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Corrigendum: Brownian molecules formed by delayed harmonic interactions (2019 *New J. Phys.* **21** 093014)

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Below equation (76) in section 6 of our article [1], we mention that the small-delay expansion of the delayed term in the linear delay Langevin equation

$$\dot{x}(t) = -\omega x(t - \tau) + \sqrt{2D}\,\eta(t) \tag{1}$$

leads to a decrease in the steady state variance $\nu_{SS} = \langle x^2 \rangle$, in contradiction to the exact expression (equation (24) in [1])

$$\nu_{SS} = \frac{D}{\omega} \frac{1 + \sin(\omega\tau)}{\cos(\omega\tau)}. \tag{2}$$

Expanding instead the full equation (1), including the noise term, up to the first order in τ, leads to the linear Langevin equation

$$\dot{x}(t) = -\frac{\omega}{1 - \omega\tau} x(t) + \sqrt{2\frac{D}{(1 - \omega_\tau\tau)^2}}\,\eta(t) \tag{3}$$

for an overdamped harmonic oscillator with frequency $\tilde{\omega} = \omega/(1 - \omega\tau)$ and diffusion coefficient $\tilde{D} = D/(1 - \omega\tau)^2$. The corresponding stationary variance is given by

$$\langle x^2 \rangle = \frac{\tilde{D}}{\tilde{\omega}} = \frac{D}{\omega} \frac{1}{1 - \omega\tau} \approx \frac{D}{\omega}(1 + \omega\tau), \tag{4}$$

where we have expanded the result to first order in τ in the final expression. The same result is obtained by expanding the exact expression (2). More details about small-delay expansions in delay Langevin equations can be found in [2].

The expansion (4) up to the first order in the time delay gives intuition why the behavior of equation (1) for small delays is reminiscent of an overdamped harmonic oscillator. Similarly, the expansion

$$\ddot{x}(t) = -2\frac{1 - \omega\tau}{\omega\tau^2}\dot{x}(t) - \frac{2}{\tau^2}x(t) + \sqrt{2\frac{4D}{\omega^2\tau^4}}\,\eta(t) \tag{5}$$

of equation (1) up to the second order in time delay gives a hint on why solutions to equation (1) for intermediate delays exhibit damped oscillations. However, one should be aware that especially the higher order Taylor expansions may be problematic [2, 3].

Equation (5) describes a noisy damped harmonic oscillator with unit mass, damping constant $\gamma = 2(1 - \omega\tau)/\omega\tau^2$, frequency $\tilde{\omega} = 2/\tau^2$, and diffusion coefficient $\tilde{D}\gamma^2 = 4D/\omega^2\tau^4$. The corresponding stationary variance reads

$$\langle x^2 \rangle = \frac{\tilde{D}\gamma}{\tilde{\omega}} = \frac{D}{\omega} \frac{1}{1 - \omega\tau} \approx \frac{D}{\omega}(1 + \omega\tau + \omega^2\tau^2) \tag{6}$$

which is the same as the result (4), obtained using the first order expansion, and which differs from the second order expansion $\frac{D}{\omega}(1 + \omega\tau + \frac{1}{2}\omega^2\tau^2)$ of the correct expression (2). In mathematical literature [3], such failures of Taylor expansions of delay differential equations are known to be a consequence of two ingredients. First, it is

not guaranteed that the remainder in the Taylor expansion is small and, second, the expansions involve higher derivatives that might not exist. In the case of the linear delay Langevin equation (1), we find that the second derivative of x involves the first derivative of the white noise, which should indeed be treated with care.

Following the mapping to the noisy damped harmonic oscillator further, we can identify the corresponding free eigenfrequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \frac{\sqrt{2}}{\tau}, \tag{7}$$

and the damping ratio

$$\chi = \frac{\gamma}{2\sqrt{mk}} = \frac{1}{\sqrt{2}}\frac{1-\omega\tau}{\omega\tau}. \tag{8}$$

These quantities control the qualitative behavior of average solutions $\langle x \rangle$ to equation (5). For $\chi < 1$, thus $\omega\tau < 1/(1+\sqrt{2}) \approx 0.41$, it exhibits overdamped behavior (pure exponential decay), and, for $\chi > 1$, $\langle x \rangle$ performs exponentially decaying oscillations. The original model (1) exhibits overdamped behavior for $\omega\tau < 1/e \approx 0.37$ and damped oscillations for $1/e < \omega\tau < \pi/2$. Both approximate models (3) and (5) are unstable for $\omega\tau > 1$ and the original model is unstable for $\omega\tau = \pi/2 \approx 1.57$.

Even though the Taylor expansions above are of limited use, the result that the behavior of overdamped delay systems is reminiscent of the behavior of corresponding underdamped harmonic oscillators seems to be robust. Delay, in general, induces oscillations into the system dynamics, and, for strong potentials, may lead to instabilities. As another example of such behavior, we refer to [4] showing that the position x described the stochastic delay equation

$$\dot{x}(t) = -\omega\frac{x(t-\tau)}{|x(t-\tau)|} + \sqrt{2D}\eta(t) \tag{9}$$

exhibits qualitatively the same behavior as the velocity $v = \dot{x}$ of the constant force oscillator described by the formula

$$\ddot{x}(t) = -\gamma\dot{x}(t) - \omega_0\frac{x(t)}{|x(t)|} + \sqrt{2D}\eta(t). \tag{10}$$

In this case, the small-delay Taylor expansion makes no sense at all due to the non-analyticity of the absolute value.

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