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To cite this article: Pei Wang 2018 *New J. Phys.* **20** 023042

New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

OPEN ACCESS

RECEIVED
28 September 2017

REVISED
11 January 2018

ACCEPTED FOR PUBLICATION
23 January 2018

PUBLISHED
16 February 2018

PAPER

Discrete Lorentz symmetry and discrete time translational symmetry

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Keywords: discrete Lorentz group, discrete time translational symmetry, spontaneous symmetry breaking

Abstract

The Lorentz symmetry and the space and time translational symmetry are fundamental symmetries of nature. Crystals are the manifestation of the continuous space translational symmetry being spontaneously broken into a discrete one. We argue that, following the space translational symmetry, the continuous Lorentz symmetry should also be broken into a discrete one, which further implies that the continuous time translational symmetry is broken into a discrete one. We deduce all the possible discrete Lorentz and discrete time translational symmetries in 1+1-dimensional spacetime, and show how to build a field theory or a lattice field theory that has these symmetries.

1. Introduction

Symmetry plays an important role in modern physics. It imposes a constraint on the physical laws and then reduces the number of candidate theories describing nature. Knowing the symmetry of a system is the prerequisite for building a theory of it. For examples, the crystals are classified by the point group and the space group according to their symmetry under rotation, translation and reflection [1], the Lorentz symmetry and its generalization the Poincaré symmetry are the basic of the relativistic quantum field theory [2], the discovery of the violation of parity symmetry [3, 4] improves our understanding of weak interaction, and the particle-hole, the time reversal and the chiral symmetry are used to classify different topological insulators and topological superconductors [5], to name just a few.

The symmetry can be spontaneously broken at low-energy states of a system. This mechanism has been used to explain the ferromagnetic–paramagnetic phase transition, the existence of crystals, or the origin of mass. While it is well known that some fundamental continuous symmetries like the space translational symmetry can be spontaneously broken into a discrete one, whether the Lorentz symmetry has such property is unexplored.

It has been long believed that, the Lorentz symmetry which is essentially important in high-energy physics does not play a role in solid-state physics, especially in the study of crystals, where the low-energy effective theories completely break the Lorentz symmetry. In crystals the continuous space translational symmetry is spontaneously broken into a discrete one [6]. A crystal does not look the same under an arbitrary spatial translation of coordinates, but only if the translation is along some specific direction with the distance being an integer times of the lattice constant. The continuous Poincaré group consists of spatial and temporal translations of arbitrary distance and Lorentz transformations of arbitrary velocity [7], which is the symmetry group of a relativistic field theory but not the symmetry group of crystals.

Disregarding the Lorentz symmetry leaves too much freedom in writing down a theory of crystals. One may ask whether the crystals can have any if not all the Lorentz symmetry which helps people to constrain the theories of them, in other words, whether something can be left after the Lorentz symmetry is spontaneously broken. In this paper we study to which extent the Lorentz symmetry may exist in a crystal, and what is the consequence of it. We argue that the continuous Lorentz symmetry contradicts the discrete space translational symmetry and then cannot exist. But a discrete Lorentz symmetry may exist, under which the physical laws stay the same for two observers who are moving at some specific velocities $\mathbf{v}$ relative to each other. $\mathbf{v}$ can only take a sequence of universal discrete values. The corresponding Lorentz transformations make up a discrete subgroup of the continuous Lorentz group. We suggest that the Lagrangian of an effective theory describing crystals should be

invariant under these discrete Lorentz transformations, just as they are invariant under the discrete spatial translations. Furthermore, the continuous time translational symmetry should also be spontaneously broken into a discrete one to be compatible with the discrete Lorentz symmetry. The discrete Lorentz transformations and the discrete temporal and spatial translations together make up a discrete Poincaré group. We discuss how to build a field theory or a lattice field theory that has the discrete Poincaré symmetry.

In 2012, Wilczek *et al* [8, 9] proposed a theory about the spontaneous breaking of the continuous time translational symmetry into a discrete one. The matter with such a broken symmetry is dubbed a 'time crystal'. Whether there exist 'time crystals' is still under debate up to now [10–17]. Our theory also predicts the breaking of the continuous time translational symmetry. But it should be distinguished from the previous theories of 'time crystals'. In our theory, the broken time translational symmetry is a result of the principle of relativity (the Lorentz symmetry).

The discrete translational symmetry that we find can be represented by a rectangular or a centered rectangular spacetime lattice, which keeps invariant under the discrete Lorentz transformations. This finding could possibly be interesting in different contexts. Some approaches to quantum gravity assumes that the spacetime is discretized instead of continuous, and how to maintain the Lorentz symmetry on a spacetime lattice has then become an important problem [18–23]. In the causal set theory, the spacetime is discretized into a random lattice by the Poisson process, which keeps invariant under continuous Lorentz transformations statistically, in the sense that one realization of the lattice has no Lorentz symmetry but the ensemble of them has [18, 19]. In the loop quantum gravity, the distance and the time interval are treated as quantum operators. The spacetime is discretized in the sense that the corresponding operators have discrete eigenvalues. And the Lorentz transformation becomes a unitary transformation acting on the operators, with the Lorentz symmetry being explained as the invariance of the eigenvalues under unitary transformations [21]. In this paper, the Lorentz symmetry on the spacetime lattice has a different meaning. On our spacetime lattice, the Lorentz symmetry is explicit but not statistical. The cost is that only a discrete symmetry is left. And the spacetime is classical and continuous with zero curvature (a Minkowski spacetime). We do not try to quantize the spacetime, nor consider any theory of quantum gravity. What we want to discuss is the symmetry group of the Minkowski spacetime after a spontaneous symmetry breaking.

The paper is organized as follows. Section 2 lists the basic hypotheses of our theory. Section 3 demonstrates why the continuous Lorentz symmetry and the discrete space translational symmetry cannot coexist, and how the latter puts restrictions on the velocity in the Lorentz transformation. In section 4, we deduce the discrete Lorentz group that is compatible with the discrete space translational symmetry. In section 5, we construct the discrete Poincaré group which includes both the translations and the Lorentz transformations. We then present a corollary of our theory—the continuous time translational symmetry is broken into a discrete one. Section 6 continues to discuss the discrete spacetime translational symmetry, and shows how its representation (the spacetime lattice) keeps invariance under discrete Lorentz transformations. We also discuss how to understand the time dilation and space contraction on the lattice. Section 7 shows the causality between events on the spacetime lattice. In section 8, we construct the field theory that has the discrete Poincaré symmetry. The possible features and problems when quantizing this theory are discussed in section 9. Section 10 is the conclusion and outlook.

2. Hypotheses of the theory

Let us recall the continuous Lorentz and Poincaré symmetry. According to the principle of relativity in special relativity [24], the physical laws must stay the same for the observers in different reference frames which are moving at a constant velocity relative to each other. The transformation connecting the space and time coordinates of an event as measured in different frames is the Lorentz transformation, which can be derived from the principle of relativity and the principle of invariant light speed. Again, the spacetime has translational symmetry, that is the physical laws stay the same in the coordinate systems which are at rest relative to each other but differ by a spatial or temporal translation of the origin. The continuous Lorentz and translational transformations together make up the Poincaré group [7], which is the symmetry group of a relativistic quantum field theory.

The crystals do not have the continuous space translational symmetry. The breaking of the continuous space translational symmetry into a discrete one signals the freezing phase transition from liquids to crystals. We imagine the whole spacetime being occupied by a perfect crystal which is infinitely large and has no boundary, thereafter, the spacetime has a discrete space translational symmetry for the observers living inside. We propose three hypotheses about such a spacetime.

The first hypothesis is a weaker version of the principle of relativity. It says that, for any observer in this spacetime, there exists another observer moving at nonzero velocity relative to him and the physical laws stay the

same for them. The principle of relativity states that all the inertial reference frames moving at arbitrary velocity are equivalent to each other in describing the physical laws. Our first hypothesis is different from the principle of relativity which is found to contradict the discrete space translational symmetry, as explained in section 3. We only suppose that there exists at least one velocity $\mathbf{v} \neq 0$ so that two reference frames are equivalent as one is moving at $\mathbf{v}$ relative to the other.

In the second hypothesis, we suppose an invariant 'light speed' which is denoted by a constant c in this paper. Note that c is not the speed of light in vacuum since the spacetime is occupied by a crystal. Instead, c is the supremum limit of the propagation speed of information and matter in the crystal. Such a limit always exists, even if it may differ from the speed of light in vacuum. We suppose that c is the same in any reference frames.

The third hypothesis is that the spacetime has a discrete space translational symmetry. For example, in the 3+1-dimensional spacetime, there exist three spatial vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. The physical laws keep invariant under a pure spatial translation of vector $\mathbf{r}$ if and only if $\mathbf{r} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$ with n_1, n_2 and n_3 being integers. Here a pure spatial translation means that the two coordinate systems are at rest relative to each other and their temporal coordinates keep the same. $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are in fact the three primitive vectors of a crystal [1]. Note that the third hypothesis not only says that the spatial translation of vector $n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$ is a symmetry transformation, but also says that the other spatial translations are not.

3. Continuous Lorentz symmetry and discrete translational symmetry do not coexist

3.1. 3+1 dimensions

Any n+1-dimensional spacetime can be equipped with the above three hypotheses. We will consider the 3+1-dimensional spacetime in this subsection and then turn to 1+1-dimensional spacetime in next.

We consider two coordinate systems, namely *Jun* and *Tao* for convenience. Or one can imagine *Jun* and *Tao* as two observers who are located at the origins of the corresponding coordinate systems, respectively. *Tao* is moving at a constant velocity $\mathbf{v}$ relative to *Jun*. And their origins differ by a four-vector $r = (r^0, r^1, r^2, r^3)^T$. The coordinate of an event measured by *Tao* is denoted by the four-vector $y' = (y'^0, y'^1, y'^2, y'^3)^T$ and that by *Jun* is denoted by y. Here the zeroth component of a four-vector denotes the time coordinate and the others denote the space coordinates. The second hypothesis says that the 'light speed' is the same in any reference frames. As is well known in special relativity [24], an invariant 'light speed', whatever its value is, leads to the Lorentz transformation between the coordinates of an event measured in different reference frames. The relation between y' and y is expressed as

$$y' = L_{\mathbf{v}}y + r, \tag{1}$$

where $L_{\mathbf{v}}$ is the 4-by-4 matrix of Lorentz transformation and r is the translation vector. Note that in the expression of $L_{\mathbf{v}}$, the speed of light in vacuum must be replaced by c, i.e. the speed limit of the propagation of matter or information in crystals, since it is c that is invariant in our hypothesis. For example, if $\mathbf{v}$ is along the x-axis with an amplitude v, the corresponding Lorentz matrix should be

$$L_{\mathbf{v}} = \begin{pmatrix} \dfrac{1}{\sqrt{1-v^2/c^2}} & \dfrac{-v/c^2}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \dfrac{-v}{\sqrt{1-v^2/c^2}} & \dfrac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Equation (1) can be reexpressed in a compact form as $y' = \Lambda(L_{\mathbf{v}}, r)\,y$ where $\Lambda(L_{\mathbf{v}}, r)$ denotes the combination of a Lorentz boost and a translation. The operator $\Lambda(L_{\mathbf{v}}, r)$ is not a matrix as $r \neq 0$. But one can still define the multiplication of Λ. Let us suppose the third observer *Pei* who is moving at a velocity $\mathbf{v}'$ relative to *Tao* and their origins differ by r'. The coordinate of the event measured by *Pei* is denoted by y'' which is $y'' = \Lambda(L_{\mathbf{v}'}, r')y' = \Lambda(L_{\mathbf{v}'}, r')\Lambda(L_{\mathbf{v}}, r)y$. Repeatedly applying equation (1) leads to $y'' = (L_{\mathbf{v}'}L_{\mathbf{v}})y + (L_{\mathbf{v}'}r + r')$, the multiplication of two Λ operators can then be expressed as

$$\Lambda(L_{\mathbf{v}'}, r')\Lambda(L_{\mathbf{v}}, r) = \Lambda(L_{\mathbf{v}'}L_{\mathbf{v}}, L_{\mathbf{v}'}r + r'). \tag{3}$$

It is straightforward to verify that the multiplication of Λ has the associative property.

Now let us consider the first hypothesis. According to it, the physical laws stay the same for two observers as one is moving at some velocity $\mathbf{v}$ relative to the other. The value of $\mathbf{v}$ is not given in the first hypothesis which only states that $\mathbf{v}$ exists. The corresponding Lorentz transformation $\Lambda(L_{\mathbf{v}}, 0)$ is an element of the symmetry group of the spacetime. Note that the translation is absent in $\Lambda(L_{\mathbf{v}}, 0)$ because we do not yet know which translations keep the physical laws invariant. Let us consider two coordinate systems K and K', while K' is moving at $\mathbf{v}$ relative to K.

K' and K are then equivalent for describing the physical laws. Since they are equivalent, K is not privileged over K'. If K is equivalent to a coordinate system moving at $\mathbf{v}$ relative to itself, so must be K'. Therefore, the coordinate system K'' that is moving at $\mathbf{v}$ relative to K' must also be equivalent to K' and then be equivalent to K, while the coordinate transformation between K'' and K is $\Lambda^2(L_{\mathbf{v}}, 0)$. Furthermore, the equivalence relation is not only transitive but also reflective. For K', its equivalent partner K is moving at the velocity $-\mathbf{v}$ relative to it, thereafter, one coordinate system moving at $-\mathbf{v}$ relative to another must also be equivalent to it. The above statements can be translated into the language of group. If $\Lambda(L_{\mathbf{v}}, 0)$ is an element of any symmetry group (not necessarily the Poincaré group), then $\Lambda^2(L_{\mathbf{v}}, 0)$ and $\Lambda^{-1}(L_{\mathbf{v}}, 0)$ must also be the elements of the symmetry group, due to the property of a group. In fact, $\Lambda^j(L_{\mathbf{v}}, 0)$ for arbitrary integer j must be an element of the group. By using the product rule equation (3), we obtain $\Lambda^2(L_{\mathbf{v}}, 0) = \Lambda(L_{\mathbf{v}}^2, 0)$ and $\Lambda^{-1}(L_{\mathbf{v}}, 0) = \Lambda(L_{\mathbf{v}}^{-1}, 0) = \Lambda(L_{-\mathbf{v}}, 0)$.

The third hypothesis can also be expressed in the language of group. A pure translation between two coordinate systems can be denoted as $\Lambda(1, r)$ where 1 is the identity matrix which is the Lorentz transformation between two coordinate systems being at rest relative to each other. We distinguish the temporal and spatial components of the four-vector r by expressing it as $r = (r^0, \mathbf{r})^T$ with $\mathbf{r}$ denoting a three-dimensional spatial vector. $\Lambda(1, r)$ with $r = (0, \mathbf{r})^T$ represents a pure spatial translation. The third hypothesis in fact says that $\Lambda(1, r)$ is an element of the symmetry group if and only if $\mathbf{r} = n_1\,\mathbf{a}_1 + n_2\,\mathbf{a}_2 + n_3\,\mathbf{a}_3$. Since the product of two translations is $\Lambda(1, r)\Lambda(1, r') = \Lambda(1, r + r')$, the spatial translations of vector $n_1\,\mathbf{a}_1 + n_2\,\mathbf{a}_2 + n_3\,\mathbf{a}_3$ by themselves make up a discrete group. It must be a subgroup of the overall symmetry group of the spacetime.

Up to now, we know that the symmetry group of the spacetime has an element $\Lambda(L_{\mathbf{v}}, 0)$ with $\mathbf{v} \neq 0$, and its subgroup for spatial translations contains only translations of vector $n_1\,\mathbf{a}_1 + n_2\,\mathbf{a}_2 + n_3\,\mathbf{a}_3$. Surprisingly, one can deduce from these properties that $\mathbf{v}$ cannot take continuous values! In detail, let us consider seven observers (or seven coordinate systems), namely $K_1, K_2, \cdots, K_6$ and K_7. K_2 is moving at the velocity $-\mathbf{v}$ relative to K_1. K_3 differs from K_2 by a spatial translation of vector $\mathbf{a}_1$. K_4 is moving at the velocity $\mathbf{v}$ relative to K_3. K_5 is moving at the velocity $\mathbf{v}$ relative to K_4. K_6 differs from K_5 by a translation of vector $\mathbf{a}_1$. Finally, K_7 is moving at the velocity $-\mathbf{v}$ relative to K_6. Obviously, due to the transitivity of equivalence relation, all these seven coordinate systems are equivalent to each other. The coordinate of an event measured by K_1 and K_7 is denoted by y_1 and y_7, respectively. And we use the notation $a_1 = (0, \mathbf{a}_1)$. The transformation from y_1 to y_7 is then

$$\begin{aligned}&\Lambda^{-1}(L_{\mathbf{v}}, 0)\Lambda(1, a_1)\Lambda^2(L_{\mathbf{v}}, 0)\Lambda(1, a_1)\Lambda^{-1}(L_{\mathbf{v}}, 0)\\ &= \Lambda(1, L_{\mathbf{v}}a_1 + L_{\mathbf{v}}^{-1}a_1).\end{aligned} \tag{4}$$

Obviously, $\Lambda\,(1, L_{\mathbf{v}}\,a_1 + L_{\mathbf{v}}^{-1}\,a_1)$ is a translation and it must be an element of the symmetry group. $L_{\mathbf{v}}$ acting on $a_1 = (0, \mathbf{a}_1)$ usually generates a four-vector with nonzero temporal component. But this temporal component exactly cancels the temporal component of $L_{\mathbf{v}}^{-1}\,a_1$, so that $L_{\mathbf{v}}\,a_1 + L_{\mathbf{v}}^{-1}\,a_1$ is in fact a four-vector with only spatial components and then $\Lambda\,(1, L_{\mathbf{v}}\,a_1 + L_{\mathbf{v}}^{-1}\,a_1)$ describes a pure spatial translation with the time coordinate keeping invariant under this transformation. To see the cancellation between the temporal components of $L_{\mathbf{v}}\,a_1$ and $L_{\mathbf{v}}^{-1}$ a_1, one can study the example of $L_{\mathbf{v}}$ in equation (2). It is straightforward to verify that $(L_{\mathbf{v}} + L_{-\mathbf{v}})\,a_1$ has no temporal component for any $\mathbf{v}$ and $\mathbf{a}_1$.

The element $\Lambda\,(1, L_{\mathbf{v}}\,a_1 + L_{\mathbf{v}}^{-1}\,a_1)$ in the symmetry group is a pure spatial translation. But we already know that for any spatial translation in the symmetry group the translation vector must be $n_1\,\mathbf{a}_1 + n_2\,\mathbf{a}_2 + n_3\,\mathbf{a}_3$. Therefore, we establish an equation

$$L_{\mathbf{v}}\,a_1 + L_{\mathbf{v}}^{-1}a_1 = (0,\ n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3)^T. \tag{5}$$

This equation puts strong restrictions on the velocity $\mathbf{v}$. Since $L_{\mathbf{v}}\,a_1 + L_{\mathbf{v}}^{-1}\,a_1$ changes continuously with $\mathbf{v}$, equation (5) indicates that $\mathbf{v}$ can only take some discrete values. The Lorentz transformation $\Lambda(L_{\mathbf{v}'},\ 0)$ is not an element of the symmetry group if $\mathbf{v}'$ does not satisfy equation (5), otherwise, the third hypothesis would be violated. If two observers are equivalent to each other, i.e., the physical laws stay the same for them, their relative velocity must be a solution of equation (5). In equation (5), different integer arrays (n_1, n_2, n_3) give different $\mathbf{v}$. Since there are infinite number of choice for (n_1, n_2, n_3), the number of solutions of equation (5) is also infinite.

The continuous Lorentz symmetry with a continuously changing $\mathbf{v}$ contradicts the discrete space translational symmetry (the third hypothesis). If there exists any Lorentz symmetry in the inhomogeneous spacetime of a crystal, it must be a discrete symmetry.

3.2. 1+1 dimensions

We have shown the contradiction between the continuous Lorentz symmetry and the discrete space translational symmetry in 3+1-dimensional spacetime. In fact, this contradiction exists in arbitrary n+1-dimensional spacetime. Next we focus on 1+1-dimensions in which the coordinate of an event is a two-vector $y = (t, x)^T$ where t and x denote the time and space coordinates, respectively. The reason that we choose $1 + 1$ dimensions is due to its simplicity. Especially, the Wigner rotation [25] is lack in $1 + 1$ dimensions so that we can easily construct the discrete Lorentz symmetry group. The presence of Wigner rotation in $2 + 1$ and $3 + 1$

dimensions makes the construction of the discrete Lorentz group more complicated. Also in experiments, a one-dimensional (1D) crystal can be realized in quantum wires. Therefore, it is reasonable to first explore the $1+1$-dimensional spacetime. The construction of the discrete Lorentz symmetry in $2+1$ and $3+1$ dimensions is left in future study.

Note that for 1D crystals a single real number a (the lattice constant) determines the discrete space translational symmetry. The crystal looks the same after a spatial translation of distance ma with m being an arbitrary integer. The generator of the spatial translation is $\Lambda(1,\ \bar{a})$ where $\bar{a} = (0,\ a)^T$. The third hypothesis in 1+1-dimensional spacetime becomes that any pure spatial translation in the symmetry group can be expressed as $\Lambda(1,\ m\bar{a})$.

The speed limit c is another important constant in our theory, thereafter, it is natural to choose $a = c = \hbar = 1$ as the unit, which will be used throughout the left paper.

In $1+1$ dimensions, the Lorentz matrix relating the coordinate of an event observed in different reference frames becomes

$$L_v = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix}, \tag{6}$$

where the relative velocity v is a signed real number satisfying $|v| \leqslant 1$ since we already set $c = 1$ to the unit of velocity. According to the first hypothesis, there exists $v \neq 0$ so that $\Lambda\ (L_v, 0)$ is an element of the symmetry group, i.e., two observers are equivalent in describing the physical laws if one is moving at the velocity v relative to the other.

$\Lambda\ (L_v, 0)$ and $\Lambda(1,\ \bar{a})$ are two elements of the symmetry group. As same as what we did in 3+1-dimensional spacetime, we use $\Lambda\ (L_v, 0)$ and $\Lambda(1,\ \bar{a})$ to construct a symmetry transformation $\Lambda(1,\ L_v\bar{a} + L_v^{-1}\bar{a})$, which will help us to obtain an equation of v. In detail, we suppose seven observers which are equivalent to each other. K_2 is moving at the velocity $-v$ relative to K_1. K_3 differs from K_2 by a spatial translation of distance a. K_4 is moving at the velocity v relative to K_3. K_5 is moving at the velocity v relative to K_4. K_6 differs from K_5 by a translation of distance a. And K_7 is moving at the velocity $-v$ relative to K_6. The relation between the coordinates of an event observed by K_7 and that by K_1 is $y_7 = \Lambda(1,\ L_v\bar{a} + L_v^{-1}\bar{a})y_1$. The symmetry transformation $\Lambda(1,\ L_v\bar{a} + L_v^{-1}\bar{a})$ is a pure spatial translation, indicating that $L_v\bar{a} + L_v^{-1}\bar{a}$ must be an integer times of $\bar{a}$. We then obtain

$$\begin{aligned} L_v\bar{a} + L_v^{-1}\bar{a} &= \begin{pmatrix} 0 \\ \frac{2}{\sqrt{1-v^2}}a \end{pmatrix} \\ &= m\begin{pmatrix} 0 \\ a \end{pmatrix}, \end{aligned} \tag{7}$$

which can be further simplified into

$$\frac{2}{\sqrt{1-v^2}} = m, \tag{8}$$

where $m = 2, 3, 4, \cdots$ is an integer larger than one. In a 1+1-dimensional spacetime that obeys our three hypotheses, the physical laws stay the same for two observers only if the relative velocity between them is $v = \pm\sqrt{1-\frac{4}{m^2}}$. $m = 2$ corresponds to $v = 0$, that is two observers are at rest relative to each other. And $|v|$ increases monotonically with m. Since m has no supremum limit, v can take infinite number of values even if $|v|$ cannot exceed the speed limit $c = 1$. Let us list a few possible values of v, which are $v = 0,\ \pm\sqrt{5}/3,\ \pm\sqrt{3}/2,\ \ldots$. Recall that the unit of v is c, and then the smallest nonzero value of $|v|$ is $\sqrt{5}/3c$. Two observers moving at a relative speed lower than $\sqrt{5}/3c$ are always not equivalent except that they are at rest relative to each other. In next text, we call the relative velocity v at which two observers are equivalent the equivalence velocity. An interesting observation is that the equivalence velocity is independent of a. It is the same in 1D crystals with different lattice constants. Once if the continuous space translational symmetry is broken into a discrete one, no matter how small a is, the equivalence velocity immediately loses its continuity.

4. Discrete Lorentz symmetry

As shown in above, the 1+1-dimensional spacetime with discrete space translational symmetry can only have a discrete Lorentz symmetry if not none at all. Equation (8) gives the necessary condition for the equivalence velocity v in the Lorentz transformation. But it is not the sufficient condition. In fact, it is impossible for the observers moving at the relative velocity $v(m) = \pm\sqrt{1-\frac{4}{m^2}}$ for arbitrary m to be equivalent to each other. In

Table 1. The integer sequence $m_j(g)$ generated by $g = 3$ (top) and $g = 4$ (bottom). The corresponding equivalence velocities $v_j(g)$ are displayed below $m_j(g)$.

m_0	$m_1(g)$	m_2	m_3	m_4	m_5	m_6
2	3	7	18	47	123	322
v_0	v_1	v_2	v_3	v_4	v_5	v_6
0	$\frac{\sqrt{5}}{3}$	$\frac{3\sqrt{5}}{7}$	$\frac{8\sqrt{5}}{18}$	$\frac{21\sqrt{5}}{47}$	$\frac{55\sqrt{5}}{123}$	$\frac{144\sqrt{5}}{322}$
m_0	$m_1(g)$	m_2	m_3	m_4	m_5	m_6
2	4	14	52	194	724	2702
v_0	v_1	v_2	v_3	v_4	v_5	v_6
0	$\frac{\sqrt{3}}{2}$	$\frac{4\sqrt{3}}{7}$	$\frac{15\sqrt{3}}{26}$	$\frac{56\sqrt{3}}{97}$	$\frac{209\sqrt{3}}{362}$	$\frac{780\sqrt{3}}{1351}$

other words, the symmetry group cannot contain all the Lorentz transformations $\Lambda(L_{v(m)}, 0)$ for $m \geqslant 2$, because such a set of $\Lambda(L_{v(m)}, 0)$ are not closed under multiplication! To see this point, let us suppose that $\Lambda(L_{v(m)},0)$ for $m = 3$ and $m = 4$ are both the elements of the symmetry group. This is to say that three observers *Jun*, *Tao* and *Pei* are equivalent to each other if *Tao* is moving at the velocity $v(3) = \sqrt{5}/3$ relative to *Jun* and *Pei* is moving at $v(4) = \sqrt{3}/2$ relative to *Tao*. The Lorentz transformation relating the coordinate observed by *Jun* to that by *Pei* is $L_{v(4)}L_{v(3)} = L_{v'}$ where v' denotes the velocity of *Pei* relative to *Jun*. By using the velocity-addition formula in special relativity which can also be derived from equation (6), we find that $v' = (v(3) + v(4))/(1 + v(3)v(4))$. However, $2/\sqrt{1 - v'^2}$ is not an integer, so that v' cannot be an equivalence velocity because this violates the third hypothesis. Therefore, the assumption of $v(m = 3)$ and $v(m = 4)$ being both the equivalence velocity must be false.

A question then arises as to which Lorentz transformations $\Lambda(L_v, 0)$ with the velocity $v = \pm\sqrt{1 - \frac{4}{m^2}}$ can be in the symmetry group which must be closed under multiplication. The answer appears to be simple. First, the identity transformation at $v = 0$ or $m = 2$ must be an element of the group. For the other elements, we strictly prove that (see appendix A for the detail) only the transformations that are generated by a single integer $g > 2$ can make up a group. All the Lorentz transformations in the group can be expressed as $\Lambda(L_{v_j(g)}, 0)$ with the relative velocity being

$$v_j(g) = \mathbf{sgn}(j)\sqrt{1 - \frac{4}{m_j^2(g)}}, \tag{9}$$

where $\mathbf{sgn}(j)$ denotes the sign of the integer j. And $m_j(g)$ is an integer sequence generated by g. For $j \geqslant 0$, $m_0 = 2$ and $m_1 = g$ are the first two integers in the sequence, and the left ones are iteratively generated according to

$$m_{j+1} = gm_j - m_{j-1}. \tag{10}$$

For $j < 0$, m_j is obtained by using the property that it is an even function, i.e. $m_j = m_{-j}$. The Lorentz transformations $\Lambda(L_{v_j(g)}, 0)$ for $j = 0, \pm 1, \cdots$ make up a cyclic group—the discrete Lorentz group which is denoted as **L. L** is uniquely determined by the integer g which is called the generator of the group. The closure of the group under multiplication can be proved by using the relations $\Lambda(L_{v_j(g)}, 0)\Lambda(L_{v_i(g)}, 0) = \Lambda(L_{v_{i+j}(g)}, 0)$ and $\Lambda^j(L_{v_1(g)}, 0) = \Lambda(L_{v_j(g)}, 0)$. The velocity-addition formula reads $v_{i+j}(g) = (v_i(g) + v_j(g))/(1 + v_i(g)v_j(g))$. Note that the Lorentz matrix in terms of m_j is expressed as

$$L_{v_j} = \begin{pmatrix} m_j/2 & \frac{-\mathbf{sgn}(j)}{2}\sqrt{m_j^2 - 4} \\ \frac{-\mathbf{sgn}(j)}{2}\sqrt{m_j^2 - 4} & m_j/2 \end{pmatrix}. \tag{11}$$

Table 1 enumerates the first few elements in the sequence $m_j(g)$ generated by $g = 3$ or $g = 4$. The corresponding equivalence velocities are also displayed. m_j increases exponentially with j, according to equation (10). A more elegant expression of m_j can be found in appendix B.

The cyclic Lorentz groups appear to be the inevitable consequence of our three hypotheses. In a spacetime where our three hypotheses stand, when an observer is writing down the equations of physical laws, he knows that the only observers who are using the same equations as him must be those who are moving at the equivalence velocity $\pm\sqrt{1 - 4/m_j(g)^2}$ relative to him. The other observers moving at different velocities have different equations for the physical laws. And $\Lambda(L_{v_j(g)}, 0)$ is the transformation relating the coordinate of an

event observed by him to those by the other equivalent observers. It is worth emphasizing that each spacetime (each crystal) has a unique generator g. But different spacetimes (different crystals) may have different generators.

5. Discrete Poincaré symmetry

We derive equation (8) by studying the coordinate transformation between seven well-designed equivalent coordinate systems, namely $K_1, \cdots, K_6$ and K_7. One may wonder whether it is possible to design some equivalent coordinate systems that finally lead to a paradox and then falsifies our three hypotheses. The answer is no! In fact, an overall symmetry group which includes the discrete Lorentz group and the discrete space translational group as its subgroups does exist. Our three hypotheses are self-consistent. We will discuss this overall symmetry group —the discrete Poincaré group in this section.

The physical laws stay the same for two observers if and only if the coordinate transformation between them is an element of the symmetry group of the spacetime. Let us use $\mathcal{P}$ to denote the overall symmetry group of the spacetime where our three hypotheses stand. According to the above discussions, the subgroup of $\mathcal{P}$ for pure boost with no translation must be the discrete Lorentz group $\{\Lambda(L_{v_j(g)}, 0)\}$. And the subgroup of $\mathcal{P}$ for pure spatial translation with no boost or temporal translation must be $\{\Lambda(1, m\bar{a})\}$ with $\bar{a} = (0, 1)^T$ and m being an integer (recall that $a = 1$ is the unit of length). In other words, two coordinate systems who have the same origins are equivalent to each other in describing the physical laws if and only if they are moving at the velocity $v_j(g)$ relative to each other. And two observers who are at rest relative to each other and use the same clock are equivalent if and only if the spatial distance between them is an integer times of $a = 1$.

We prove that (see appendix B for the detail) the group $\mathcal{P}$ exists and is determined by the generator g. The elements of $\mathcal{P}$ can be generally expressed as $\Lambda(L_{v_j(g)}, Y)$, where $L_{v_j(g)}$ is a Lorentz matrix of velocity $v_j(g)$ with $j = 0, \pm1, \pm2, \cdots$ being an arbitrary integer. And Y is a discrete translation of spacetime. It is a combination of spatial and temporal translations, being expressed as

$$Y = N_1\begin{pmatrix}0\\1\end{pmatrix} + N_2\begin{pmatrix}\frac{1}{2}\sqrt{g^2-4}\\ \frac{1}{2}g\end{pmatrix}, \tag{12}$$

where $N_1, N_2 = 0, \pm1, \pm2, \cdots$ are arbitrary integers. $\mathcal{P}$ can be written as

$$\mathcal{P} = \{\Lambda(L_{v_j(g)}, Y_{N_1N_2}(g))|\, j, N_1, N_2 = 0, \pm1, \pm2, \cdots\}. \tag{13}$$

The group $\mathcal{P}$ is a discrete subgroup of the continuous Poincaré group.

It is easy to verify that the subgroup of $\mathcal{P}$ for pure boost is the discrete Lorentz group $\mathbf{L}$. Not only the Lorentz boost but also the translation depends on g. The subgroup of $\mathcal{P}$ for pure translations includes the elements $\Lambda(1, Y_{N_1N_2}(g))$ which are the translations of vector $Y_{N_1N_2}(g)$. We use $\mathbf{Y}$ to denote the translational group $\{\Lambda(1, Y_{N_1N_2}(g))\}$ which is closed under multiplication since $\Lambda(1, Y_{N_1,N_2})\Lambda(1, Y_{N_1',N_2'}) = \Lambda(1, Y_{N_1+N_1',N_2+N_2'})$. $\mathbf{Y}$ includes not only the pure spatial translation, but also the temporal translation and the combination of spatial and temporal translations. The vectors $Y_{N_1N_2}(g)$ form a lattice including the origin in 1+1-dimensional spacetime, which is the characteristic lattice of $\mathbf{Y}$. This characteristic lattice is generated by two primitive vectors: $(0, 1)^T$ and $\left(\frac{1}{2}\sqrt{g^2-4}, \frac{1}{2}g\right)^T$. The first vector $(0, 1)^T$ corresponds to a minimum spatial translation, while the second one corresponds to a combination of spatial and temporal translations since $\frac{1}{2}\sqrt{g^2-4}$ and $\frac{1}{2}g$ are both nonzero. The subgroup of $\mathbf{Y}$ for pure spatial translations is $\{\Lambda(1, m\bar{a})\}$, as we expected.

Let us see the properties of the characteristic lattice for an odd g. At $g = 3$, the two primitive vectors are $(0, 1)^T$ and $\left(\frac{\sqrt{5}}{2}, \frac{3}{2}\right)^T$. We plot the characteristic lattice of the translational symmetry for $g = 3$ in figure 1. Note that the first component of the vector $Y_{N_1N_2}$ denotes the time t which is the label of the vertical axis, while the second component denotes the space x which is the label of the horizontal axis. As g is an odd number, the characteristic lattice is always a centered rectangular lattice (the red rectangular in figure 1 represents the unit cell). It includes vectors that lie in the direction of t-axis. In other words, $\mathbf{Y}$ includes pure temporal translations.

The period of the characteristic lattice in temporal and spatial directions are incommensurate. Since the unit of time is a/c according to our choice, the period of the lattice in the temporal direction is $T = \sqrt{5}\,a/c$ for $g = 3$. The pure temporal translation of an integer times of $\sqrt{5}\,a/c$ is a symmetry transformation, but those of the other periods are not. In other words, the physical laws stay the same for two observers who are at rest relative to each other and located at the same spot if and only if their clocks differ by an integer times of $\sqrt{5}\,a/c$. Or equivalently, for a specific observer, the physical laws change periodically with time and the period is $T = \sqrt{5}\,a/c$. For a general odd generator g, the period of the characteristic lattice in the temporal direction is $T = \sqrt{g^2-4}\,a/c$.

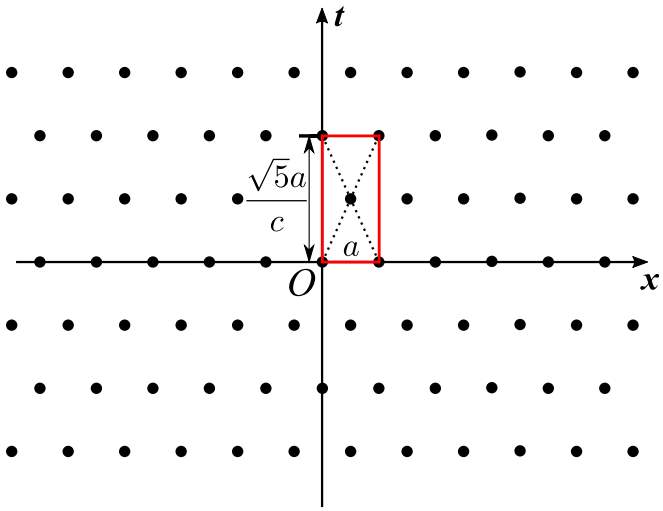

Figure 1. The characteristic lattice of the translational group generated by $g = 3$. The red rectangular represents the unit cell of the lattice.

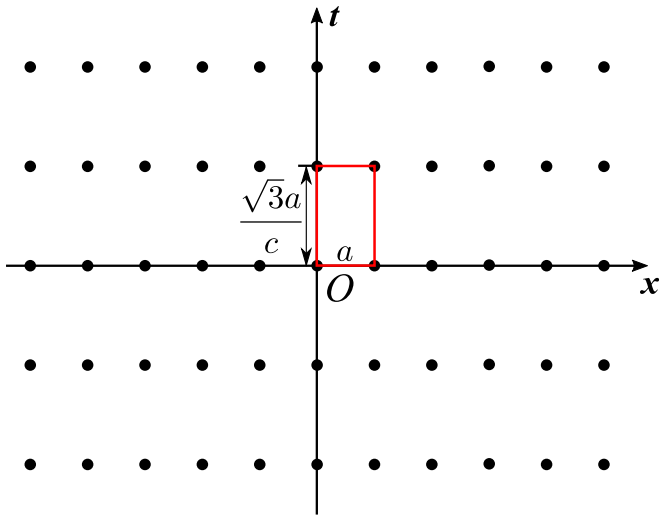

Figure 2. The characteristic lattice of **Y** generated by $g = 4$. The red rectangular represents the unit cell.

Following the breaking of continuous space translational symmetry, the time translational symmetry must also be broken into a discrete one. This is not a surprise, since the first hypothesis is a weaker version of the principle of relativity, according to which the time cannot be separated from the space. According to Noether's theorem [2], the breaking of the continuous space (time) translational symmetry indicates that the momentum (energy) is not conserved. In a spacetime where our three hypotheses stands, both the momentum and the energy are not conserved quantities. But according to the Bloch theorem [26] or the Floquet theorem [27], there exist quasi-momentum or quasi-energy which are conserved as the system has a discrete space or time translational symmetry, respectively.

The characteristic lattice for an even g has a different shape. Figure 2 plots the characteristic lattice of the translational group **Y** generated by $g = 4$. For an even g, the characteristic lattice is always a rectangular lattice. And its period in the temporal direction is $\sqrt{g^2 - 4}\,a/2c$.

The lack of the continuous time translational symmetry and then the energy conservation law sound strange, since it is generally believed that an isolated system should have conserved energy. But one should not forget that our three hypotheses stand in a spacetime where the continuous space translational symmetry has already been spontaneously broken into a discrete one. $\mathcal{P}$ is in fact the symmetry group of an effective theory that describes a system living in such an inhomogeneous spacetime. Just as the electron cannot conserve its momentum when moving within the periodic potential of crystals, but the whole crystal as an isolated system keeps its momentum invariant. We should understand the lack of energy conservation in the similar way as we understand the lack of

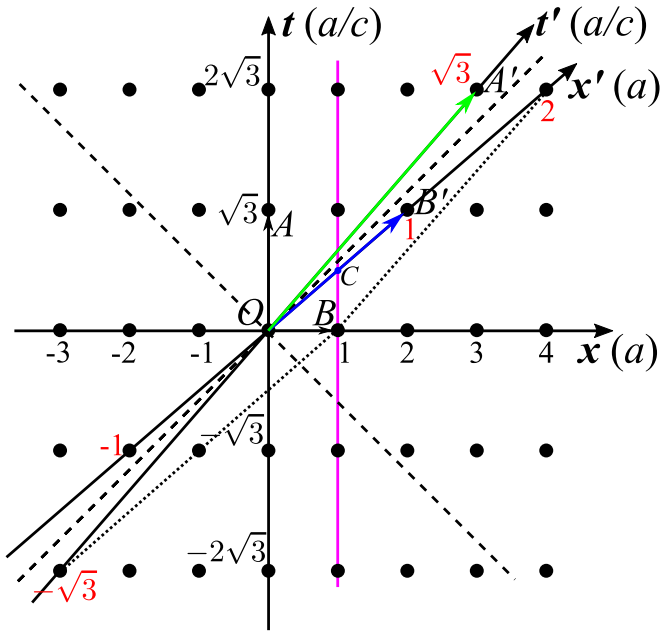

Figure 3. Invariance of the characteristic lattice under the Lorentz transformation. t' and x' are the time and space axes of a frame K' moving at the velocity $\sqrt{3}/2c$ relative to K. The black and red numbers denote the spacetime coordinates of an event in the frames K and K', respectively. The green ($\overrightarrow{OA'}$) and blue vectors ($\overrightarrow{OB'}$) are the two primitive vectors of the lattice in K'. The dotted lines guides the decomposition of $\overrightarrow{OB}$ into $\overrightarrow{OA'}$ and $\overrightarrow{OB'}$. The pink line is the world line of a particle that is static in K. Finally, the dashed lines represent the light cone of O.

momentum conservation of an electron. The system that the effective theory describes is not a real isolated system.

6. Invariance of the characteristic lattice under the discrete Lorentz transformation, time dilation and length contraction

We already know from above that the discrete spacetime translational symmetry can be represented by the characteristic lattice. In this section, we show that the characteristic lattice $\{Y_{N_1N_2}\}$ keeps invariant under a discrete Lorentz transformation L_{v_j}, even if each site is transformed into another one on the same lattice. This is expected since the spacetime translational symmetry is independent of observers.

Let us consider an observer K. The physical laws for him are not the same everywhere in the continuous spacetime, but change periodically. In next section, we will express the physical laws in terms of the Lagrangian of a field theory. We can then understand the physical laws changing as that the coupling parameters in the theory varies in the spacetime. Since the coupling parameters change periodically, the observer K can find a group of spacetime points which form a lattice (the characteristic lattice), and on this lattice the coupling parameters stay the same. In other words, the spacetime has a discrete translational symmetry represented by this lattice. Now a second observer K' is moving at the velocity v_j relative to K. What does this spacetime lattice look like in the eye of K'? The answer is: exactly the same! Under the transformation from K to K', a site $Y_{N_1N_2}$ on the lattice is transformed into

$$Y_{N_1'N_2'} = L_{v_j} Y_{N_1N_2}, \tag{14}$$

which is another site on the same lattice (the proof is given in appendix B where the relation between the integers N_1', N_2', N_1 and N_2 is presented).

Figure 3 explains why the characteristic lattice keeps invariant under the Lorentz transformation. We choose a spacetime with $g = 4$ as an example and set $v_j = \sqrt{3}/2$ which is the lowest positive equivalence velocity in this spacetime. The characteristic lattice of $g = 4$ is a rectangular lattice which can be seen as created by two primitive vectors $(0,\ 1)^T$ and $(\sqrt{3},\ 0)^T$, i.e., $\overrightarrow{OB}$ and $\overrightarrow{OA}$ in figure 3, respectively. Every lattice site can be expressed as $(n_1\overrightarrow{OB} + n_2\overrightarrow{OA})$ with n_1, n_2 being arbitrary integers. Notice that there are infinite ways of choosing primitive vectors. For the observer K', the time and space axes are oriented in different directions, denoted by t' and x', respectively. It is not an accident that t' and x' cross not only the origin but also some other points (A' and B') on the lattice. $\overrightarrow{OA'}$ (the green vector) and $\overrightarrow{OB'}$ (the blue vector) can be seen as a new pair of primitive vectors of the

lattice, that is every lattice site can also be uniquely expressed as $(n_1'\ \overrightarrow{OB'} + n_2'\ \overrightarrow{OA'})$ with two new integers n_1' and n_2'. For example, we have $\overrightarrow{OB} = 2\ \overrightarrow{OB'} - \overrightarrow{OA'}$ and $\overrightarrow{OA} = -3\ \overrightarrow{OB'} + 2\ \overrightarrow{OA'}$. And in the reference frame K', the length of $\overrightarrow{OB'}$ and $\overrightarrow{OA'}$ is a and $\sqrt{3}\,a/c$, respectively, as same as the length of $\overrightarrow{OB}$ and $\overrightarrow{OA}$ in the K reference frame, respectively. Therefore, in the eye of K', the characteristic lattice is exactly the same rectangular lattice as that in the eye of K.

Now let us discuss the time dilation and the length contraction. Imagine a clock staying at rest relative to K', i.e., moving on the t' axis relative to K. The world line of this clock during one period is $\overrightarrow{OA'}$ with $t'_{A'} = \sqrt{3}\,a/c$ being the period of the time translational symmetry. But for another clock staying at rest relative to K, the event A' happens at the time $t_{A'} = 2\sqrt{3}\,a/c$. This reflects the fact that the clock at rest runs twice as fast as the moving clock. In spite of the time dilation, a time interval of integer periods keeps an interval of integer periods in any equivalent reference frames.

On the other hand, the length contraction seems to contradict the discrete translational symmetry at the first sight. Let us choose two static points in the reference frame K, say the points $x = 0$ and $x = a$. Because a is the proposed period of space translational symmetry, the coupling parameters of physical laws always keep the same at these two points. But in the reference frame K', the distance between the two points contracts to $a/2$, which seems to cause a paradox since a is set to the shortest distance for a symmetric space translation. There is in fact no paradox. The world lines of the two points are shown in figure 3, which are the t axis and the pink line, respectively. One must remember that the coupling parameters also change with time. Their changes are synchronized in the reference frame K, but are not in K'. The coupling parameters at the spacetime points O and B are the same, but they are different from the parameters at the point C. When the distance is measured in K', the concerned points are O and C which are simultaneous in K'. The spatial distance between O and C is $a/2$ in K', but the coupling parameters at these two points are different. In previous literatures, researchers were used to take it for granted that the Lorentz contraction forbids a spacetime lattice to have any Lorentz symmetry in the traditional meaning. The above argument clarifies that the Lorentz contraction can coexist with the discrete Lorentz symmetry on a proper spacetime lattice.

7. Causality

The characteristic lattice is constituted of spacetime points (events) arranged periodically. It is therefore interesting to discuss the causality between different events. According to the causal set theory, the causal structure of a spacetime lattice can be used to determine the geometry of the background manifold into which the lattice is embedded [18].

We use the symbol $\prec$ to denote the causal relation [19]. O and A are two events. $O \prec A$ if and only if A is in the future of O and $A \neq O$. In figure 4, the relation $O \prec A$ is represented by an arrow pointing from O to A. In our units, the light cone of O is the 45° and 135° lines (the dashed lines). Therefore, an arrow is a relation if and only if it is along the positive-t direction and the angle between it and the t-axis is less than 45°. As is well known in special relativity, the causal relation is invariant under the Lorentz transformation. Once if $O \prec A$ stands, it stands in any reference frames. There are infinite relations. One then defines the elementary relations—the links. The relation $O \prec A$ is a link if there exists no intervening event X so that $O \prec X \prec A$. The causal structure of the spacetime lattice is totally determined by the links. The links are represented by the red arrows in figure 4. The blue arrow $(\overrightarrow{OD})$ is a relation but not a link, because it can be decomposed into $\overrightarrow{OA} + \overrightarrow{AD}$ which means $O \prec A \prec D$.

The characteristic lattice has periodicity. All the sites on the lattice are equivalent to each other. Therefore, we only need to find out the links starting from a specific point, e.g., the origin O. And the lattice has a mirror symmetry with respect to the t axis. We then focus on the links between the t-axis and the 45° line (in the quadrant I). One can verify that $\overrightarrow{OA}$, $\overrightarrow{OC}$, $\overrightarrow{OA'}$ and $\overrightarrow{OC'}$ are links. In fact, at $x = na$ for each integer n, there exists at most one site that is the end point of the link from O. For example, $\overrightarrow{OC}$ is the link pointing to $x = a$. And the arrows pointing to all the other sites above C at $x = a$ cannot be the links because they can be decomposed into $\overrightarrow{OC}$ and an arrow parallel to the t-axis. Similarly, $\overrightarrow{OA'}$ is the unique link pointing to $x = 3a$. And there is no link pointing to $x = 2a$ or $x = 4a$.

Among $\overrightarrow{OA}$, $\overrightarrow{OC}$, $\overrightarrow{OA'}$ and $\overrightarrow{OC'}$, $\overrightarrow{OA}$ and $\overrightarrow{OC}$ are obviously links. $\overrightarrow{OA'}$ and $\overrightarrow{OC'}$ are links because they can be obtained by Lorentz transforming $\overrightarrow{OA}$ and $\overrightarrow{OC}$, respectively. Recall that $\overrightarrow{OA'}$ is the primitive vector of the lattice for the observer K' (see figure 3). Under the Lorentz transformation from K to K', the primitive vectors transform as $\overrightarrow{OA} \to \overrightarrow{OA'}$ and $\overrightarrow{OB} \to \overrightarrow{OB'}$, and then $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$ transforms into $\overrightarrow{OC'} = \overrightarrow{OA'} + \overrightarrow{OB'}$. The

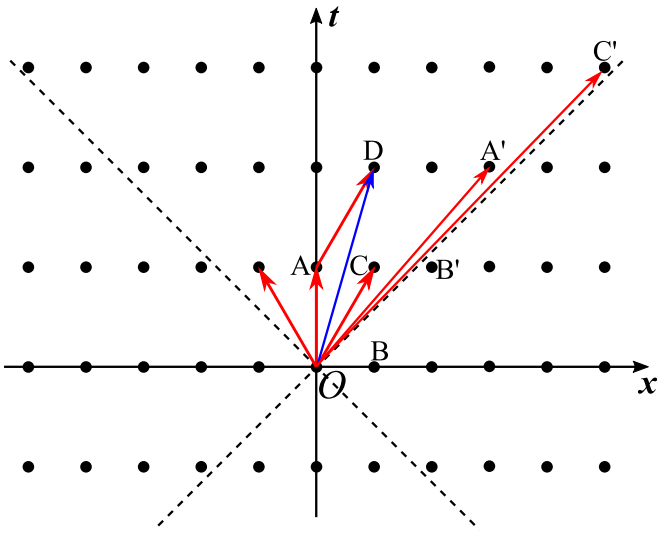

Figure 4. The characteristic lattice of $g = 4$ as a causal set.

relation 'link' is invariant under the discrete Lorentz transformations. Therefore, $\overrightarrow{OC'}$ and $\overrightarrow{OA'}$ must also be links.

Invariance of the relation 'link' can be proved by contradiction. Suppose that a relation $\overrightarrow{OA}$ is a link in the frame K but not a link in K'. But $\overrightarrow{OA}$ is still a causal relation in K', because the causal relation is invariant under arbitrary Lorentz transformation. We then suppose that $\overrightarrow{OA}$ can be decomposed into $\overrightarrow{OX} + \overrightarrow{XA}$ with $\overrightarrow{OX}$ and $\overrightarrow{XA}$ being the causal relations in K'. But $\overrightarrow{OX}$ and $\overrightarrow{XA}$ must also be causal relations in K. Therefore, $\overrightarrow{OA}$ is also not a link in K, which causes a paradox.

We have infinite discrete Lorentz transformations L_{v_j} with $j = 1, 2, 3, \cdots$ that keep the lattice invariant. L_{v_j} for arbitrary j acting on $\overrightarrow{OA}$ and $\overrightarrow{OC}$ produces a new pair of links. There are then infinite possible links starting from O with their end points being just above the 45° line.

8. Field theory that has the discrete Poincaré symmetry

In this section, we discuss how to construct a field theory with the discrete Poincaré symmetry. Such a theory is expected to be the effective theory describing a system in which the space translational symmetry is spontaneously broken into a discrete one.

We will first discuss the continuous field theory in the section 8.1. The continuous field theory is defined in a continuous spacetime, in which the time derivative and the space derivative of the field are present in the Lagrangian density. In the section 8.2, we turn to the lattice field theory, which is defined on the characteristic lattice of the discrete Poincaré group. In the lattice field theory, the coupling between different lattice sites take the place of the derivative operators. Both field theories are invariant under the discrete Poincaré transformations.

8.1. Field theory

Let us recall how to write down a relativistic field theory that has the continuous Poincaré symmetry. To guarantee the Poincaré symmetry, the Lagrangian density $\mathcal{L}(y)$ must be a scalar (rank-0 tensor). It is made up of constants, scalar fields, or the contraction of higher-rank tensors. For example, the Klein–Gordon Lagrangian density for spinless particles is expressed as [2]

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{1}{2}m^2\phi^2, \tag{15}$$

where ϕ is a real scalar field, m is a constant, and $\partial_{\mu}\phi$ and $\partial^{\mu}\phi$ are the covariant and contravariant vectors (rank-1 tensors), respectively. Here $\mu = 0, 1$ denotes the temporal and spacial components, respectively. The metric signature is chosen to $(+, -)$. The contraction of a covariant and a contravariant vectors leaves a scalar. Therefore, the Lagrangian density (15) is a scalar which keeps invariant under arbitrary Poincaré transformation. In other words, the Lagrangian density in two coordinate systems K and K' satisfies $\mathcal{L}(y) = \mathcal{L}'(y')$.

Let us see how to modify equation (15) to obtain a new Lagrangian density that loses the continuous Poincaré symmetry but keeps only the discrete Poincaré symmetry $\mathcal{P}$. In other words, we want to construct a Lagrangian density that keeps invariant under a Poincaré transformation Λ if and only if Λ is in $\mathcal{P}$. There is only a single tunable parameter in the Lagrangian density (15), which is the mass m. Alternatively, one can treat $M = m^2$ as the tunable parameter. To break the continuous Poincaré symmetry, we replace the constant M by a function $M(y)$. The Lagrangian density in the coordinate system K becomes

$$\mathcal{L}(y) = \frac{1}{2}\partial^{\mu}\phi(y)\partial_{\mu}\phi(y) - \frac{1}{2}M(y)\phi^2(y). \tag{16}$$

To obtain the Lagrangian density in a different coordinate system K', we need to replace y and ϕ by y' and ϕ', respectively. We obtain $\mathcal{L}'(y') = \frac{1}{2}\partial^{\mu}\phi'(y')\partial_{\mu}\phi'(y') - \frac{1}{2}M(y')\phi'^2(y')$. Note that ϕ and $\partial^{\mu}\phi\partial_{\mu}\phi$ are scalars which keep invariant under an arbitrary Poincaré transformation, that is $\phi(y) = \phi'(y')$ and $\partial^{\mu}\phi(y)\partial_{\mu}\phi(y) = \partial^{\mu}\phi'(y')\partial_{\mu}\phi'(y')$. Therefore, $\mathcal{L}'(y') = \mathcal{L}(y)$ if and only if $M(y) = M(y')$. The Lagrangian density (16) keeps invariant under a Poincaré transformation if and only if the function $M(y)$ keeps invariant under this transformation. In order that the Lagrangian density (16) has the symmetry $\mathcal{P}$, we must choose a function M that satisfies

$$M(y) = M(y'), \tag{17}$$

where $y' = \Lambda y$ for arbitrary Λ in $\mathcal{P}$. But $M(y) \neq M(\Lambda y)$ if Λ is not in the group $\mathcal{P}$.

Let us see how to construct the function $M(y)$. The detailed derivation is given in appendix C. Here we only give the results. $M(y)$ is a periodic function in the 1+1-dimensional spacetime and has the same periodicity as the characteristic lattice of $\mathcal{P}$. We define two reciprocal primitive vectors which are $k_{\mu}^{(1)} = \left(\frac{-2\pi g}{\sqrt{g^2-4}}, 2\pi\right)$ and $k_{\mu}^{(2)} = \left(\frac{4\pi}{\sqrt{g^2-4}}, 0\right)$. M must be expressed as the Fourier transformation

$$M(y) = \sum_{n_1,n_2=-\infty}^{\infty} M_{n_1n_2}\mathrm{e}^{\mathrm{i}(n_1k_{\mu}^{(1)}+n_2k_{\mu}^{(2)})y^{\mu}}, \tag{18}$$

where n_1 and n_2 are integers, y^{μ} is the coordinate vector ($y^0 = t$ and $y^1 = x$), and $M_{n_1n_2}$ is the coefficient of the Fourier transformation. The Einstein summation convention has been used in equation (18). Furthermore, the coefficients $M_{n_1n_2}$ should satisfy

$$M_{n_1n_2} = M_{n_1'n_2'} \tag{19}$$

for arbitrary integer pairs $(n_1,\ n_2)$ and $(n_1',\ n_2')$ that have the relation

$$\begin{cases} n_1' = z_{j+1}n_1 - z_jn_2 \\ n_2' = z_jn_1 - z_{j-1}n_2 \end{cases}. \tag{20}$$

Here z_j is an integer sequence generated by g (see appendix B for a detailed discussion about z_j). For $j \geqslant 0$, the first two elements of the sequence are $z_0 = 0$ and $z_1 = 1$, and the left ones are generated according to

$$z_{j+1} = gz_j - z_{j-1}. \tag{21}$$

And $z_{-j} = -z_j$ is an odd function of j. According to the properties of z_j, all the integer pairs (n_1, n_2) which are related to each other by equation (20) make up an equivalence class. All the coefficients $M_{n_1n_2}$ with $(n_1,\ n_2)$ being in the same class must be the same. But if (n_1, n_2) and $(n_1',\ n_2')$ are in different classes, $M_{n_1n_2}$ and $M_{n_1'n_2'}$ are not necessarily the same. For example, as $g = 3$, $(0, 0)$ is by itself a class, and $(1, 1)$, $(2, 1)$, $(1, 2)$, $(5, 2)$, $(2, 5)$, $\cdots$ are in the same class, but $(1, 3)$ is not in this class. Therefore, we have $M_{1,1} = M_{2,1} = M_{1,2} = M_{5,2} = M_{2,5} = \cdots$, but $M_{0,0}$, $M_{1,1}$ and $M_{1,3}$ may be different from each other. Equations (18)–(20) provide a method of constructing any function $M(y)$ that is invariant under $\mathcal{P}$. In order that $M(y)$ is a real function, we need further require $M_{n_1,n_2}^* = M_{-n_1,-n_2}$. A special example of $M(y)$ can be obtained by setting $M_{n_1n_2} = \bar{M}$ to a constant, that is $M_{n_1n_2}$ in different classes are all the same. The corresponding $M(y)$ is

$$M(y) = \bar{M}\frac{\sqrt{g^2-4}}{2}\sum_{N_1,N_2}\delta(x - Y_{N_1N_2}^1)\delta(t - Y_{N_1N_2}^0), \tag{22}$$

where $Y_{N_1N_2}^0$ and $Y_{N_1N_2}^1$ denote the temporal and the spatial components of the vector $Y_{N_1N_2}$ (see equation (12)), respectively. $M(y)$ is a Dirac-δ function centered at the sites of the characteristic lattice. In the derivation of equation (22) we used the relation $\sum_n \mathrm{e}^{\mathrm{i}nx} = 2\pi\sum_N \delta(2\pi N - x)$.

By using the above approach, we can change any relativistic field theory (e.g., the theory of vector fields and spinor fields) into a theory that has the discrete Poincaré symmetry $\mathcal{P}$. We start from a theory that keeps invariant under arbitrary continuous Poincaré transformations. We then replace the constants (e.g., the coupling or the mass) in the theory into the functions like $M(y)$. Since $M(y)$ keeps invariant only under $\mathcal{P}$, so is the new Lagrangian density.

The charge conjugation (C), the parity (P), and the time reversal (T) symmetries are frequently considered in the study of field theories. The charge conjugation concerns the transformation in the internal space, therefore, it is independent of whether the theory has a continuous or a discrete Poincaré symmetry. The parity and the time reversal symmetries concern the coordinate transformation, just like the Poincaré symmetry. If we need the Lagrangian density (16) to have the PT symmetry, we need to impose a further constraint on the function $M(y)$, that is $M(y) = M(-y)$. According to equation (18), this can be realized by demanding $M_{n_1,n_2} = M_{-n_1,-n_2}$. The function (22) has this property. Because it is centered on the characteristic lattice, and the characteristic lattice for arbitrary g has the PT symmetry (see figures 1 and 2).

8.2. Lattice field theory

Sometimes we hope to simulate the field theory by using computers and then need to discretize the spacetime. It is impossible to discretize a spacetime without breaking the continuous Poincaré symmetry. But one can maintain the discrete Poincaré symmetry $\mathcal{P}$ of a theory when discretizing the spacetime into the characteristic lattice of $\mathcal{P}$. Because the characteristic lattice keeps invariant under the transformations in $\mathcal{P}$. Next we discuss how to build a lattice field theory [28] that has the symmetry $\mathcal{P}$.

Let us write down a general noninteracting lattice model whose action is

$$S = \sum_{P_1P_2Q_1Q_2} h_{P_1P_2,Q_1Q_2} \times \phi_{P_1P_2} \times \phi_{Q_1Q_2}, \tag{23}$$

where $\phi_{P_1P_2}$ denotes the value of the field ϕ at the lattice site $Y_{P_1P_2}$, and $h_{P_1P_2,Q_1Q_2}$ denotes the coupling between the sites $Y_{P_1P_2}$ and $Y_{Q_1Q_2}$ with P_1, P_2, Q_1 and Q_2 being all integers. The characteristic lattice by itself is invariant under an arbitrary transformation $\Lambda \in \mathcal{P}$. But each lattice site changes into another one after the transformation. Suppose that, under a transformation Λ, the sites $Y_{P_1P_2}$ and $Y_{Q_1Q_2}$ change into $Y_{P_1'P_2'}$ and $Y_{Q_1'Q_2'}$, respectively, i.e. $Y_{P_1'P_2'} = \Lambda Y_{P_1P_2}$ and $Y_{Q_1'Q_2'} = \Lambda Y_{Q_1Q_2}$. The action S keeps invariant under the transformation Λ if and only if

$$h_{P_1P_2,Q_1Q_2} = h_{P_1'P_2',Q_1'Q_2'}. \tag{24}$$

For S being invariant under the group $\mathcal{P}$, we need the coupling function h to be invariant under $\mathcal{P}$. As proved in appendix C, such a coupling function depends only upon the difference between the sites $Y_{P_1P_2}$ and $Y_{Q_1Q_2}$, i.e.

$$h_{P_1P_2,Q_1Q_2} = h(P_1 - Q_1,\, P_2 - Q_2). \tag{25}$$

Furthermore, if we use the notation $Y_{N_1N_2} = Y_{P_1P_2} - Y_{Q_1Q_2}$, or equivalently, $N_1 = P_1 - Q_1$ and $N_2 = P_2 - Q_2$, the function $h(N_1,\, N_2)$ must satisfy

$$h(N_1,\, N_2) = h(N_1',\, N_2') \tag{26}$$

for arbitrary integer pairs (N_1, N_2) and $(N_1',\, N_2')$ that have the relation

$$\begin{cases} N_1' = z_{j+1}N_1 + z_jN_2 \\ N_2' = -z_jN_1 - z_{j-1}N_2 \end{cases}. \tag{27}$$

Here the integer sequence z_j appears again. Similarly, all the integer pairs (N_1, N_2) which are related to each other by equation (27) make up an equivalence class. And equation (26) in fact states that $h(N_1, N_2)$ in each class has a unique value. One should notice the difference between equations (27) and (20). As (N_1, N_2) denotes a site on the characteristic lattice, (n_1, n_2) denotes a site on the reciprocal lattice.

Equations (25)–(27) give the conditions of the coupling function h. Equipped with a coupling function satisfying these conditions, the lattice field theory must have the symmetry $\mathcal{P}$.

9. Conservation of quasi-energy and quasi-momentum

We will not quantize the field theories in this paper, which is left for future study. But it is interesting to discuss some possible features of the quantized theory. Equation (16) describes a physical system in a periodic potential which varies both with time and space. The consequence of such a potential is well known to condensed matter community. The electrons moving in a crystal feel a typical periodic potential varying with space. While the electrons in an irradiated material are often treated as moving in the time-periodic electromagnetic potential.

The momentum and energy are not conserved in the presence of periodic potentials. They are not good quantum numbers any more. But according to the Floquet theorem and the Bloch theorem, the quasi-energy and the quasi-momentum are good quantum numbers instead, which are defined as the energy or the momentum modulo $2\pi\hbar/T$ or $2\pi\hbar/a$, respectively, where T and a are the periods of the potential in the temporal and spatial directions, respectively. Our analysis in section 5 has already established the relation between T and a.

Due to the spatial periodic potential, the single-particle spectrum should include a series of Bloch bands. The energy levels are distinguished by the quasi-momentum and the band label. If we further consider the potential being temporally periodic, the solutions of the Schrödinger equation must be further distinguished by the quasi-energy and the Floquet band label. The quasi-energy, the quasi-momentum, the Bloch band and the Floquet band together determine a solution.

Recent study showed that an isolated generic Floquet system will be heated up until it reaches the infinite-temperature state [29]. But we should not forget that our three hypotheses stand in a system where the continuous translational symmetry has been spontaneously broken. The Lagrangian (16) is the effective theory for the symmetry-breaking state, describing something like the electrons moving in a crystal. Therefore, the system is in fact an open system, to which the argument in [29] does not apply.

10. Conclusions and outlook

In summary, we propose a theory about the Lorentz and Poincaré symmetries in a spacetime with discrete space translational symmetry based on three hypotheses. In solid-state physics, the spacetime occupied by crystals is expected to have these properties. We describe the whole symmetries of the spacetime which include the discrete Lorentz, space translational and time translational symmetries, and show how to construct a Lagrangian or action under these symmetries. It is worth emphasizing that our hypotheses and results are expected to stand in the case of the continuous space translational symmetry being spontaneously broken but not being broken by periodic external fields.

It is worth mentioning the difference between our hypotheses and those hiding behind the effective models of crystals (such as the Hubbard model [30]) that were frequently used up to now. In these effective models, the kinetic energy is either expressed as $p^2/2m$ with p and m denoting the momentum and the mass, respectively, or expressed as the hopping energy between neighbor sites on the crystal lattice. In the language of quantum field theory, the kinetic part of the Hamiltonian looks like $\psi(x)(-\hbar^2\partial_x^2/2m)\psi(x)$ in a field theory or $c_i^\dagger c_{i+1}$ in a lattice field theory where $\psi(x)$ and c_i denote the field operators of particles (e.g. electrons) in the crystal. The discrete space translational symmetry is considered, but the Lorentz symmetry is not under consideration in these models whose Lagrangians or Hamiltonians always change as transformed from one reference frame to the other moving at a different velocity. The complete loss of Lorentz symmetry can be viewed as a special case of our theory as the generator of the spacetime is $g = 2$ and then the discrete Lorentz group $\mathbf{L}$ contains only a single element $\Lambda(1, 0)$ (the identity element). On the other hand, the general version of our theory assumes $g > 2$, and then there exist infinite number of Lorentz transformations with nonzero velocities in the symmetry group. Our theory involves more symmetries than the models in previous studies. According to our theory, the Lagrangian of a model can only take some specific form, which the previous models do not have.

Our theory is built on three hypotheses. Recall that the first hypothesis is similar to but weaker than the principle of relativity, while the second one states an invariant speed c which represents the speed limit of the propagation of information or matter in crystals. These two hypotheses are not deduced from any known principles. Whether they are true should be examined by experiments. We would like to point out some results coming out from these hypotheses that could be checked by experiments. One is the breaking of the continuous time translational symmetry. In a spacetime with an odd generator g, the time translational symmetry has a period $T = \sqrt{g^2 - 4}\,a/c$, while in that with an even g the period is $T = \sqrt{g^2 - 4}\,a/2c$. If the spacetime has a discrete time translational symmetry, the local observables should change periodically with time, just as they vary periodically within the space of the crystal lattice. Let us estimate the magnitude of the temporal period T which depends on a/c. The lattice constant of a crystal is typically at the nanoscale. c is distinguished from the light speed in vacuum but is expected to be at the same magnitude as it. We choose $a = 1$ nm and $c = 3 \times 10^8\ \mathrm{m\,s^{-1}}$, and find $a/c = 3.3 \times 10^{-18}$ s. The temporal period is only a few attoseconds (too short), which maybe explains why such a periodicity has not been observed up to now. An alternate way is to examine the absorption spectrum of crystals. According to the Floquet theorem (see [27] for a recent review), the energy of a time-periodic system is not conserved, and should be replaced by the quasi-energy which has a period of $\hbar\,2\pi/T$. A resonance happens between the quasi-energy levels whose difference is an integer times of $\hbar\,2\pi/T$, which may cause a peak at the frequency $1/T$ in the absorption spectrum of crystals. Note that $1/T \sim 10^{18}$ Hz is in the frequency range of x-rays.

It is worth mentioning that the time-periodic oscillation of observables has been predicted in the theory of 'time crystals' [8]. But whether there exists a 'time crystal' is still under debate. It was argued that a time-periodic oscillation of observables cannot happen in an equilibrium state described by the Gibbs ensemble [12]. On the other hand, our theory indicates that the Lagrangian of a model for crystals should be time-periodic, in which case the idea of describing the equilibrium states by Gibbs ensemble should be reexamined since it does not put space and time on an equal footing.

Finally, we would like to mention the open problems that are expected to be solved in the future. These include the construction of the discrete Lorentz symmetry in 2+1-dimensional and 3+1-dimensional spacetimes, and the quantization of a field theory that has the discrete Poincaré symmetry.

Acknowledgments

This work is supported by NSFC under Grant No. 11304280 and 11774315.

Appendix A. The discrete Lorentz group

We have proved in section 3.2 that the velocity of a Lorentz transformation in the symmetry group must be $v = \pm\sqrt{1-\frac{4}{m^2}}$. In other words, the symmetry group of pure Lorentz transformations is a subset of

$$\mathcal{V} = \left\{\Lambda(L_v, 0)|v = \pm\sqrt{1-\frac{4}{m^2}}, \quad m = 2, 3, 4\cdots\right\}. \tag{A1}$$

In this section, we prove that such a group must be a cyclic group generated by some integer g, as defined in section 4. Recall that a group is closed with respect to multiplication and the Lorentz matrix is expressed as

$$L_v = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix}. \tag{A2}$$

The multiplication between two Lorentz transformations is $\Lambda(L_v, 0)\Lambda(L_{v'}, 0) = \Lambda(L_vL_{v'}, 0)$ where $L_vL_{v'}$ is the product of two matrices.

The simplest group that is a subset of $\mathcal{V}$ is the trivial group containing only the identity transformation at $v = 0$ or $m = 2$. It is a special cyclic group. On the other hand, according to our first hypothesis, the symmetry group should contain at least one Lorentz transformation with $v \neq 0$. Let us suppose that except for the identity element the symmetry group contains an element $\Lambda(L_{v(g)}, 0)$ with $v(g) = \sqrt{1-\frac{4}{g^2}}$ for some integer $g > 2$. Note that supposing $v(g) > 0$ does not lose the generality since $\Lambda(L_{v(g)}, 0)$ and $\Lambda^{-1}(L_{v(g)}, 0) = \Lambda(L_{-v(g)}, 0)$ must be the elements of a group simultaneously. Once if $\Lambda(L_{v(g)}, 0)$ is an element, according to the property of a group, $\Lambda^j(L_{v(g)}, 0) = \Lambda(L^j_{v(g)}, 0)$ must be an element of the group for arbitrary integer $j = 0, \pm1, \pm2,\cdots$. The physical meaning of $\Lambda^j(L_{v(g)}, 0)$ is clear. We choose an observer K_0, and call who is moving at the velocity $v(g)$ relative to K_0 the observer K_1. Similarly, the observer K_{j+1} is moving at the velocity $v(g)$ relative to K_j. Therefore, the coordinate transformation from K_0 to K_j is $\Lambda^j(L_{v(g)}, 0)$. The set $\mathbf{L} = \{\Lambda^j(L_{v(g)}, 0)|j = 0, \pm1, \pm2,\cdots\}$ satisfies all the properties of a group. It is a cyclic group. The set of observers $\{K_j|j = 0, \pm1, \pm2,\cdots\}$ are all equivalent to each other in describing the physical laws.

Next we prove that $\mathbf{L}$ is a subset of $\mathcal{V}$, i.e., any element $\Lambda^j(L_{v(g)}, 0)$ of $\mathbf{L}$ is also in $\mathcal{V}$. Let us denote the velocity of K_j relative to K_0 as v_j. By definition, we have $v_0(g) = 0$ and $v_1(g) = v(g) = \sqrt{1-\frac{4}{g^2}}$, and the Lorentz matrix relating K_0 to K_j is $L_{v_j(g)} = L^j_{v_1(g)}$. Similarly, the velocity of K_j relative to K_i is denoted as v_{j-i} which satisfies $L_{v_{j-i}(g)} = L^{j-i}_{v_1(g)}$, and the velocity of K_i relative to K_0 is v_i satisfying $L_{v_i(g)} = L^i_{v_1(g)}$. Note that $L^i_{v_1(g)}$ is the ith power of $L_{v_1(g)}$. We then have $L^j_{v_1(g)} = L^i_{v_1(g)}L^{j-i}_{v_1(g)}$ or $L_{v_j(g)} = L_{v_i(g)}L_{v_{j-i}(g)}$. By using the expression of L_v in equation (A2), we obtain

$$v_j(g) = \frac{v_i(g) + v_{j-i}(g)}{1 + v_i(g)v_{j-i}(g)}. \tag{A3}$$

Equation (A3) is the velocity-addition formula which is as same as that in special relativity, because both are derived from the Lorentz transformation. It is easy to verify that v_i is an odd function of i, i.e., $v_{-i} = -v_i$, and $v_{j\pm i} = \frac{v_j \pm v_i}{1 \pm v_iv_j}$ according to equation (A3).

Now we define $m_j(g) = \frac{2}{\sqrt{1-v_j(g)^2}}$ for each velocity $v_j(g)$. One can easily see $m_0 = 2$ and $m_1(g) = g$ from $v_0 = 0$ and $v_1(g) = \sqrt{1-\frac{4}{g^2}}$, respectively. By definition, $m_j(g) = m_{-j}(g)$ is an even function of j. To prove that $\Lambda^j(L_{v(g)}, 0) = \Lambda(L_{v_j(g)}, 0)$ is an element of $\mathcal{V}$, we only need to prove that $m_j(g)$ is an integer. This is done by finding an iterative formula for $m_j(g)$. Expressing $m_{j\pm i}$ by using $v_{j\pm i}$ and then by v_i and v_j, we obtain

$$m_{i+j} + m_{j-i} = m_i m_j. \tag{A4}$$

Choosing $i = 1$, we have

$$m_{j+1} = g m_j - m_{j-1}. \tag{A5}$$

Since we already know m_0 and m_1, equation (A5) can be used to calculate m_j iteratively. For example, we find $m_2 = g^2 - 2$, $m_3 = g^3 - 3g, \cdots$. The numbers $m_0, m_1, m_2, \cdots$ make up an infinite sequence. And because $m_0 = 2$ and $m_1 = g$ are both integers, m_j for arbitrary j in the sequence must be an integer according to equation (A5). Therefore, $\Lambda^j(L_{v(g)}, 0)$ for arbitrary j and g is an element of $\mathcal{V}$, and $\mathbf{L}$ is a subset of $\mathcal{V}$.

Up to now, we proved that the cyclic group $\mathbf{L}$ generated by an integer g is a subset of $\mathcal{V}$. Next we prove that $\mathbf{L}$ is the only possible group that is a subset of $\mathcal{V}$. We will construct a proof by contradiction. We assume that there exists a group $\mathcal{G}$ which is included in $\mathcal{V}$ but not a cyclic group. By definition, $\mathcal{G}$ must include at least two elements $\Lambda(L_{v(g)}, 0)$ and $\Lambda(L_{v(g')}, 0)$ where g, $g' > 2$ are both integers and not in the same sequence m_j generated by an integer. Especially, g and g' are not in the sequence generated by each other. Without loss of generality, we suppose $2 < g < g'$. Remember that g' is not in the sequence generated by g, i.e. $g' \neq m_j(g)$ for arbitrary j. According to the property of group, $\Lambda(L_{v''}, 0) = \Lambda(L_{v(g')}, 0)\Lambda^{-1}(L_{v(g)}, 0)$ is an element of $\mathcal{G}$. The velocity-addition formula reads $v'' = (v(g') - v(g))/(1 - v(g)v(g'))$. We can deduce $v'' < v(g')$ since $v(g) > 0$, and also $v'' \neq v(g)$, otherwise, we have $v(g') = 2v(g)/(1 + v(g)^2)$ and then $g' = m_2(g)$ which contradicts $g' \neq m_j(g)$ for arbitrary j. Since $\Lambda(L_{v''}, 0)$ is in $\mathcal{G}$, it must be also in $\mathcal{V}$, thereafter, $g'' = 2/\sqrt{1 - v''^2}$ is an integer satisfying $g'' \neq g$ and $g'' < g'$, deduced from $v'' \neq v(g)$ and $v'' < v(g')$. We can also deduce $g'' > 2$, since $g'' = 2$ indicates $v'' = 0$ and then $v(g) = v(g')$ or $g = g'$ which contradicts our assumption. In consequence, we constructed an integer g'' which is different from both g and g' and is less than the max of them. In the same way, we can use g'' and the smaller one of g and g'(g in this case) to construct a new integer g''' that is different from g or g'' and less than the max of them. We can do this because g (g'') is not in the sequence m_j generated by g'' (g), otherwise, we can deduce that g' is also in the sequence m_j generated by g'' (g) which contradicts the assumption that g and g' cannot be in the same sequence. The process of constructing new integers can be repeated for infinite number of times. Every time we choose the smallest two in the integers that we already obtained to construct a new one. The sequence of integers (g, g'', g''',$\cdots$) that we obtain are all different to each other and all less than g' and larger than 2. But this is impossible, because there only exist finite number of integers between 2 and g'. Our assumption must be false. The only possible groups included in $\mathcal{V}$ are cyclic groups. This finishes the proof.

Appendix B. The discrete Poincaré group

Our hypotheses infer that the overall symmetry group of the spacetime should have next properties: its subgroup for pure Lorentz transformations is $\mathbf{L} = \{\Lambda(L_{v_j(g)}, 0)\}$, and its subgroup for pure spatial translations is $\mathbf{A} = \{\Lambda(1, m\bar{a})\}$ with m an integer and $\bar{a} = (0, 1)^T$. In this section, we prove that the minimum group that has these properties is

$$\mathcal{P} = \{\Lambda(L_{v_j(g)}, Y_{N_1N_2}(g)) | j, N_1, N_2 = 0, \pm 1, \pm 2, \cdots\}, \tag{B1}$$

where

$$Y_{N_1N_2} = N_1\begin{pmatrix}0\\1\end{pmatrix} + N_2\begin{pmatrix}\frac{1}{2}\sqrt{g^2-4}\\ \frac{1}{2}g\end{pmatrix}, \tag{B2}$$

and

$$L_{v_j} = \begin{pmatrix} m_j/2 & \frac{-\mathbf{sgn}(j)}{2}\sqrt{m_j^2-4} \\ \frac{-\mathbf{sgn}(j)}{2}\sqrt{m_j^2-4} & m_j/2\end{pmatrix}. \tag{B3}$$

And any group that has these properties must contain $\mathcal{P}$ as the subgroup. The proof is divided into two steps. First, we prove that $\mathcal{P}$ is a group, i.e., $\mathcal{P}$ is closed under multiplication, and $\mathcal{P}$ has the above-mentioned properties. Second, we prove that a group that has these properties must contain $\mathcal{P}$ by proving that each element in $\mathcal{P}$ can be expressed as a product of the elements in $\mathbf{L}$ and $\mathbf{A}$.

Let us list some important properties of the integer sequence m_j which will be used in the proof. The iterative formula (A5) can be reexpressed as

$$m_{j+1} - \frac{g - \sqrt{g^2 - 4}}{2} m_j = \frac{g + \sqrt{g^2 - 4}}{2} \left(m_j - \frac{g - \sqrt{g^2 - 4}}{2} m_{j-1} \right), \tag{B4}$$

from which we derive an expression of m_j:

$$m_j = \left(\frac{g - \sqrt{g^2 - 4}}{2} \right)^j + \left(\frac{g + \sqrt{g^2 - 4}}{2} \right)^j. \tag{B5}$$

For convenience of presentation, we define a new sequence

$$z_j = \mathbf{sgn}(j) \frac{\sqrt{m_j^2 - 4}}{\sqrt{g^2 - 4}}. \tag{B6}$$

It is straightforward to prove that z_j can be expressed as

$$\sqrt{g^2 - 4} z_j = \left(\frac{g + \sqrt{g^2 - 4}}{2} \right)^j - \left(\frac{g - \sqrt{g^2 - 4}}{2} \right)^j. \tag{B7}$$

The iterative formula of z_j is as same as that of m_j, being

$$z_{j+1} = g z_j - z_{j-1}. \tag{B8}$$

The first two elements of z_j are $z_0 = 0$ and $z_1 = 1$ which are both integers, thereafter, z_j must be also an integer sequence just like m_j! z_j for arbitrary j is an integer, and $z_j = -z_{-j}$ is an odd function of j. The useful formulas involving z_j and m_j are

$$\begin{cases} z_{j+1} = \frac{g}{2} z_j + \frac{m_j}{2} \\ z_{j-1} = \frac{g}{2} z_j - \frac{m_j}{2} \end{cases}. \tag{B9}$$

And a generalized iterative formula for z_j is

$$z_{i+j+1} = z_{i+1} z_{j+1} - z_i z_j, \tag{B10}$$

which can also be expressed in a matrix form as

$$\begin{pmatrix} z_{i+1} & z_i \\ -z_i & -z_{i-1} \end{pmatrix} \begin{pmatrix} z_{j+1} & z_j \\ -z_j & -z_{j-1} \end{pmatrix} = \begin{pmatrix} z_{i+j+1} & z_{i+j} \\ -z_{i+j} & -z_{i+j-1} \end{pmatrix}. \tag{B11}$$

Especially, by taking $i = -j$ we obtain

$$\begin{pmatrix} -z_{j-1} & -z_j \\ z_j & z_{j+1} \end{pmatrix} \begin{pmatrix} z_{j+1} & z_j \\ -z_j & -z_{j-1} \end{pmatrix} = 1. \tag{B12}$$

B.1. $\mathcal{P}$ is a group

According to equation (3), the product of arbitrary two elements in $\mathcal{P}$ is

$$\begin{aligned} &\Lambda(L_{v_j}, Y_{P_1 P_2}) \Lambda(L_{v_i}, Y_{N_1 N_2}) \\ &= \Lambda(L_{v_j} L_{v_i}, Y_{P_1 P_2} + L_{v_j} Y_{N_1 N_2}) \\ &= \Lambda(L_{v_{i+j}}, Y_{P_1 P_2} + L_{v_j} Y_{N_1 N_2}), \end{aligned} \tag{B13}$$

where i, j, N_1, N_2, P_1 and P_2 are all integers. To prove that $\mathcal{P}$ is closed with respect to multiplication, we need to prove that $\Lambda(L_{v_{i+j}}, Y_{P_1 P_2} + L_{v_j} Y_{N_1 N_2})$ is in $\mathcal{P}$. This is equivalent to prove that $L_{v_j} Y_{N_1 N_2} = Y_{N_1' N_2'}$ is a vector in the characteristic lattice for arbitrary L_{v_j} and $Y_{N_1 N_2}$ in the lattice. By using the expression of L_{v_j} (see equations (B3) and (B9)), we obtain

$$\begin{cases} N_1' = z_{j+1} N_1 + z_j N_2 \\ N_2' = -z_j N_1 - z_{j-1} N_2 \end{cases}. \tag{B14}$$

Since z_j for arbitrary j is an integer, N_1' and N_2' must be integers. Therefore, $Y_{N_1' N_2'}$ is a vector in the characteristic lattice, and then $\mathcal{P}$ is closed with respect to multiplication.

In section 5, we already showed that **L** and **A** are the subgroups of $\mathcal{P}$ for pure Lorentz transformations and pure spatial translations, respectively. We conclude that $\mathcal{P}$ is a group that satisfies the conditions of the overall symmetry group.

B.2. The symmetry group cannot be smaller than $\mathcal{P}$

In this subsection, we prove that each element in $\mathcal{P}$ can be expressed as a product of the elements in **L** and **A**.

Recall that the characteristic lattice $\{Y_{N_1N_2}\}$ has two primitive vectors: $Y_{1,0} = (0,\ 1)^T$ and $Y_{0,1} = \left(\frac{1}{2}\sqrt{g^2-4},\ \frac{1}{2}g\right)^T$. By using the expression of $L_{v_{-1}}$ in terms of g (see equation (B3)), we express the second primitive vector as $Y_{0,1} = L_{v_{-1}} Y_{1,0}$. We then obtain

$$\Lambda(1,\ Y_{0,1}) = \Lambda(L_{v_{-1}},\ 0)\Lambda(1,\ Y_{1,0})\Lambda(L_{v_1},\ 0). \tag{B15}$$

$\Lambda(1,\ Y_{1,0})$ denotes the minimum spatial translation which is an element of **A**, and $\Lambda(L_{v_{\pm 1}},\ 0)$ are the elements of **L**. Therefore, $\Lambda(1,\ Y_{N_1,N_2}) = \Lambda(1,\ Y_{1,0})^{N_1}\Lambda(1,\ Y_{0,1})^{N_2}$ can be expressed as a product of the elements in **L** and **A** for arbitrary N_1 and N_2.

For the element $\Lambda(L_{v_j},\ Y_{N_1'N_2'})$ in $\mathcal{P}$, we can factorize it into

$$\Lambda(L_{v_j},\ Y_{N_1'N_2'}) = \Lambda(L_{v_j},\ 0)\Lambda(1,\ Y_{N_1N_2}), \tag{B16}$$

where $Y_{N_1'N_2'} = L_{v_j} Y_{N_1N_2}$ or $Y_{N_1N_2} = L_{v_{-j}} Y_{N_1'N_2'}$. $(N_1',\ N_2')$ and $(N_1,\ N_2)$ satisfy the relation (B14). According to equation (B12), this relation is invertible and its inverse is

$$\begin{cases} N_1 = -z_{j-1}N_1' - z_j N_2' \\ N_2 = z_j N_1' + z_{j+1}N_2' \end{cases}. \tag{B17}$$

For arbitrary $(N_1',\ N_2')$, we can find integers N_1 and N_2 that satisfy equation (B16). This means that each element in $\mathcal{P}$ can be expressed as the product of an element in **L** and $\Lambda(1,\ Y_{N_1N_2})$. But the latter has been proved to be a product of the elements in **L** and **A**. Therefore, each element in $\mathcal{P}$ can be expressed as a product of the elements in **L** and **A**.

Appendix C. Theories that has the discrete Poincaré symmetry $\mathcal{P}$

C.1. The field theory

In this subsection, we explain how to construct the function $M(y)$ which satisfies

$$M(y) = M(\Lambda y) \tag{C1}$$

for arbitrary $\Lambda \in \mathcal{P}$. In other words, $M(y)$ is invariant under $\mathcal{P}$. Each element of $\mathcal{P}$ can be factorized into $\Lambda(L_{v_j(g)},\ Y_{N_1N_2}) = \Lambda(1,\ Y_{N_1N_2})\Lambda(L_{v_j(g)},\ 0)$ where $\Lambda(1,\ Y_{N_1N_2})$ and $\Lambda(L_{v_j(g)},\ 0)$ are also the elements of $\mathcal{P}$. Therefore, $M(y)$ is invariant under $\mathcal{P}$ if and only if $M(y)$ is invariant under the transformations $\Lambda(L_{v_j(g)},\ 0)$ and $\Lambda(1,\ Y_{N_1N_2})$, i.e., $M(y)$ is invariant under the discrete Lorentz group **L** and the discrete translational group **Y**.

We notice that $\Lambda(1,\ Y_{N_1N_2})y = y + Y_{N_1N_2}$. Because M is invariant under **Y**, we obtain $M(y) = M(y + Y_{N_1N_2})$ for arbitrary N_1 and N_2. This means that M is a periodic function in the 1+1-dimensional spacetime, and has the same periodicity as the characteristic lattice $\{Y_{N_1N_2}\}$. Such a periodic function can be expressed as a Fourier transformation. The characteristic lattice has two primitive vectors: $Y_{1,0}$ and $Y_{0,1}$. For convenience of presentation, in this subsection we rename them as $Y^{(1)} = (0,\ 1)^T$ and $Y^{(2)} = \left(\frac{1}{2}\sqrt{g^2-4},\ \frac{1}{2}g\right)^T$. Each vector in the characteristic lattice can be expressed as $Y_{N_1N_2} = N_1 Y^{(1)} + N_2 Y^{(2)}$. The reciprocal lattice has also two primitive vectors which are found to be $k^{(1)} = \left(\frac{-2\pi g}{\sqrt{g^2-4}},\ 2\pi\right)$ and $k^{(2)} = \left(\frac{4\pi}{\sqrt{g^2-4}},\ 0\right)$. The inner product between the primitive vectors of the characteristic lattice and the reciprocal lattice satisfies $k^{(a)} \cdot Y^{(b)} = 2\pi\delta_{a,b}$ where $a, b = 1, 2$ and $\delta_{a,b}$ is the Kronecker delta function. For the momentum vector $k = n_1 k^{(1)} + n_2 k^{(2)}$ with n_1 and n_2 being integers, we have $e^{ik\cdot y} = e^{ik\cdot(y+Y_{N_1N_2})}$. $\{e^{ik\cdot y}\}$ at different (n_1, n_2) form a basis of the periodic functions on the characteristic lattice. Therefore, M must be expressed as

$$M(y) = \sum_{n_1,n_2} M_{n_1n_2} e^{i(n_1k^{(1)}+n_2k^{(2)})\cdot y}, \tag{C2}$$

where $M_{n_1n_2}$ is the coefficient of the Fourier transformation.

$M(y)$ should also be invariant under **L**, which imposes a constraint on the coefficients M_{n_1,n_2}. Substituting equation (C2) into the condition $M(y) = M(\Lambda(L_{v_j},\ 0)y)$, we obtain

$$\sum_{n_1,n_2} M_{n_1n_2} e^{i(n_1k^{(1)}+n_2k^{(2)})\cdot y} = \sum_{n_1,n_2} M_{n_1n_2} e^{i(n_1k^{(1)}L_{v_j}+n_2k^{(2)}L_{v_j})\cdot y}, \tag{C3}$$

where we used the properties of the inner product and $L_{v_j}^T = L_{v_j}$. We notice that

$$n_1k^{(1)}L_{v_j} + n_2k^{(2)}L_{v_j} = n_1'k^{(1)} + n_2'k^{(2)}, \tag{C4}$$

where

$$\begin{cases} n_1' = z_{j+1}n_1 - z_jn_2 \\ n_2' = z_jn_1 - z_{j-1}n_2 \end{cases}. \tag{C5}$$

Therefore, equation (C3) stands if and only if the coefficients $M_{n_1n_2}$ satisfy

$$M_{n_1n_2} = M_{n_1'n_2'} \tag{C6}$$

for the integer pairs $(n_1,\ n_2)$ and $(n_1',\ n_2')$ that are related to each other by equation (C5). The relation (C5) is in fact an equivalence relation which is reflexive, symmetric and transitive. The reflexivity, symmetry and transitivity can be easily proved by using the properties of z_j given in equations (B11) and (B12). The integer pairs $(n_1,\ n_2)$ that are related to each other by equation (C5) form an equivalence class. All the coefficients $M_{n_1n_2}$ with (n_1, n_2) being in the same class must be the same.

C.2. The lattice field theory

In this subsection, we explain how to construct the coupling function h in a lattice field theory that is invariant under $\mathcal{P}$. h must satisfy

$$h_{P_1P_2,Q_1Q_2} = h_{P_1'P_2',Q_1'Q_2'} \tag{C7}$$

with $Y_{P_1'P_2'} = \Lambda Y_{P_1P_2}$ and $Y_{Q_1'Q_2'} = \Lambda Y_{Q_1Q_2}$ for arbitrary $\Lambda \in \mathcal{P}$.

Again, each element of $\mathcal{P}$ can be factorized into $\Lambda(L_{v_j(g)},\ Y_{N_1N_2}) = \Lambda(1,\ Y_{N_1N_2})\Lambda(L_{v_j(g)},\ 0)$. The coupling function h is invariant under $\mathcal{P}$ if and only if it is invariant under the transformations $\Lambda(1,\ Y_{N_1N_2})$ and $\Lambda(L_{v_j(g)},\ 0)$. Since h is invariant under $\Lambda(1,\ Y_{N_1N_2})$, we have $h_{P_1,P_2,Q_1,Q_2} = h_{P_1+N_1,P_2+N_2,Q_1+N_1,Q_2+N_2}$ for arbitrary integers N_1 and N_2. This means that $h_{P_1P_2,Q_1Q_2}$ depends only upon the difference between $(P_1,\ P_2)$ and $(Q_1,\ Q_2)$. We can then reexpress the coupling function as

$$h_{P_1P_2,Q_1Q_2} = h(P_1 - Q_1,\ P_2 - Q_2). \tag{C8}$$

Let us use the notation $Y_{N_1N_2} = Y_{P_1P_2} - Y_{Q_1Q_2}$, or equivalently, $N_1 = P_1 - Q_1$ and $N_2 = P_2 - Q_2$. The coupling function $h_{P_1P_2,Q_1Q_2}$ should be invariant under the Lorentz transformation $\Lambda(L_{v_j},\ 0)$, under which we have $Y_{P_1'P_2'} = \Lambda(L_{v_j},\ 0)\,Y_{P_1P_2}$ and $Y_{Q_1'Q_2'} = \Lambda(L_{v_j},\ 0)\,Y_{Q_1Q_2}$. We then find $Y_{N_1'N_2'} = Y_{P_1'P_2'} - Y_{Q_1'Q_2'} = L_{v_j}Y_{N_1N_2}$. The integer pairs $(N_1,\ N_2)$ and $(N_1',\ N_2')$ have the next relation:

$$\begin{cases} N_1' = z_{j+1}N_1 + z_jN_2 \\ N_2' = -z_jN_1 - z_{j-1}N_2 \end{cases}. \tag{C9}$$

Substituting equation (C8) into equation (C7), we obtain

$$h(N_1,\ N_2) = h(N_1',\ N_2'). \tag{C10}$$

The coupling function must satisfy equation (C10) for being invariant under $\mathcal{P}$. Again, the integer pairs that are related to each other by equation (C9) form an equivalence class. Equation (C10) says that $h(N_1,\ N_2)$ with $(N_1,\ N_2)$ being in the same class must be the same.

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