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Analytic solution for a joint Bohm sheath and pre-sheath potential profile

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Abstract

An analytic solution is presented in this paper for the electric potential near a wall in a confined plasma. This is well fitted for both the sheath and pre-sheath regions. In the sheath region, the potential is well adapted to the differential equation proposed by Bohm. In the pre-sheath region, the potential is also well suited, decaying to zero electric field in the plasma, which is a physical condition. The potential is also valid for any value of the parameter K measuring the dimensionless Bohm velocity.

Keywords: sheath potential, Bohm sheath, pre-sheath potential, sheath analytic solution

(Some figures may appear in colour only in the online journal)

1. Introduction

Plasma is separated by a wall containing a peculiar region called a plasma sheath, which is followed by a pre-sheath region. The model most commonly used to describe the plasma over this region is the so-called Bohm plasma sheath [1]. This is extensively described in most plasma physics books [2–4], and also in those related to industrial applications [5]. There are also some complementary and review papers [6–10]. It is also interesting to see the work of Riemann, where a new approximation of the plasma sheath has been derived [11]. An interesting discussion of the problem including an electron sheath and multiple ion species can be found in [12]. Analysis using kinetic theory has also been performed by several authors—see for instance [13]. Sheaths in magnetized plasma have also been considered [14]. Plasma sheath formation in low-pressure discharges has also been studied [15]. In a later paper, the authors also discuss the importance of collisions in the Bohm criterion [16]. Finally, an interesting topical review of several aspects of these matters has been made by Robertson [17]. In this paper, an analysis is made in the simplest case, where magnetic fields, kinetic effects and more complicated situations are not considered.

There is a potential drop between neutral plasma and the entrance of the sheath. In the sheath region the number of ions is higher than the electron density, due in part to the large reflection of electrons by the negative potential of the wall. The ion density is determined by the continuity equation as well as the energy conservation equation, where the cold ion approximation is considered. Thus, for the electrons, it is assumed that they follow a Boltzmann distribution, and for the ions the cold fluid approach is adopted. In the sheath region the potential is the solution of Poisson's equation with suitable boundary conditions for the dynamical equations of ions and electrons. For the electron fluid the gradient pressure is dominant over the momentum term, and since the ions are cold, the momentum term is more important than the pressure gradient. For this reason, the electron density is described by the Maxwell–Boltzmann factor

$$n_e(x) = n_0 \exp\left[e\varphi(x)/k_B T_e\right]$$

with the boundary conditions $V(x \to \infty) = 0$ and $n_e(x \to \infty) = n_0$, where n_0 is the density in the neutral plasma, T_e is the temperature of the electrons, $\varphi(x)$ is the electrostatic potential and k_B is the Boltzmann constant. By integrating the ion moment equation and the continuity

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equation, the density of ions is obtained for plasma of a single species, or for hydrogen plasma:

$$n_{\mathrm{i}}(x) = n_0\left[1 - \frac{2e\varphi(x)}{mv^2}\right]^{-1/2}$$

where v is the characteristic Bohm plasma velocity at the edge of the plasma, m is the ion mass and n_0 is also the density of ions in the plane plasma. In this way Poisson's equation is written as

$$\frac{\mathrm{d}^2\varphi(x)}{\mathrm{d}x^2} = \frac{n_0 e}{\epsilon_0}\left[\exp\left(\frac{e\varphi(x)}{k_{\mathrm{B}}T_{\mathrm{e}}}\right) - \left(1 - \frac{2e\varphi(x)}{mv^2}\right)^{-1/2}\right].$$

The well-known differential equation in dimensionless variables is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} = -\left[e^{-\phi} - \frac{1}{\sqrt{1+\phi/K}}\right] \tag{1}$$

where dimensionless units are taken, and thus

$$\phi = -\frac{e\varphi}{k_{\mathrm{B}}T_{\mathrm{e}}}.$$

Here $(-e)$ is the electron charge, k_{B} is the Boltzmann constant and T_{e} is the electron temperature in the plain plasma. The dimensionless distance is $\xi = x/\lambda_{\mathrm{D}}$, where x is the distance measured from the wall and λ_{D} is the plasma Debye length. Finally the parameter K is a dimensionless quantity measuring the characteristic Bohm plasma velocity v, that is,

$$\lambda_{\mathrm{D}} = \sqrt{\frac{\epsilon_0 k_{\mathrm{B}}T_{\mathrm{e}}}{n_{\mathrm{e}}e^2}}, \quad K = \frac{\frac{1}{2}mv^2}{k_{\mathrm{B}}T_{\mathrm{e}}}, \tag{2}$$

where n_{e} is the electron density. It is assumed that ions enter the sheath region with the velocity of sound; then $K = 1/2$ [1]. However, it is interesting that Bohm obtains a relation with an inequality sign [1], $v \geqslant v_{\mathrm{Bohm}} = (k_{\mathrm{B}}T_{\mathrm{e}}/m)^{1/2}$.

Here the dimensionless potential ϕ is positive because of the minus sign in its definition. On the other hand, the distance x is measured from the wall, and not from the plane separating the sheath and pre-sheath. The size of the sheath is usually considered to be d, which is the place where the derivative of the potential coming from equation (1) becomes zero.

In this paper, treatment is performed in the simplest way. Thus, the magnetic field for instance is not included, neither is hot plasma flow, and the wall is considered a flat surface. One of the problems with the solution of the Bohm equation for the sheath region is that this does not verify the condition for zero electric field in the plasma. Therefore, it has been widely agreed to provide a second description for the so-called artificially created pre-sheath region, in which it is recovered as a smooth continuation of the potential profile, as it has been experimentally observed to be [18, 19]. There are two main goals in this work: The first is to find a solution to the Bohm sheath equation, which is straightforward to calculate, and one with high accuracy in the region where the solution has physical meaning, that is, near the wall. However, there is a second purpose, which is to avoid problems in the region where the solution is unphysical, in such a way that now the solution could be joined smoothly to the plasma potential. This can be done first by looking for an adequate form of the solution, and second by determining the right parameters mainly for the conditions in the region near the wall. The precise connection between the plasma sheath and the plasma, whether or not using a pre-sheath region, is a very complex problem—see for instance [20, 21]. There is no general agreement on or solution to this problem. This is the reason why we consider a solution for the sheath and pre-sheath regions in which the adjustment to the good part of the well-known sheath solution is adapted. Furthermore, the electric field with respect to the plasma region is zero. Here, we consider Maxwellian distributions which are commonly assumed, yet there are other views such as the truncated bi-Maxwellian distribution [22]; this case could be considered in the future. This work is organized as follows: In section 2 we discuss the solution for the sheath potential and a new solution is also proposed. Moreover, the proposed solution is presented as a joint solution for both the sheath and pre-sheath regions. This section also includes several figures for different plasma conditions. Finally, a conclusion section is presented.

2. Theoretical treatment and discussion

Equation (1) can be integrated once to obtain a first-order differential equation, but after that it must be solved numerically. Yet, in the numerical solution x is obtained as a function of ϕ, but it is more convenient to obtain ϕ as a function of x. Several attempts have been made to perform this inversion [7, 11]. However, an analytic solution for both the sheath and pre-sheath regions has not been achieved yet. This is the reason why it is useful to obtain a complete accurate approximation for ϕ as a function of ξ, as shown here. The simplest approximation is one of the exponential type $\phi = \phi_{\mathrm{w}}\exp^{-\alpha\xi}$ as shown in the literature [4], but as K approaches the Bohm limit, $K = 1/2$, the decay distance becomes too large. A first integration of equation (1), considering ϕ as the integration variable and $\mathrm{d}\phi/\mathrm{d}\xi = 0$ for $\phi = 0$, leads to

$$\frac{1}{\sqrt{2}}\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = -\sqrt{F(\phi)} \tag{3}$$

with

$$F(\phi) = 2K\sqrt{1+\frac{\phi}{K}} - 2K + e^{-\phi} - 1 = \frac{2K-1}{4K}\phi^2 + \frac{3-4K^2}{24K^2}\phi^3 + \ldots. \tag{4}$$

The simplest approximation is obtained by keeping only the first term of the series expansion [4]. However, the accuracy of this approximation is not good, and furthermore the approximation fails for $K \leqslant 1/2$, as the so-called Bohm limit

$K = 1/2$. Better approximations can be obtained using more terms of the series [11, 23]. The accuracy of these approximations is better than that obtained by using the first term of the second expansion in equation (4). In our last approximation [23], all the parameters are functions of K only. This is not convenient because the numerical calculation shows that the slope at the origin is also a function of the wall potential ϕ_w. In the present treatment, all the parameters will be functions of K and ϕ_w. The form of the approximation to be determined will be

$$\Phi = \phi_w \frac{e^{-\lambda\xi}}{1 + \beta(e^{-\lambda\xi} - 1)} \tag{5}$$

where ϕ_w is the wall potential, that is, $\widetilde{\phi}(0) = \phi_w$. In this way, we are certain that for large values of ξ, the potential will be the plasma potential, which must be zero for $\phi(\infty)$. The parameters to be determined in this approximation are λ and β. Considering the derivatives of equation (1), the following are obtained:

$$\frac{d\Phi}{d\xi} = \phi_w \frac{\lambda(\beta - 1)e^{-\lambda\xi}}{[1 + \beta(e^{-\lambda\xi} - 1)]^2} \tag{6}$$

and

$$\frac{d^2\Phi}{d\xi^2} = \phi_w \frac{\lambda^2(\beta - 1)^2 e^{-\lambda\xi} + \lambda^2\beta(\beta - 1)e^{-2\lambda\xi}}{[1 + \beta(e^{-\lambda\xi} - 1)]^3}. \tag{7}$$

As in the wall, it is found that

$$\left.\frac{d\Phi}{d\xi}\right|_{\xi=0} = \lambda(\beta - 1)\phi_w \tag{8}$$

and

$$\left.\frac{d^2\phi}{d\xi^2}\right|_{\xi=0} = \lambda^2(\beta - 1)(2\beta - 1)\phi_w. \tag{9}$$

On the other hand, it is known by equations (3) and (1) that

$$\begin{aligned}\frac{1}{\sqrt{2}}\left.\frac{d\phi}{d\xi}\right|_{\xi=0} &= -\sqrt{F(\phi_w)}\\ &= -\sqrt{2K\left(1 + \frac{\phi_w}{K}\right)^{1/2} - 2K + e^{-\phi_w} - 1}\end{aligned} \tag{10}$$

and

$$\left.\frac{d^2\phi}{d\xi^2}\right|_{\xi=0} = \frac{1}{\sqrt{1 + \frac{\phi_w}{K}}} - e^{-\phi_w} = \phi_w f(\phi_w). \tag{11}$$

Now by equalizing equations (8) and (10), and equations (9) and (11), we obtain

$$\lambda(\beta - 1) = -\frac{\sqrt{2F(\phi_w)}}{\phi_w} \tag{12}$$

and

$$\lambda^2(\beta - 1)(2\beta - 1) = f(\phi_w). \tag{13}$$

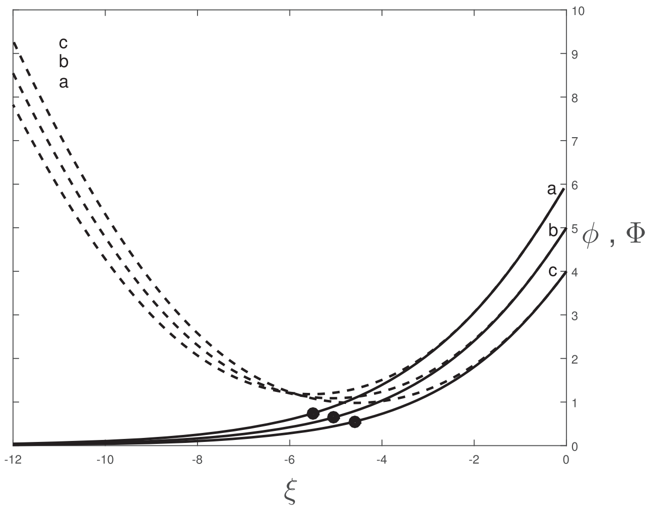

Figure 1. Numerical solution ϕ (dashed line) of the differential equation, equation (1), and proposed analytic function Φ (full line), equation (5), for $K = 1/2$ and $\phi_w = 5$, 6, and 4 as a function of the dimensionless distance ξ. The edge point between the sheath and pre-sheath is marked with a black point, and letters a, b, and c follow the change in the function order.

From these the values of λ and β are obtained, giving

$$\beta = \frac{\phi_w^2 f(\phi_w) - 2F(\phi_w)}{\phi_w^2 f(\phi_w) - 4F(\phi_w)} \tag{14}$$

and

$$\lambda = \frac{4F(\phi_w) - \phi_w^2 f(\phi_w)}{\phi_w\sqrt{2F(\phi_w)}}. \tag{15}$$

These parameters are now introduced in equation (5), and in this way a new approximation is obtained.

A plot of the solutions of equation (1) (dashed lines) and the present solutions (full lines) is shown in figure 1, for $K = 1/2$ and $\phi_w = 5$, 4, and 6. These values have been considered because they are coincident with those appearing in [17]—see figures 2, 11 and 13 there. It is clear that each pair of curves are coincident near the wall, for each potential. However, they become different when the solutions of equation (1) are near the minimum of each curve. This is true just when the solutions are not valid because the potentials begin to increase. The new solution has a better adjustment to the physical system, just where the solution of equation (1) does not show the desired behavior. In figures 1, 2, and 3, the position corresponding to each minimum potential is denoted as a black point in the corresponding new solutions of equation (5) (full line). These points show the edge or transition region of the corresponding sheath region—see, for instance, figure 8 in [17]. These regions increase with the wall potential, notwithstanding that the total sizes of both the sheath and pre-sheath regions are almost equal independently of the wall potentials. The minimum of the solution coming from the differential equations for $\phi = 6$ presents a better behavior than that for $\phi_w = 5$, because the minimum of the curve is closer to the ξ-axis. The solutions now proposed have better behavior, and the electric field as well as the potential is zero in the plasma. Although the Bohm value of $K = 1/2$ is the

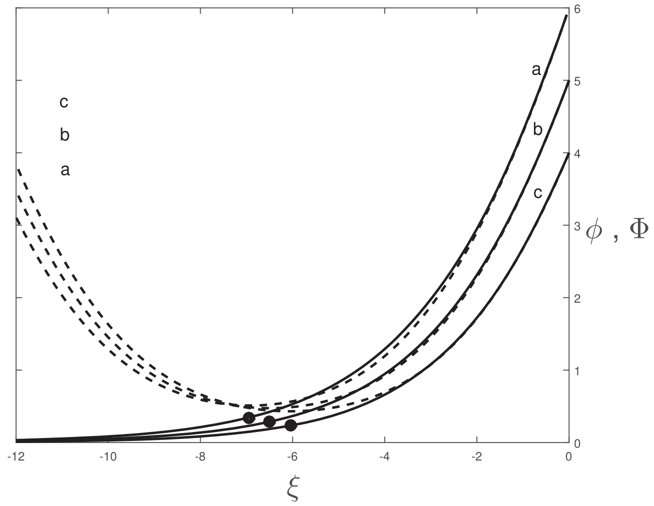

Figure 2. Numerical solution ϕ (dashed line), equation (1), and proposed analytic function Φ (full line), equation (5), for $K = 0.58$ and $\phi_w = 4$, 5, and 6 as a function of the dimensionless distance ξ. The edge point between the sheath and pre-sheath is marked with a black point, and letters *a*, *b*, and *c* follow the change in the function order.

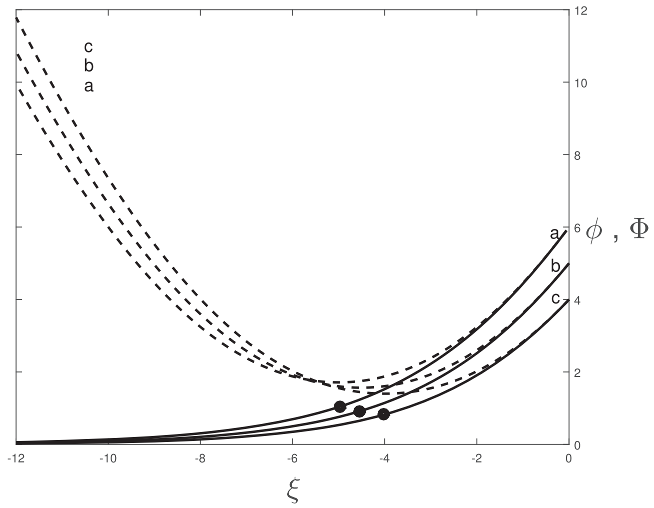

Figure 3. Numerical solution ϕ (dashed line), equation (1), and proposed analytic function Φ (full line), equation (5), for $K = 0.42$ and $\phi_w = 4$, 5, and 6 as a function of the dimensionless distance ξ. The edge point between the sheath and pre-sheath is marked with a black point, and letters *a*, *b*, and *c* follow the change in the function order.

most important one, it is also worthwhile to look for values nearby, such as $K = 0.58$ and $K = 0.42$. The results are shown in figures 2 and 3 for the same values of the wall potential in figure 1. That is, different values of K are analyzed, when $\phi_w = 4$, 5 and 6. In the case of $K = 0.58$, figure 2 shows that the minimum of the differential equation solution is just near zero for $\phi_w = 6$, but an increasing potential after that is also present. In the case of $K = 0.42$, the Bohm criterion is not achieved, that is, it does not correspond to any appropriate physical case. However, this case has been included in figure 3 because it seems interesting to check the behavior of the solution in this case. The solution for $K = 0.42$ does not present any problems, and the potential and electric fields become zero at the plasma.

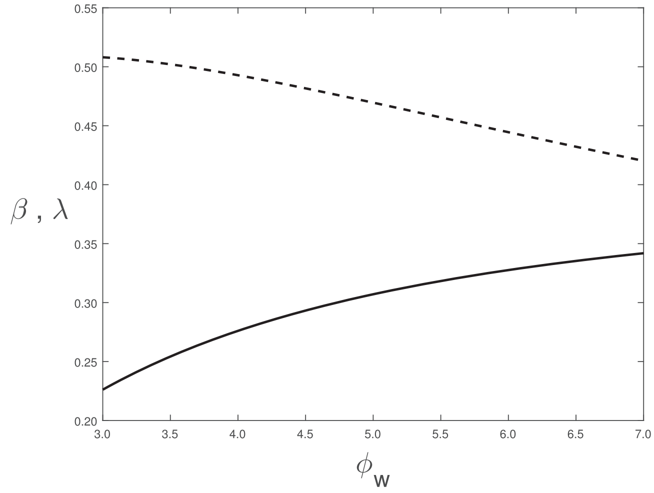

Figure 4. Analytic functions β (full line) and λ (dashed line) for $K = 1/2$, as a function of the wall potential.

Table 1. Values of the parameters λ and β for $K = 1/2$, 0.58, 0.42, and dimensionless potentials $\phi_w = 5$, 4, 6.

	ϕ_w	β	λ
$K = 1/2$	5	0.307 1	0.469 6
	4	0.276 0	0.492 8
	6	0.327 1	0.444 5
$K = 0.58$	5	0.312 1	0.494 2
	4	0.283 8	0.521 9
	6	0.331 2	0.466 1
$K = 0.42$	5	0.299 0	0.439 1
	4	0.263 6	0.456 3
	6	0.321 9	0.418 1

The values of the parameters λ and β used in the above figures are shown in table 1. From the table, it is clear that the slope λ at the wall increases with K and decreases with the value of ϕ_w. On the other hand, the parameter β increases with K and ϕ_w.

The approximation here presented is obtained using the usual equation for sheath or Bohm treatment. It is clear that the approximation and the numerical solution are almost coincident in the sheath region, which is the most important region. However, later the numerical solution of the equation is not right since $\frac{d\phi}{d\xi}$ becomes zero and the solution increases instead of decreasing. However, the behavior of the present solution is now different from the actual function far from the wall. There it behaves like the potential in the pre-sheath region.

It seems interesting to show the behavior of β and λ with the dimensionless wall potential ϕ_w and characteristic Bohm velocity K. This is shown in figures 4 and 5 respectively. From figure 4 it is clear that β (full line) increases with the Bohm potential, but λ (dashed line) decreases. In figure 5, it is shown that both parameters increase with K but the increase of λ is much greater than that of β.

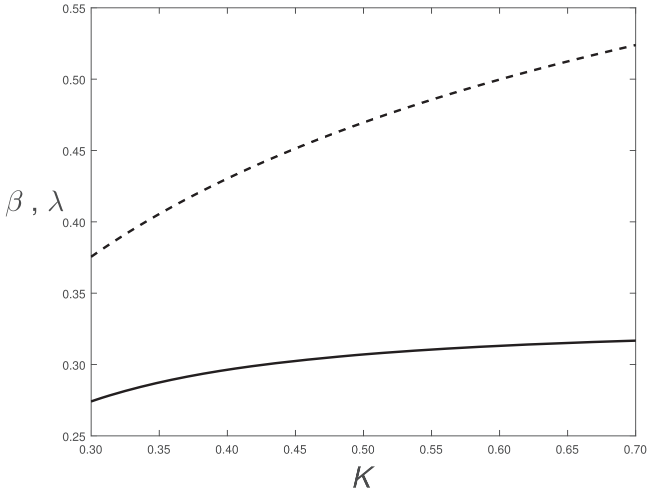

Figure 5. Analytic functions β (full line) and λ (dashed line) for $\phi_w = 5$, as a function of the K parameter.

3. Conclusion

A new analytic solution for the Bohm sheath potential has been presented, whose advantages can be summarized as follows: (1) The accuracy of the new solution is better than that of previously published approximations—see [11, 23]. (2) It is a simple approach, with only two parameters to be determined, which are functions of K as well as ϕ_w (this is an important advantage with respect to approximations where the parameters to be determined are only functions of K). (3) The sheath and pre-sheath are included in a unique or joint solution. The present treatment has also been performed using the most basic model. However, a more elaborate treatment, which may include the magnetic field, could be considered in future work. (4) There is no need to discriminate between the sheath and pre-sheath anymore, and because of this comparing theory with experiment can be done in a direct way.

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