bound of the cardinality.

Nowadays, nonlocal multipartite orthogonal product states continue to be extensively researched. However, the research mentioned above focused on performing local measurements on a single quantum subsystem, and researchers did not consider the case of local measurements on two or multiple subsystems. Hence, we investigate whether local measurements can be performed on two subsystems. In this paper, we provide a new partitioning method in multipartite quantum systems. Under this division scheme, two subsystems

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constitute a composite system. We handle the two subsystems together, improving work efficiency. Thus, this method is very practical and effective. Orthogonal quantum states that involve entanglement will increase the difficulty of distinguishing them using LOCC. Therefore, the without entanglement non-local orthogonal product states are based on the partitioning condition. We construct a set of orthogonal product states in specific quantum systems, including$(\mathbb{C}^3)^{\otimes 6}$ and $(\mathbb{C}^4)^{\otimes 6}$. Then we separately discuss the situations in $(\mathbb{C}^d)^{\otimes 2n}$ where d is odd and even.

The organization of the paper is as follows. In section 2, we provide a new partitioning method in multipartite quantum systems. In section 3, we construct a set of orthogonal product states in specific quantum systems: $(\mathbb{C}^3)^{\otimes 6}$ and $(\mathbb{C}^4)^{\otimes 6}$. In section 4, under the divide method, we construct a set of orthogonal product states in $(\mathbb{C}^d)^{\otimes 2n}$ which is indistinguishable by LOCC. We conclude the paper with a brief summary in section 5.

2. Preliminaries

In this section, we present a novel partitioning scheme for multipartite quantum systems. Subsequently, we provide a comprehensive review of the relevant knowledge used in this paper.

Consider a composite quantum system $\mathcal{H} = \bigotimes_{k=1}^{2n} \mathcal{H}_k$, where $n \geqslant 3$, and $dim\ \mathcal{H}_k \geqslant 3$ for $k = 1, 2, \ldots, 2n$. We divide those $2n$ subsystems into n parts two by two, i.e.,

$$\underbrace{(\mathcal{H}\otimes\mathcal{H})\otimes(\mathcal{H}\otimes\mathcal{H})\otimes\cdots\otimes(\mathcal{H}\otimes\mathcal{H})\otimes(\mathcal{H}\otimes\mathcal{H})}_{2n}. \tag{1}$$

For convenience, the above formula is written as:

$$(\mathcal{H}\otimes\mathcal{H})_1\otimes(\mathcal{H}\otimes\mathcal{H})_2\otimes\cdots\otimes(\mathcal{H}\otimes\mathcal{H})_{n-1}\otimes(\mathcal{H}\otimes\mathcal{H})_n. \tag{2}$$

We take the computational basis $\{|i\rangle\}_{i=0}^{d-1}$ for each subsystem $\mathcal{H}_k$. On the composite system $\mathcal{H}\otimes\mathcal{H}$, we denote the basis $\{|i\rangle\otimes|j\rangle\}_{i,j=0}^{d-1}$ as $\{|ij\rangle\}_{i,j=0}^{d-1}$. Throughout this paper, we exclusively focus on pure states, and we do not normalize states for simplicity.

A positive operator valued measure (POVM) on Hilbert space $\mathcal{H}$ is a set of positive operators $\{E_m = M_m^\dagger M_m\}$ such that $\sum_m E_m = \mathbb{I}_m$, where each E_m is called a POVM element, and $\mathbb{I}$ is the identity operator on $\mathcal{H}$.

A measurement is nontrivial if not all the POVM elements are proportional to the identity operator. Otherwise, the measurement is trivial. If all the POVMs are trivial, the set of orthogonal quantum states cannot be distinguished by LOCC.

Based on this particular division, we perform local POVM measurements on two subsystems. Consider a pair of multipartite orthogonal product states $\{|\phi_i\rangle, |\phi_j\rangle\}_{i,j=0}^{d-1}$ whose $n-1$ composite subsystems are not mutually orthogonal except for the tth composite subsystem. A local POVM is preformed on the tth composite subsystem and identity operators are preformed on the rest of the $n-1$ composite subsystems. The postmeasurement states $\{\mathbb{I}_1\otimes\cdots\otimes M_t\otimes\cdots\mathbb{I}_n|\phi_i\rangle\}$ and $\{\mathbb{I}_1\otimes\cdots\otimes M_t\otimes\cdots\mathbb{I}_n|\phi_j\rangle\}$ should be mutually orthogonal, we have

$$\langle\phi_i|\left(\mathbb{I}_1\otimes\cdots\otimes M_t^\dagger M_t\otimes\cdots\mathbb{I}_n\right)|\phi_j\rangle = 0. \tag{3}$$

Each POVM element $M_t^\dagger M_t$ can be expressed as a $d^2\times d^2$ matrix E_t in the computational basis of $\{|ij\rangle\}_{i,j=0}^{d-1}$. From equation (3) , we obtain the elements of matrix E_t. Furthermore, we can determine whether the POVM measurement is trivial. This approach enables us to proficiently analyze and manipulate the composite system as a higher-dimensional. This is pivotal for comprehending the characteristics of multipartite quantum systems and holds implications for diverse quantum information processing tasks. Consequently, our research predominantly operates within this division.

3. Construction of nonlocal states in $(\mathbb{C}^3)^{\otimes 6}$ and$(\mathbb{C}^4)^{\otimes 6}$

Under the partitioning method described in section 2, composite quantum system $(\mathbb{C}^3)^{\otimes 6}$ and $(\mathbb{C}^4)^{\otimes 6}$ divided into

$$\left(\mathbb{C}^3\otimes\mathbb{C}^3\right)_1\otimes\left(\mathbb{C}^3\otimes\mathbb{C}^3\right)_2\otimes\left(\mathbb{C}^3\otimes\mathbb{C}^3\right)_3 \tag{4}$$

and

$$\left(\mathbb{C}^4\otimes\mathbb{C}^4\right)_1\otimes\left(\mathbb{C}^4\otimes\mathbb{C}^4\right)_2\otimes\left(\mathbb{C}^4\otimes\mathbb{C}^4\right)_3. \tag{5}$$

We first construct a nonlocal set of orthogonal product states in $(\mathbb{C}^3\otimes\mathbb{C}^3)_1\otimes(\mathbb{C}^3\otimes\mathbb{C}^3)_2\otimes(\mathbb{C}^3\otimes\mathbb{C}^3)_3$.

Theorem 1. *The set of orthogonal product quantum states is distinguishable by LOCC in* $(\mathbb{C}^3\otimes\mathbb{C}^3)_1\otimes(\mathbb{C}^3\otimes\mathbb{C}^3)_2\otimes(\mathbb{C}^3\otimes\mathbb{C}^3)_3$ *:*

$$\begin{aligned}
|\alpha^1_{\pm i,0}\rangle &= |i0\rangle_1(|00\rangle\pm|i0\rangle)_2|00\rangle_3,\\
|\alpha^2_{\pm i,0}\rangle &= |00\rangle_1|i0\rangle_2(|00\rangle\pm|i0\rangle)_3,\\
|\alpha^3_{\pm i,0}\rangle &= (|00\rangle\pm|i0\rangle)_1|00\rangle_2|i0\rangle_3,\\
|\beta^1_{\pm m,i}\rangle &= |mi\rangle_1(|m0\rangle\pm|mi\rangle)_2|00\rangle_3,\\
|\beta^2_{\pm m,i}\rangle &= |00\rangle_1|mi\rangle_2(|m0\rangle\pm|mi\rangle)_3,\\
|\beta^3_{\pm m,i}\rangle &= (|m0\rangle\pm|mi\rangle)_1|00\rangle_2|mi\rangle_3,\\
|\gamma^1_{\pm i,1}\rangle &= (|i1\rangle\pm|i2\rangle)_1|00\rangle_2|00\rangle_3,\\
|\gamma^2_{\pm i,1}\rangle &= |00\rangle_1(|i1\rangle\pm|i2\rangle))_2|00\rangle_3,\\
|\gamma^3_{\pm i,1}\rangle &= |00\rangle_1|00\rangle_2(|i1\rangle\pm|i2\rangle)_3,
\end{aligned} \tag{6}$$

where $0\leqslant m\leqslant 2,\ 1\leqslant i\leqslant 2.$

Proof. In order to prove the nonlocality of the set of multipartite orthogonal product states, we perform local POVM on these states such that the postmeasurement states remain orthogonal. Due to the symmetric of the states, we only need to prove that any orthogonality-preserving local POVM performed on the first composite subsystem must be trivial. We perform a local POVM measurement on the first composite subsystems and apply identity operator to the second and third composite subsystems. Let the POVM measurement operator

$\mathbf{\Pi}_1$ describe a general orthogonality-preserving measurement on the first composite subsystem. Each POVM element consists of nine block matrices. Assume one element in $\mathbf{\Pi}_1$ is as follows,

$$\left(\begin{array}{ccc} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{array}\right). \tag{7}$$

Each block matrix $A_{pq}(0 \leqslant p,q \leqslant 2)$ can be written as a 3×3 matrix on the $\{|0\rangle, |1\rangle, |2\rangle\}$ basis,

$$\left(\begin{array}{ccc} a_{pq,00} & a_{pq,01} & a_{pq,02} \\ a_{pq,10} & a_{pq,11} & a_{pq,12} \\ a_{pq,20} & a_{pq,21} & a_{pq,22} \end{array}\right). \tag{8}$$

First, we consider the zero entries of matrices $A_{pq}(p=q)$. A pair of multipartite orthogonal product states, denoted as $|\beta^1_{+0,i}\rangle$and $|\beta^2_{+0,i}\rangle$, where the first composite subsystem are mutually orthogonal, while the second and third composite subsystems are not mutually orthogonal. Since the postmeasurement states $M_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 |\beta^1_{+0,i}\rangle$ and $M_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 |\beta^2_{+0,i}\rangle$ should be mutually orthogonal, we have $\langle \beta^1_{0,i}|(\mathbf{\Pi}_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3)|\beta^2_{0,i}\rangle = 0$, which implies $a_{00,0i} = a_{00,i0} = 0$. For $|\beta^1_{+m,i}\rangle, |\beta^1_{+m,j}\rangle$, we have the ability to acquire $a_{mm,ij} = a_{mm,ji} = 0$, where $1 \leqslant i \neq j \leqslant 2$. For $|\alpha^1_{+i,0}\rangle, |\gamma^1_{+i,1}\rangle$, we can obtain $a_{ii,01} = a_{ii,10} = 0, a_{ii,02} = a_{ii,20} = 0$.

Second, we seek the zero elements of matrices $A_{pq}(p \neq q \neq 0)$. Considering the states $|\alpha^1_{+i,0}\rangle$ and $|\alpha^1_{+j,0}\rangle$ $(1 \leqslant i \neq j \leqslant 2)$, we directly get $a_{ij,00} = a_{ji,00} = 0$. For $|\gamma^1_{+i,1}\rangle, |\gamma^1_{+j,1}\rangle$, we are able to obtain

$$\begin{aligned} &a_{ij,11} = a_{ji,11} = 0, a_{ij,22} = a_{ji,22} = 0, \\ &a_{ij,21} = a_{ji,12)} = 0, a_{ij,12} = a_{ji,21} = 0. \end{aligned} \tag{9}$$

For $|\alpha^1_{+i,0}\rangle, |\gamma^1_{+j,1}\rangle$, we know

$$a_{ij,01} = a_{ji,10} = 0, a_{ij,02} = a_{ji,20} = 0, \tag{10}$$

where $1 \leqslant i \neq j \leqslant 2$.

Third, we want to find the zero elements of matrices A_{p0}, A_{0p} $(p \neq 0)$. For state $|\beta^1_{+0,i}\rangle, |\alpha^1_{+k,0}\rangle$ $(1 \leqslant i,k \leqslant 2)$, $a_{0k,i0} = a_{k0,0i} = 0$ can be deduced. From states $|\alpha^1_{+i,0}\rangle$ and $|\alpha^2_{+i,0}\rangle$, we directly deduce that $a_{0i,00} = a_{i0,00} = 0$. For state $|\beta^1_{+i,1}\rangle, |\beta^1_{+i,2}\rangle, |\gamma^2_{+i,1}\rangle$, we can obtain

$$a_{0i,01} = a_{i0,10} = 0, a_{0i,02} = a_{i0,20} = 0. \tag{11}$$

For state $|\beta^1_{+0,i}\rangle, |\gamma^1_{+k,1}\rangle$ $(1 \leqslant i,k \leqslant 2)$, we have

$$a_{0k,i1} = a_{k0,1i} = 0, a_{0k,i2} = a_{k0,2i} = 0. \tag{12}$$

All the discussion above shows that all off-diagonal elements are equal to 0.

Last, we have the relations among the diagonal entries of $\mathbf{\Pi}_1$. For instance, from the states $|\alpha^3_{\pm i,0}\rangle$, we know

$$\langle \alpha^3_{+i,0}|\mathbf{\Pi}_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3|\alpha^3_{-i,0}\rangle = 0, \tag{13}$$

Table 1. Off-diagonal elements of K_1.

States	Off-diagonal elements	Range
$\lvert\beta^1_{+0,i}\rangle, \lvert\beta^2_{+0,i}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\zeta^1_{+i,1}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\varphi^1_{+i,1}\rangle$	$a_{00,0i}, a_{00,i0}$ $a_{ii,02}, a_{ii,20}, a_{ii,03}, a_{ii,30}$ $a_{ii,01}, a_{ii,10}$	$1 \leqslant i \leqslant 3$
$\lvert\beta^1_{+m,i}\rangle, \lvert\beta^1_{+m,j}\rangle$	$a_{mm,ij}, a_{mm,ji}$	$1 \leqslant i \neq j \leqslant 3$
$\lvert\alpha^1_{+i,0}\rangle, \lvert\alpha^1_{+j,0}\rangle$ $\lvert\zeta^1_{+i,1}\rangle, \lvert\zeta^1_{+j,1}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\zeta^1_{+j,1}\rangle$ $\lvert\zeta^1_{+i,1}\rangle, \lvert\varphi^1_{+j,1}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\varphi^1_{+j,1}\rangle$ $\lvert\varphi^1_{+i,1}\rangle, \lvert\varphi^1_{+j,1}\rangle$	$a_{ij,00}, a_{00,ji}$ $a_{ij,22}, a_{ji,22}, a_{ij,33}, a_{ji,33}$ $a_{ij,23}, a_{ji,32}, a_{ij,32}, a_{ji,23}$ $a_{ij,02}, a_{ji,20}, a_{ij,03}, a_{ji,30}$ $a_{ij,21}, a_{ji,12}, a_{ij,13}, a_{ji,31}$ $a_{ij,01}, a_{ji,10}$ $a_{ij,11}, a_{ji,11}$	$1 \leqslant i \neq j \leqslant 3$
$\lvert\beta^1_{+i,1}\rangle, \lvert\varphi^2_{+i,1}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\alpha^2_{+i,0}\rangle$ $\lvert\beta^1_{+i,2}\rangle, \lvert\beta^1_{+i,3}\rangle, \lvert\zeta^2_{+i,1}\rangle$	$a_{0i,01}, a_{i0,10}$ $a_{0i,00}, a_{i0,00}$ $a_{0i,02}, a_{i0,20}, a_{0i,03}, a_{i0,30}$	$1 \leqslant i \leqslant 3$
$\lvert\beta^1_{+0,i}\rangle, \lvert\zeta^1_{+k,1}\rangle$ $\lvert\beta^1_{+0,i}\rangle, \lvert\varphi^1_{+k,1}\rangle$ $\lvert\beta^1_{+0,i}\rangle, \lvert\alpha^1_{+k,0}\rangle$	$a_{0k,i2}, a_{k0,2i}, a_{0k,i3}, a_{k0,3i}$ $a_{0k,i1}, a_{k0,1i}$ $a_{0k,i0}, a_{k0,0i}$	$1 \leqslant i,k \leqslant 3$

which results in $a_{00,00} = a_{ii,00}$. For states $|\beta^3_{\pm m,i}\rangle$, we can get

$$\langle \beta^3_{+m,i}|\mathbf{\Pi}_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3|\beta^3_{-m,i}\rangle = 0, \tag{14}$$

which implies $a_{mm,00} = a_{mm,ii}$. That is, all the diagonal elements are equal. Thus, nobody can start with a nontrivial measurement on any subsystem. This completes the proof. $\square$

Then,we consider another quantum system $(\mathbb{C}^4)^{\otimes 6}$ and propose a nonlocal set of orthogonal product states in $(\mathbb{C}^4 \otimes \mathbb{C}^4)_1 \otimes (\mathbb{C}^4 \otimes \mathbb{C}^4)_2 \otimes (\mathbb{C}^4 \otimes \mathbb{C}^4)_3$.

Theorem 2. *The set of orthogonal product quantum states in* $(\mathbb{C}^4 \otimes \mathbb{C}^4)_1 \otimes (\mathbb{C}^4 \otimes \mathbb{C}^4)_2 \otimes (\mathbb{C}^4 \otimes \mathbb{C}^4)_3$ *cannot be perfectly distinguished by LOCC:*

$$\begin{aligned} |\varphi^1_{\pm i,1}\rangle &= |i1\rangle_1 |00\rangle_2 |00\rangle_3, \\ |\varphi^2_{\pm i,1}\rangle &= |00\rangle_1 |i1\rangle_2 |00\rangle_3, \\ |\varphi^3_{\pm i,1}\rangle &= |00\rangle_1 |00\rangle_2 |i1\rangle_3, \\ |\alpha^1_{\pm i,0}\rangle &= |i0\rangle_1 \left(|00\rangle \pm |i0\rangle\right)_2 |00\rangle_3, \\ |\alpha^2_{\pm i,0}\rangle &= |00\rangle_1 |i0\rangle_2 \left(|00\rangle \pm |i0\rangle\right)_3, \\ |\alpha^3_{\pm i,0}\rangle &= \left(|00\rangle \pm |i0\rangle\right)_1 |00\rangle_2 |i0\rangle_3, \\ |\beta^1_{\pm m,i}\rangle &= |mi\rangle_1 \left(|m0\rangle \pm |mi\rangle\right)_2 |00\rangle_3, \\ |\beta^2_{\pm m,i}\rangle &= |00\rangle_1 |mi\rangle_2 \left(|m0\rangle \pm |mi\rangle\right)_3, \\ |\beta^3_{\pm m,i}\rangle &= \left(|m0\rangle \pm |mi\rangle\right)_1 |00\rangle_2 |mi\rangle_3, \\ |\zeta^1_{\pm i,1}\rangle &= \left(|i2\rangle \pm |i3\rangle\right)_1 |00\rangle_2 |00\rangle_3, \\ |\zeta^2_{\pm i,1}\rangle &= |00\rangle_1 \left(|i2\rangle \pm |i3\rangle\right)_2 |00\rangle_3, \\ |\zeta^3_{\pm i,1}\rangle &= |00\rangle_1 |00\rangle_2 \left(|i2\rangle \pm |i3\rangle\right)_3, \end{aligned} \tag{15}$$

where $0 \leqslant m \leqslant 3$, $1 \leqslant i \leqslant 3$.

Proof. Let the POVM K_1 describe a general orthogonality-preserving measurement on the first composite subsystem. By employing the proof methodology outlined in theorem 1, we can demonstrate that all off-diagonal elements of K_1 are equal to 0 (further details are provided in table 1).

From the states $|\alpha^3_{\pm i,0}\rangle$, we know

$$\langle\alpha^3_{+i,0}|K_1\otimes\mathbb{I}_2\otimes\mathbb{I}_3|\alpha^3_{-i,0}\rangle=0, \tag{16}$$

which implies $a_{mm,00}=a_{mm,ii}$. For state $|\beta^3_{\pm m,i}\rangle$, we know

$$\langle\beta^3_{+m,i}|K_1\otimes\mathbb{I}_2\otimes\mathbb{I}_3|\beta^3_{-m,i}\rangle=0, \tag{17}$$

which implies $a_{00,00}=a_{ii,00}$. In conclusion, all parties cannot start with a nontrivial measurement. The set of orthogonal product states is indistinguishable by LOCC. □

4. Construction of nonlocal states in $(\mathbb{C}^d)^{\otimes 2n}$

According to theorems 1 and 2, we propose the general construction on $(\mathbb{C}^d)^{\otimes 2n}$. When the dimension of the subsystem is odd or even, the constructed quantum state will also exhibit differences. We first show our construction in $(\mathbb{C}^d\otimes\mathbb{C}^d)_1\otimes\cdots\otimes(\mathbb{C}^d\otimes\mathbb{C}^d)_n$, where d is odd.

Theorem 3. *The following orthogonal product states in $(\mathbb{C}^d\otimes\mathbb{C}^d)_1\otimes\cdots\otimes(\mathbb{C}^d\otimes\mathbb{C}^d)_n$ cannot be perfectly distinguished by LOCC:*

$$\begin{aligned}
|\alpha^w_{\pm i,0}\rangle&=|00\rangle_1|00\rangle_2\cdots|00\rangle_{w-1}|i0\rangle_w\left(|00\rangle\pm|i0\rangle\right)_{w+1}\cdots|00\rangle_{n-1}|00\rangle_n,\\
|\beta^w_{\pm m,i}\rangle&=|00\rangle_1|00\rangle_2\cdots|00\rangle_{w-1}|mi\rangle_w\left(|m0\rangle\pm|mi\rangle\right)_{w+1}\cdots|00\rangle_{n-1}|00\rangle_n,\\
|\gamma^w_{\pm i,l}\rangle&=|00\rangle_1|00\rangle_2\cdots|00\rangle_{w-1}\left(|il\rangle\pm|i\,(l+1)\rangle\right)_w|00\rangle_{w+1}\cdots|00\rangle_{n-1}|00\rangle_n,\\
|\alpha^n_{\pm i,0}\rangle&=\left(|00\rangle\pm|i0\rangle\right)_1|00\rangle_2\cdots|00\rangle_{n-1}|i0\rangle_n,\\
|\beta^n_{\pm m,i}\rangle&=\left(|m0\rangle\pm|mi\rangle\right)_1|00\rangle_2\cdots|00\rangle_{n-1}|mi\rangle_n,\\
|\gamma^n_{\pm i,l}\rangle&=|00\rangle_1|00\rangle_2\cdots|00\rangle_{n-1}\left(|il\rangle\pm|i\,(l+1)\rangle\right)_n,
\end{aligned} \tag{18}$$

where $0\leqslant m\leqslant d-1$, $1\leqslant i\leqslant d-1, 1\leqslant w\leqslant n-1, n\geqslant 3, 1\leqslant l\leqslant d-2$, *and l is odd.*

Proof. We exploit the triviality of an orthogonality-preserving POVM to infer quantum nonlocality. It is noteworthy that the sets remain invariant under cyclic permutation. So we only need to prove that the measurement applied to the $(\mathbb{C}^d\otimes\mathbb{C}^d)_1$ system is trivial. Let B_1 be a general orthogonal-preserving measurement opertor on $(\mathbb{C}^d\otimes\mathbb{C}^d)_1$. Assume one element in B_1 is as follows:

$$\begin{pmatrix} A_{00} & \cdots & A_{0(d-1)}\\ \vdots & \ddots & \vdots\\ A_{(d-1)0} & \cdots & A_{(d-1)(d-1)}\end{pmatrix}. \tag{19}$$

The matrix B_1 is composed of block matrix A_{pq} $(0\leqslant p,q\leqslant d-1)$. Each block matrix A_{pq} can be written as a $(d-1)\times(d-1)$ matrix on the $\{|0\rangle,|1\rangle,\ldots,|d-1\rangle\}$ basis :

$$\begin{pmatrix} a_{pq,00} & \cdots & a_{pq,0(d-1)}\\ \vdots & \ddots & \vdots\\ a_{pq,(d-1)0} & \cdots & a_{pq,(d-1)(d-1)}\end{pmatrix}. \tag{20}$$

Since one performs local POVM on these states such that the post measurement states remain orthogonal, all off-diagonal elements of B_1 are equal to zero. The details are presented in table 2.

Furthermore, from the states $|\alpha^3_{\pm i,0}\rangle$, we know $a_{00,00}=a_{ii,00}$. For state $|\beta^3_{\pm m,i}\rangle$, we can get $a_{mm,00}=a_{mm,ii}$. That is, all diagonal elements in B_1 are equal.

In a word, B_1 is proportional to the identity matrix, which implies that one cannot start with a nontrivial measurement on the first combined subsystem. Thus, the states cannot be distinguished by LOCC. □

For even-dimensional subsystems, the constructed quantum state is different from that in the case of odd dimension. Next, we need more complex construction methods to construct quantum states in $(\mathbb{C}^d\otimes\mathbb{C}^d)_1\otimes\cdots\otimes(\mathbb{C}^d\otimes\mathbb{C}^d)_n$, where d is even.

Theorem 4. *The following set of orthogonal product states in $(\mathbb{C}^d\otimes\mathbb{C}^d)_1\otimes\cdots\otimes(\mathbb{C}^d\otimes\mathbb{C}^d)_n$ cannot be perfectly distinguished by LOCC,*

Table 2. Off-diagonal elements of B_1 (d is odd).

States	Zero elements	Range
$\lvert\beta^1_{+0,i}\rangle, \lvert\beta^2_{+0,i}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\gamma^1_{+i,l}\rangle$	$a_{00,0i}, a_{00,i0}$ $a_{ii,0l}, a_{ii,l0}, a_{ii,0(l+1)}, a_{ii,(l+1)0}$	$1 \leqslant i \leqslant d-1$
$\lvert\beta^1_{+m,i}\rangle, \lvert\beta^1_{+m,j}\rangle$	$a_{mm,ij}, a_{mm,ji}$	$1 \leqslant i \neq j \leqslant d-1$
$\lvert\alpha^1_{+i,0}\rangle, \lvert\alpha^1_{+j,0}\rangle$	$a_{ij,00}, a_{00,ji}$	$1 \leqslant i \neq j \leqslant d-1$
$\lvert\gamma^1_{+i,l}\rangle, \lvert\gamma^1_{+j,h}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\alpha^2_{+i,0}\rangle$ $\lvert\beta^1_{+i,l}\rangle, \lvert\beta^1_{+i,l+1}\rangle, \lvert\gamma^2_{+i,l}\rangle$	$a_{ij,lh}$, $a_{ji,hl}$, $a_{ij,(l+1)(h+1)}$, $a_{ji,(h+1)(l+1)}$ $a_{ij,l(h+1)}$, $a_{ji,(h+1)l}$, $a_{ij,(l+1)h}$, $a_{ji,h(l+1)}$ $a_{0i,00}, a_{i0,00}$ $a_{0i,0l}, a_{i0,l0}, a_{0i,0(l+1)}, a_{i0,(l+1)0}$	$1 \leqslant i \neq j \leqslant d-1$ l,h is odd $1 \leqslant i \leqslant d-1$
$\lvert\beta^1_{+0,i}\rangle, \lvert\gamma^1_{+k,l}\rangle$ $\lvert\beta^1_{+0,i}\rangle, \lvert\alpha^1_{+k,0}\rangle$	$a_{0k,il}, a_{k0,li}, a_{0k,i(1+1)}, a_{k0,(1+1)i}$ $a_{0k,i0}, a_{k0,0i}$	$1 \leqslant i,k \leqslant d-1$

Table 3. Off-diagonal elements of B_1 (d is even).

States	Zero elements	Range
$\lvert\beta^1_{+0,i}\rangle, \lvert\beta^2_{+0,i}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\zeta^1_{+i,t}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\varphi^1_{+i,1}\rangle$	$a_{00,0i}, a_{00,i0}$ $a_{ii,0(t+1)}, a_{ii,(t+1)0}, a_{ii,0(t+2)}, a_{ii,(t+2)0}$ $a_{ii,01}, a_{ii,10}$	$1 \leqslant i \leqslant d-1$
$\lvert\beta^1_{+m,i}\rangle, \lvert\beta^1_{+m,j}\rangle$	$a_{mm,ij}, a_{mm,ji}$	$1 \leqslant i \neq j \leqslant d-1$
$\lvert\alpha^1_{+i,0}\rangle, \lvert\alpha^1_{+j,0}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\zeta^1_{+j,t}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\varphi^1_{+j,1}\rangle$ $\lvert\varphi^1_{+i,1}\rangle, \lvert\varphi^1_{+j,1}\rangle$ $\lvert\zeta^1_{+i,t}\rangle, \lvert\varphi^1_{+j,1}\rangle$	$a_{ij,00}, a_{00,ji}$ $a_{ij,0(t+2)}, a_{ji,(t+2)0}, a_{ij,0(t+1)}, a_{ji,(t+1)0}$ $a_{ij,01}, a_{ji,10}$ $a_{ij,11}, a_{ji,11}$ $a_{ij,(t+1)1}, a_{ji,1(t+1)}, a_{ij,1(t+2)}, a_{ji,(t+2)1}$	$1 \leqslant i \neq j \leqslant d-1$
$\lvert\zeta^1_{+i,t}\rangle, \lvert\zeta^1_{+j,s}\rangle$	$a_{ij,(t+2)(s+2)}$, $a_{ji,(s+2)(t+2)}$ $a_{ij,(t+1)(s+1)}$, $a_{ji,(s+1)(t+1)}$ $a_{ij,(t+2)(s+1)}$, $a_{ji,(s+1)(t+2)}$ $a_{ij,(t+1)(s+2)}$, $a_{ji,(s+2)(t+1)}$	$1 \leqslant i \neq j \leqslant d-1$ t,s is odd
$\lvert\beta^1_{+i,1}\rangle, \lvert\varphi^2_{+i,1}\rangle$ $\lvert\alpha^1_{+i,0}\rangle, \lvert\alpha^2_{+i,0}\rangle$ $\lvert\beta^1_{+i,(t+1)}\rangle, \lvert\beta^1_{+i,(t+2)}\rangle, \lvert\zeta^2_{+i,t}\rangle$	$a_{0i,01}, a_{i0,10}$ $a_{0i,00}, a_{i0,00}$ $a_{0i,0t}, a_{i0,t0}, a_{0i,0(t+1)}, a_{i0,(t+1)0}$	$1 \leqslant i \leqslant d-1$
$\lvert\beta^1_{+0,i}\rangle, \lvert\zeta^1_{+k,t}\rangle$ $\lvert\beta^1_{+0,i}\rangle, \lvert\varphi^1_{+k,1}\rangle$ $\lvert\beta^1_{+0,i}\rangle, \lvert\alpha^1_{+k,0}\rangle$	$a_{0k,i(t+1)}, a_{k0,(t+1)i}, a_{0k,i(t+2)}, a_{k0,(t+2)i}$ $a_{0k,i1}, a_{k0,1i}$ $a_{0k,i0}, a_{k0,0i}$	$1 \leqslant i,k \leqslant d-1$

$$
\begin{aligned}
|\varphi^w_{\pm i,1}\rangle &= |00\rangle_1|00\rangle_2\cdots|00\rangle_{w-1}|i1\rangle_w|00\rangle_{w+1}\cdots|00\rangle_{n-1}|00\rangle_n,\\
|\alpha^w_{\pm i,0}\rangle &= |00\rangle_1|00\rangle_2\cdots|00\rangle_{w-1}|i0\rangle_w\left(|00\rangle\pm|i0\rangle\right)_{w+1}\cdots|00\rangle_{n-1}|00\rangle_n,\\
|\beta^w_{\pm m,i}\rangle &= |00\rangle_1|00\rangle_2\cdots|00\rangle_{w-1}|mi\rangle_w\left(|m0\rangle\pm|mi\rangle\right)_{w+1}\cdots|00\rangle_{n-1}|00\rangle_n,\\
|\zeta^w_{\pm i,t}\rangle &= |00\rangle_1|00\rangle_2\cdots|00\rangle_{w-1}\left(|i(t+1)\rangle\pm|i(t+2)\rangle\right)_w|00\rangle_{w+1}\cdots|00\rangle_{n-1}|00\rangle_n,\\
|\varphi^n_{\pm i,1}\rangle &= |00\rangle_1|00\rangle_2\cdots|00\rangle_{n-1}|i1\rangle_n,\\
|\alpha^n_{\pm i,0}\rangle &= \left(|00\rangle\pm|i0\rangle\right)_1|00\rangle_2\cdots|00\rangle_{n-1}|i0\rangle_n,\\
|\beta^n_{\pm m,i}\rangle &= \left(|m0\rangle\pm|mi\rangle\right)_1|00\rangle_2\cdots|00\rangle_{n-1}|mi\rangle_n,\\
|\zeta^n_{\pm i,t}\rangle &= |00\rangle_1|00\rangle_2\cdots|00\rangle_{n-1}\left(|i(t+1)\rangle\pm|i(t+2)\rangle\right)_n,
\end{aligned}
\tag{21}
$$

where $0 \leqslant m \leqslant d-1, 1 \leqslant i \leqslant d-1, 1 \leqslant w \leqslant n-1, \quad n \geqslant 3, 0 < t < d-2$, *t is odd and d is even.*

Proof. It is important to note that these sets remain invariant under cyclic permutation. So we only need to prove that the measurement on the $(\mathbb{C}^d \otimes \mathbb{C}^d)_1$ system is trivial.

Similar to the proof of theorem 3, one performs local POVM on these states such that the postmeasurement states remain orthogonal, all off-diagonal elements of B_1 are equal to zero. The details are presented in table 3.

Furthermore, we shows that all diagonal elements in B_1 are equal. From the states $|\alpha^3_{\pm i,0}\rangle$, we can get $a_{00,00} = a_{ii,00}$. For state $|\beta^3_{\pm m,i}\rangle$, we obtain $a_{mm,00} = a_{mm,ii}$. That is, all diagonal elements in B_1 are equal.

In conclusion, no individual party can commence with a nontrivial measurement. The set of orthogonal product states is indistinguishable by LOCC. □

5. Conclusion

In summary, the majority of papers investigating local indistinguishability focuses on performing POVM local measurement on a single quantum subsystem. In contrast, this paper proposed a novel partitioning method for multipartite quantum systems. Our results deepen our understanging of the structures of nonlocal sets. Under this partitioning scheme, we conducted POVM local measurement on composite multiple quantum systems. This approach had the potential to significantly reduce workload by half, thereby enhancing operational efficiency. Furthermore, we provided explicit expressions for nonlocal orthogonal product states in $(\mathbb{C}^3)^{\otimes 6}$ and $(\mathbb{C}^4)^{\otimes 6}$. Then, we presented a construction of nonlocal quantum states orthogonal product states in $(\mathbb{C}^d)^{\otimes 2n}$ (d is odd), and built orthogonal product states in $(\mathbb{C}^d)^{\otimes 2n}$ (d is even) which have been proved to be nonlocal. In the future, we will explore the construction of other nonlocal states in multipartite quantum systems using this method.

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References

[1] Zhang Z C, Gao F, Qin S J, Yang Y H and Wen Q Y 2015 *Phys. Rev.* A **92** 012332
[2] Walgate J, Short A J, Hardy L and Vedral V 2000 *Phys. Rev. Lett.* **85** 4972
[3] Fan H 2004 *Phys. Rev. Lett.* **92** 177905
[4] Cohen S M 2007 *Phys. Rev.* A **75** 052313
[5] Zhang X, Tan X, Weng J and Li Y 2016 *Sci. Rep.* **6** 28864
[6] Bai C M, Zhang S J and Liu L 2022 *Quantum Inf. Process.* **21** 377
[7] Modi K, Pati A K, Sen A and Sen U 2018 *Phys. Rev. Lett.* **120** 230501
[8] Li M S and Wang Y L 2018 *Phys. Rev.* A **98** 062306
[9] Bennett C H, DiVincenzo D P, Fuchs C A, Mor T, Rains E, Shor P W, Smolin J A and Wootters W K 1999 *Phys. Rev.* A **59** 1070
[10] Xin Y and Duan R 2008 *Phys. Rev.* A **77** 012315
[11] Shi F, Ye Z, Chen L and Zhang X 2022 *Phys. Rev.* A **105** 022209
[12] Halder S, Banik M, Agrawal S and Bandyopadhyay S 2019 *Phys. Rev. Lett.* **122** 040403
[13] Yang Y H, Gao F, Tian G J, Cao T Q and Wen Q Y 2013 *Phys. Rev.* A **88** 024301
[14] Xu G B, Yang Y H and Wen Q Y 2016 *Sci. Rep.* **6** 31048
[15] Wang Y L, Li M S, Zheng Z J and Fei S M 2015 *Phys. Rev.* A **92** 032313
[16] Chen J and Johnston N 2015 *Commun. Math. Phys.* **333** 351–65
[17] Xiong Z X, Li M S, Zheng Z J and Li L 2023 *Phys. Rev.* A **108** 022405
[18] Zhou H, Gao T and Yan F 2022 *Phys. Rev.* A **106** 052209
[19] Zhou H Q, Gao T and Yan F L 2023 *Phys. Rev.* A **107** 042214
[20] Xu G B, Zhu Y Y and Jiang D H 2023 *Physica* A **619** 128734
[21] Wang Y L, Li M S and Zheng Z J 2017 *Quantum Inf. Process.* **16** 1–13
[22] Halder S 2018 *Phys. Rev.* A **98** 022303
[23] Jiang D H and Xu G B 2020 *Phys. Rev.* A **102** 032211
[24] Rout S, Maity A G, Mukherjee A, Halder S and Banik M 2021 *Phys. Rev.* A **104** 052433
[25] Li M S, Wang Y L and Shi F 2021 *J. Phys. A: Math. Theor.* **54** 445301
[26] Zhu Y Y, Jiang D H and Xu G B 2023 *Physica* A **624** 128956
[27] Zhang Z C, Gao F, Cao Y, Qin S J and Wen Q Y 2016 *Phys. Rev.* A **93** 012314
[28] Zhang Z C, Zhang K J, Gao F, Wen Q Y and Oh C H 2017 *Phys. Rev.* A **95** 052344
[29] Zhen X F, Fei S M and Zuo H J 2022 *Phys. Rev.* A **106** 062432
[30] Cao H Q, Li M S and Zuo H J 2023 *Phys. Rev.* A **108** 012418