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Classical information capacity of a class of quantum channels

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Abstract. We consider the additivity of the minimal output entropy and the classical information capacity of a class of quantum channels. For this class of channels, the norm of the output is maximized for the output being a normalized projection. We prove the additivity of the minimal output Renyi entropies with entropic parameters $\alpha \in [0, 2]$, generalizing an argument by Alicki and Fannes, and present a number of examples in detail. In order to relate these results to the classical information capacity, we introduce a weak form of covariance of a channel. We then identify various instances of weakly covariant channels for which we can infer the additivity of the classical information capacity. Both additivity results apply to the case of an arbitrary number of different channels. Finally, we relate the obtained results to instances of bi-partite quantum states for which the entanglement cost can be calculated.

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1. Introduction

The study of capacities is at the heart of essentially any quantitative analysis of the capabilities to store or transmit quantum information. This includes the case of transmission of quantum states through noisy channels modelling decohering transmission lines, such as fibres or waveguides in quantum-optical settings. Capacities and entropic quantities characterizing the specifics of a given quantum channel come in several flavours: for each resource that is allowed for, one may define a certain asymptotic rate that can be achieved. A question that is of key interest here—and a notoriously difficult one—is whether the respective quantities are generally additive. In other words, if we encode quantum information before transmitting it through a quantum channel, can it potentially be an advantage to use entangled inputs over several invocations of the channel? This question is particularly interesting for two central concepts characterizing quantum channels: the minimal output entropy and the classical information capacity.

The classical information capacity specifies the capability of a noisy channel to transmit classical information encoded in quantum states [1, 2]. The question of the classical information capacity is then the one of the asymptotic efficiency of sending classical information from sender to receiver, assuming the capability of encoding data in a coherent manner. This capacity is one of the central notions in the study of quantum channels to assess their potential for communication purposes. The minimal output entropy in turn is a measure for the decoherence accompanied with invocations of the channel. It specifies the minimal entropy of any output that can be achieved by optimizing over all channel inputs [3]. The conjectures on general additivity of both quantities have been linked to each other, in that they are either both true or both false [4]–[6].

The purpose of this paper is to investigate the additivity properties of a class of quantum channels for which the output norm is maximized if the output state is (up to normalization) a projection. For such channels, we prove additivity of the minimal output α -entropies in the interval $\alpha \in [0, 2]$. This further exploits an idea going back to Alicki and Fannes in [7] and Matsumoto and Yura in [8]. For all weakly covariant instances of the considered channels, the additivity is shown to extend to the classical information capacity. Both additivity results are proved for the case of an arbitrary number of different channels. So on the one hand, this paper provides several new instances of channels for which the additivity of the minimal output entropy and the classical information capacity is known. On the other hand, it further substantiates the conjecture that this additivity might be generally true. Finally, following the ideas of [5], we relate

the obtained additivity results to the additivity of the entanglement of formation, for instances of bipartite quantum states. We will begin with an introduction of basic notions and related results in section 2 and the characterization of the considered class of quantum channels in section 3.

2. Preliminaries

Consider a quantum channel, i.e., a completely positive trace-preserving map $T : \mathcal{S}(\mathbb{C}^d) \to \mathcal{S}(\mathbb{C}^d)$ taken to have input and output Hilbert spaces of dimension *d*. The minimal output entropy of the channel, measured in terms of the Renyi α -entropy [9], is given by

$$\nu_{\alpha}(T) := \inf_{\rho} (S_{\alpha} \circ T)(\rho), \qquad S_{\alpha}(\rho) := \frac{1}{1 - \alpha} \log \operatorname{tr}[\rho^{\alpha}], \tag{1}$$

 $\alpha \ge 0$. The α -Renyi entropies are generalizations of the von Neumann entropy defined as $S(\rho) = -\text{tr}[\rho \log \rho]$, which is obtained in the limit $\alpha \to 1$. Therefore, we consistently define $S_1(\rho) := S(\rho)$. Physically, ν_{α} can be interpreted as a measure of decoherence induced by the channel when acting on pure input states. The minimal output α -entropy is said to be additive [10] if for arbitrary $N \in \mathbb{N}$

$$\frac{1}{N}\nu_{\alpha}(T^{\otimes N}) = \nu_{\alpha}(T).$$
⁽²⁾

It is known that additivity of ν_{α} does not hold in general for $\alpha > 4.79$ [11]. For smaller values of α , however, no counterexample is known so far and in particular in the interval $\alpha \in [1, 2]$, where the function $x \mapsto x^{\alpha}$ becomes operator convex, additivity might be conjectured to hold in general.

The classical information capacity of a quantum channel can be inferred from its Holevo capacity [1]. The Holevo capacity of the channel T is defined as

$$C(T) := \sup\left[S\left(\sum_{i=1}^{n} p_i T(\rho_i)\right) - \sum_{i=1}^{n} p_i (S \circ T)(\rho_i)\right],\tag{3}$$

 $n \leq d2$, where the supremum is taken over pure states $\rho_1, \ldots, \rho_n \in \mathcal{S}(\mathbb{C}^d)$ and all probability distributions (p_1, \ldots, p_n) . The classical information capacity is according to the Holevo–Schumacher–Westmoreland theorem [1, 2] given by

$$C_{\rm Cl}(T) := \lim_{N \to \infty} \frac{1}{N} C(T^{\otimes N}), \tag{4}$$

so is the asymptotic version of the above Holevo capacity. Unfortunately, as such, to evaluate the quantity in equation (4) is intractable in practice, being in general an infinite-dimensional non-convex optimization problem. However, in instances where one can show that

$$\frac{1}{N}C(T^{\otimes N}) = C(T),$$
(5)

for all $N \in \mathbb{N}$, then equation (3) already gives the classical information capacity. That is, to know the single-shot quantity in equation (3) is then sufficient to characterize the channel with

respect to its capability of transmitting classical information. A stronger version of the additivity statements in equations (2) and (5) is the one where equality is not only demanded for N instances of the same channel but for N different channels $\bigotimes_{i=1}^{N} T_i$. We will refer to this form of additivity as 'strong additivity'.

The additivity of the Holevo capacity in the sense of the general validity of equation (5) or the additivity of the minimal output entropy is one of the key open problems in the field of quantum information theory—despite a significant research effort to clarify this issue. In the case $\alpha = 1$, the two additivity statements in equations (2) and (5) were shown to be equivalent in their strong version in the sense that if one is true for all channels (including those with different input and output dimensions), then so is the other [4]–[6]. For a number of channels, additivity of the minimal output entropy for $\alpha = 1$ [7], [12]–[15], [17]–[19] and additivity of the Holevo capacity [16]–[21] are known. For integer α , the minimal output α -entropy is more accessible than for values close to 1 [22, 23]. Notably, for the case $\alpha = 2$, a number of additivity statements have been derived [24], and the minimal output entropy can be assessed with relaxation methods from global optimization [25]. For covariant channels, one can indeed infer the additivity of the Holevo capacity from the additivity of the minimal output von Neumann entropy [26]. In fact, as we will discuss in section 5, a much weaker assumption already suffices for this implication.

A paradigmatic and well-known representative of the class of channels we consider in this paper is the Werner–Holevo channel [11], which is of the form

$$T(\rho) = \frac{\mathbb{1}_d - \rho^T}{d - 1}.$$
(6)

This channel serves as a counterexample for the additivity of the minimal output α -entropy for $\alpha > 4.79$. However, for ν_{α} with $\alpha \in [1, 2]$ and for the Holevo capacity, additivity has been proven in [7, 8]. In the following we will generalize these additivity results to a much larger class of channels.

3. Characterization of the class of quantum channels

We will consider a class of channels with a remarkable property: for this class of quantum channels, one can relate the problem of additivity of the minimal output entropy to that of another Renyi- α entropy. The first key observation is the following:

Lemma 1 (Basic property). Let T be a quantum channel for which

$$\nu_{\alpha}(T) = \nu_{\beta}(T), \qquad \alpha > \beta \ge 0. \tag{7}$$

Then the additivity of the minimal output α -entropy implies the additivity for the minimal output β -entropy.

Proof. This statement follows immediately from the fact $S_{\alpha}(\rho) \leq S_{\beta}(\rho)$ for all $\rho \in \mathcal{S}(\mathbb{C}^d)$ and all $\alpha \geq \beta \geq 0$ [27], and the inequality chain

$$\nu_{\beta}(T) = \nu_{\alpha}(T) = \frac{1}{N}\nu_{\alpha}(T^{\otimes N}) = \frac{1}{N}\inf_{\rho}(S_{\alpha} \circ T^{\otimes N})(\rho) \leqslant \frac{1}{N}\inf_{\rho}(S_{\beta} \circ T^{\otimes N})(\rho),$$
(8)

for $N \in \mathbb{N}$. Since on the other hand $\nu_{\beta}(T) \ge \nu_{\beta}(T^{\otimes N})/N$ equality has to hold in equation (8).

Surprisingly, the property required in equation (7) does not restrict the channels to the extent that only trivial examples can be found. Quite to the contrary, a fairly large class of channels has this property. A simple example of a class of channels for which condition (7) is satisfied is the generalization of the Werner–Holevo channel:

Example 1. Consider a channel $T : \mathcal{S}(\mathbb{C}^d) \to \mathcal{S}(\mathbb{C}^d)$ of the form

$$T(\rho) = \frac{1_d - M(\rho)}{d - 1},$$
(9)

where $M : S(\mathbb{C}^d) \to S(\mathbb{C}^d)$ is a linear, trace-preserving positive map (not necessarily a channel) which has the property that there exists an input state leading to a pure output state. Then for all $\alpha > 0$

$$\nu_{\alpha}(T) = \log\left(d - 1\right). \tag{10}$$

Proof. Let us first note that $\rho \mapsto \operatorname{tr}[(\mathbb{1}_d - \rho)^{\alpha}]$ is convex for any $\alpha \ge 1$ and concave for $0 \le \alpha < 1$. Hence, the sought extremum over the convex set of all states is attained at an extreme point, i.e. a pure state. Moreover, all pure states will give the same value. Exploiting this together with the fact that there exists an output under *M* which is pure, and inserting into $S_{\alpha}(\rho) = (\log \operatorname{tr}[\rho^{\alpha}])/(1-\alpha)$ yields equation (10).

The class of channels in example 1 has the property that $\nu_{\alpha}(T)$ is independent of α and therefore condition (7) is trivially satisfied. However, it is not yet the most general class of channels for which ν_{α} is constant. In fact, all quantum channels fulfilling this condition can easily be characterized. This will be the content of the next theorem, which will make use of a lemma that we state subsequently. The following channels are the ones investigated in this paper:

Theorem 1 (Characterization of channels). Let $T : S(\mathbb{C}^d) \to S(\mathbb{C}^d)$ be a quantum channel. Then the following three statements are equivalent:

- 1. The minimal output α -entropy is independent of α . That is, for all $\alpha > \beta \ge 0$ we have $\nu_{\alpha}(T) = \nu_{\beta}(T)$.
- 2. The channel is of the form

$$T(\rho) = \frac{\mathbb{1}_d - mM(\rho)}{d - m},\tag{11}$$

where *M* is a positive, linear and trace-preserving map for which there exists an input state ρ_0 such that $mM(\rho_0)$ is a projection of rank *m*.

3. The maximal output norm $\sup_{\rho} || T(\rho) ||_{\infty}$ is attained for an output state being a normalized projection.

Proof. $1 \to 2$: Since in general $\mathbb{R}^+ \ni \alpha \longmapsto S_{\alpha}(\rho)$ is a non-increasing function for all $\rho \in S(\mathbb{C}^d)$, there exists a state ρ_0 which gives rise to the minimum in ν_{α} for all values of α . Then, by lemma 2, $T(\rho_0)$ has to be a projection except from normalization. In particular, $\sup_{\rho} || T(\rho) ||_{\infty} \leq || T(\rho_0) ||_{\infty} = 1/m_0$, where $m_0 := \operatorname{rank}(T(\rho_0))$. This means that the map $M_0 : S(\mathbb{C}^d) \to S(\mathbb{C}^d)$ defined as

$$M_0(\rho) := \frac{1}{m_0} \mathbb{1}_d - T(\rho)$$
(12)

is positive and has the property that $M_0(\rho_0)$ is except from normalization a projection of rank $m = d - m_0$. Due to the fact that *T* is trace-preserving, the map $M : \mathcal{S}(\mathbb{C}^d) \to \mathcal{S}(\mathbb{C}^d)$,

$$M(\rho) := \frac{m_0}{d - m_0} M_0(\rho),$$
(13)

is also trace-preserving. Hence, the channel T has indeed a representation of the form claimed above.

 $2 \to 3$: We want to argue that $\sup_{\rho} || 1 - mM(\rho) ||_{\infty}$ is attained if $R := mM(\rho)$ is a projection. To this end, note that R is an element of the convex set

$$C := \{r \ge 0 \mid \operatorname{tr}[r] = m, r \le 1\},\tag{14}$$

whose extreme points are projections of rank m. Remember further that the maximum of a convex function (as the largest eigenvalue of a positive matrix) over a closed convex set is attained at an extreme point. When optimizing over the entire set C, the maximum is thus attained for R being a projection of rank m, which is indeed accessible due to the assumed property of M.

 $3 \rightarrow 1$: This follows immediately from $\mathbb{R}^+ \ni \alpha \longmapsto S_\alpha$ being a non-increasing function together with the fact that for any normalized projection ρ_{out} , $S_\alpha(\rho_{out}) = \log \operatorname{rank}(\rho_{out})$ is independent of α .

Lemma 2. Let $\rho \in S(\mathbb{C}^d)$ be a state for which $S_{\alpha}(\rho) = S_{\alpha'}(\rho)$, for some $\alpha' > \alpha \ge 0$. Then ρ is except from normalization a projection and for all $\beta \ge 0$ we have

$$S_{\beta}(\rho) = \log \operatorname{rank}(\rho). \tag{15}$$

Proof. The function $\mathbb{R}^+ \ni \beta \longmapsto S_{\beta}(\rho)$ is a convex and non-increasing function [27]. Hence, the assumption in the lemma immediately implies that $S_{\beta}(\rho) = S_{\alpha}(\rho) =: c$ for all $\beta \ge \alpha$, i.e.

$$\operatorname{tr}[\rho^{\beta}] = 2^{c(1-\beta)},\tag{16}$$

for all $\beta \ge \alpha$. Taking the β th root on both sides and then the limit $\beta \to \infty$ leads to $2^{-c} = \|\rho\|_{\infty}$ and thus

$$tr[(\rho/\|\rho\|_{\infty})]^{\beta} = \|\rho\|_{\infty}^{-1}.$$
(17)

Considering again the limit $\beta \to \infty$ yields that the multiplicity of the largest eigenvalue of ρ is equal to $\|\rho\|_{\infty}^{-1}$, such that ρ has indeed to be a normalized projection.

4. Additivity of the minimal output entropy

For a class of channels of the form in theorem 1, we find the additivity of the minimal output α -Renyi entropy for $\alpha \in [0, 2]$. We exploit lemma 1 for these channels in the simple case, where $\alpha = 2$ and $\beta \in [0, 2]$. What then remains to be shown is the additivity of the minimal output 2-entropy. This can, however, be done in the same way as has been done in [7] for the specific case $M(\rho) = \rho^T$, except that more care has to be taken due to the fact that the involved projections are not necessarily one dimensional.

Theorem 2 (Strong additivity of the minimal output entropy). Consider channels T_1, \ldots, T_N of the form in equation (11) such that $\bigotimes_{i=1}^N M_i$ is a positive map. Then the minimal output α -entropy is strongly additive for all $\alpha \in [0, 2]$, i.e.

$$\nu_{\alpha}\left(\bigotimes_{i=1}^{N} T_{i}\right) = \sum_{i=1}^{N} \nu_{\alpha}(T_{i}) = \sum_{i=1}^{N} \log\left(d_{i} - m_{i}\right)$$
(18)

for $T_i : \mathcal{S}(\mathbb{C}^{d_i}) \to \mathcal{S}(\mathbb{C}^{d_i})$ as in equation (11).

Proof. We can express with $T_i(\rho) = (\mathbb{1}_{d_i} - m_i M_i(\rho))/(d_i - m_i)$, the action of the tensor product channel $T := \bigotimes_{i=1}^N T_i$ as

$$T(\rho) = \prod_{i=1}^{N} \frac{1}{d_j - m_i} \sum_{\Lambda \subset \{1, \dots, N\}} (\omega_\Lambda \otimes \mathbb{1}_{\Lambda^C}) \prod_{k \in \Lambda} (-m_k),$$
(19)

where Λ^C denotes the complement of Λ , $\omega := (M_1 \otimes \cdots \otimes M_N)(\rho)$, and ω_{Λ} denotes the reduced density matrix of ω with respect to the systems labelled with Λ . Hence, we obtain

$$\operatorname{tr}[(T(\rho))^{2}] = \prod_{i=1}^{N} \frac{1}{(d_{i} - m_{i})^{2}} \sum_{\Lambda,\Lambda' \subset \{1,\dots,N\}} \prod_{k \in \Lambda} \prod_{l \in \Lambda'} (-m_{k})(-m_{l})\operatorname{tr}[\omega_{\Lambda \cap \Lambda'}^{2}] \prod_{k \in (\Lambda \cup \Lambda')^{C}} d_{k}$$

$$= \prod_{i=1}^{N} \frac{1}{(d_{i} - m_{i})^{2}} \sum_{\Gamma \subset \{1,\dots,N\}} \operatorname{tr}[\omega_{\Gamma}^{2}] \sum_{\Delta \subset \Gamma^{C}} \sum_{\Lambda' \subset \Gamma^{C} \setminus \Delta} \prod_{k \in \Delta \cup \Gamma} (-m_{k}) \prod_{l \in \Delta' \cup \Gamma} (-m_{l}) \prod_{j \in \Gamma^{C} \setminus \Delta \setminus \Delta'} d_{j}$$

$$= \prod_{i=1}^{N} \frac{1}{(d_{i} - m_{i})^{2}} \sum_{\Gamma \subset \{1,\dots,N\}} \operatorname{tr}[\omega_{\Gamma}^{2}] \prod_{k \in \Gamma} m_{k}^{2} \prod_{j \in \Gamma^{C}} (d_{j} - 2m_{j}).$$

Now, exploiting the subsequently stated lemma 3, we have tr $[\omega_{\Gamma}^2] \leq \prod_{i \in \Gamma} m_i^{-1}$ and thus

$$\operatorname{tr}[(T(\rho))^2] \leq \prod_{i=1}^N \frac{1}{d_i - m_i}.$$
 (20)

Together with the fact that $v_2(T_i) = \log (d_i - m_i)$, this means finally that we obtain

$$\nu_2(T) \ge \prod_{i=1}^N \log (d_i - m_i) = \sum_{i=1}^N \nu_2(T_i) \ge \nu_2(T),$$
(21)

implying by lemma 1 the claimed additivity in the entire interval $\alpha \in [0, 2]$.

Lemma 3. Let $M_i : S(\mathbb{C}^{d_i}) \to S(\mathbb{C}^{d_i})$, i = 1, ..., N, be trace-preserving linear maps, for which there exist positive numbers $m_i \in \mathbb{N}$ such that $\rho \mapsto (\mathbb{1}_{d_i} \operatorname{tr}[\rho] - m_i M_i(\rho))$ is completely positive. If in addition $\bigotimes_{i=1}^N M_i$ is a positive map, then

$$\forall \rho \in \mathcal{S}(\mathbb{C}^{\prod_i d_i}) : \operatorname{tr}\left[\left(\left(\bigotimes_{i=1}^N M_i\right)(\rho)\right)^2\right] \leqslant \prod_{i=1}^N m_i^{-1}.$$
(22)

Proof. Let M_i^* be the adjoint map defined by $tr[M_i^*(A)B] = tr[AM_i(B)]$. Then the complete positivity condition is equivalent to the validity of

$$(M_i^* \otimes \mathbb{1}_{d_i})(P_{12}) \leqslant \frac{\mathbb{1}_{d_i}}{m_i} \otimes \operatorname{tr}_1[P_{12}]$$
(23)

for all positive operators $P_{12} \in \mathcal{S}(\mathbb{C}^{d_i^2})$. In order to apply this inequality, we exploit some of the properties of the flip operator \mathbb{F}_d : $|\Phi\rangle\otimes|\Psi\rangle\mapsto|\Psi\rangle\otimes|\Phi\rangle$ for $|\Psi\rangle, |\Phi\rangle\in\mathbb{C}^d$. Recall that $\operatorname{tr}[A^2] = \operatorname{tr}[(A\otimes A)\mathbb{F}_d]$ and $\mathbb{F}_d^{T_2} = \sum_{i,j=1}^d |i,i\rangle\langle j,j|$. Hence,

$$\operatorname{tr}\left[\left(\left(\bigotimes_{i=1}^{N} M_{i}\right)(\rho)\right)^{2}\right] = \operatorname{tr}\left[\left[\rho \otimes \left(\bigotimes_{i} M_{i}\right)(\rho)\right]\left[\bigotimes_{i} (M_{i}^{*} \otimes \mathbb{1}_{d_{i}})(\mathbb{F}_{d_{i}})\right]\right]$$
(24)

$$= \operatorname{tr}\left[\left[\rho \otimes \left(\left(\bigotimes_{i} M_{i}\right)(\rho)\right)^{T}\right]\left[\bigotimes_{i} (M_{i}^{*} \otimes \mathbb{1}_{d_{i}})(\mathbb{F}_{d_{i}}^{T_{2}})\right]\right]$$
(25)

$$\leq \operatorname{tr}\left[\rho \otimes \left(\left(\bigotimes_{i} M_{i}\right)(\rho)\right)^{T}\right] \prod_{j} m_{j}^{-1}$$
(26)

$$=\prod_{j}m_{j}^{-1}.$$
(27)

Lemma 3 and therefore theorem 2 require the assumption that $\bigotimes_i M_i$ is a positive map. Although the presented proof depends on this property, at present we do not know of any channel of the form in equation (11) for which equation (22) is not valid. In fact, all the following examples are such that $M_i = \Xi_i \circ \theta$, where each Ξ_i is completely positive and θ is the transposition. For all these cases, $\bigotimes_i M_i$ is evidently positive.

Obviously, theorem 2 implies in particular that for any channel $T : S(\mathbb{C}^d) \to S(\mathbb{C}^d)$ of the considered form, we have for all $\alpha \in [0, 2]$

$$\frac{1}{N}\nu_{\alpha}(T^{\otimes N}) = \nu_{\alpha}(T).$$
(28)

As mentioned earlier, the most prominent example of channels in the considered class is the Werner–Holevo channel itself for which $M(\rho) = \rho^T$. For this channel, the additivity of the minimal output entropy has been shown in [8], and with inequivalent methods in [12, 28]. The following list includes further instances of channels for which we find additivity of the minimal output entropy as a consequence of theorem 2. As stated above, all examples are such that the corresponding *M* is a concatenation of a completely positive map and the transposition.

Example 2 (Stretching). For ω being a pure state, consider

$$M(\rho) = \lambda \rho^{I} + (1 - \lambda)\omega, \qquad m = 1.$$
⁽²⁹⁾

Complete positivity is a consequence of this channel being a convex combination of the completely positive Werner–Holevo channel and the channel $\rho \mapsto (\mathbb{1}_d - \omega)/(d - 1)$. Obviously, $\rho_0 = \omega^T$ leads to a normalized projection at the output.

Example 3 (Weyl shifts). Consider the set of unitaries $W_i = \sum_{j=1}^d |j+i \mod d\rangle \langle j|$ and take

$$M(\rho) = \frac{1}{d} \sum_{i=1}^{d} W_i \rho^T W_i^{\dagger}, \quad m = 1.$$
 (30)

Complete positivity of the respective channel T follows from the fact that it is a composition of the Werner–Holevo channel with another completely positive map. The state ρ_0 with $\langle i | \rho_0 | j \rangle = 1/d$ for all i, j = 1, ..., d is an example for an appropriate pure input state for which $M(\rho_0) = \rho_0$.

Example 4 (Pinching). Let $\{P_i\}$ be a set of orthogonal projections yielding a resolution of the identity, i.e., $\sum_i P_i = \mathbb{1}_d$. Then take

$$M(\rho) = \sum_{i} P_i \rho^T P_i, \quad m = 1.$$
(31)

Again the respective channel *T* is a composition of two completely positive maps and thus itself completely positive. Moreover, any pure state ρ_0 for which ρ_0^T is in the support of any P_i gives rise to a normalized projection at the output of *T*.

So far the examples were restricted to the case m = 1. The following examples show explicitly that all larger values of m are possible as well:

Example 5 (Casimir channel for a reducible representation). This example is based on a Casimir channel $T' : S(\mathbb{C}^4) \longrightarrow S(\mathbb{C}^4)$ (see section 5) for a reducible representation of SU(2),

$$T'(\rho) = \sum_{i=1}^{3} A_i \rho A_i^{\dagger},$$
(32)

where $A_i = (4/3)^{1/2} \pi(J_i)$, with

$$\pi(J_1) = \frac{i}{2} \left(|2\rangle\langle 3| + |4\rangle\langle 1| - |1\rangle\langle 4| - |3\rangle\langle 2| \right), \tag{33}$$

$$\pi(J_2) = \frac{i}{2} \left(|3\rangle \langle 1| + |4\rangle \langle 2| - |1\rangle \langle 3| - |2\rangle \langle 4| \right), \tag{34}$$

$$\pi(J_3) = \frac{i}{2} \left(|1\rangle\langle 2| + |4\rangle\langle 3| - |2\rangle\langle 1| - |3\rangle\langle 4| \right).$$
(35)

The operators $\pi(J_1)$, $\pi(J_2)$ and $\pi(J_3)$ form the generators of a four-dimensional reducible representation of the Lie algebra of the group SU(2). As an example for m = 2, consider the channel

$$T(\rho) = \frac{3T'(\rho) + \rho}{4}.$$
(36)

This map is clearly completely positive by construction. We find M to be given by

$$M(\rho) = \mathbb{1}_4/2 - T(\rho). \tag{37}$$

An appropriate input ρ_0 for which the output is a two-dimensional projection $M(\rho_0) = (|3\rangle\langle 3| + |4\rangle\langle 4|)/2$ up to normalization is given by

$$\rho_0 = (|1\rangle\langle 1| + i|1\rangle\langle 4| - i|4\rangle\langle 1| + |4\rangle\langle 4|)/2.$$
(38)

Finally, *M* is a positive map, as it can actually be written as a transposition θ , followed by a completely positive map Ξ , that is, $M = \Xi \circ \theta$. To show that this is indeed the case, consider

$$(M \otimes \mathrm{id})(\Omega^{T_1}) = \frac{1}{2} \otimes \frac{1}{4} - \frac{3}{4}(T' \otimes \mathrm{id})(\Omega^{T_1}) - \frac{1}{4}\Omega^{T_1} \ge 0,$$
(39)

where Ω is the maximally entangled state with state vector $|\Omega\rangle = \frac{1}{2} \sum_{i=1}^{4} |i, i\rangle$.

Example 6 (Shifts and pinching). Let W_k be defined as in example 3 and $K \subset \{1, \ldots, d\}$:

$$M(\rho) = \frac{1}{|K|} \sum_{k \in K} \sum_{i=1}^{d} |i\rangle \langle i| (W_k^{\dagger} \rho W_k) |i\rangle \langle i|, \quad m = |K|.$$

$$\tag{40}$$

In fact, T is an entanglement-breaking channel (cf [13, 20]) which can be written as

$$T(\rho) = \frac{1}{d - |K|} \sum_{i=1}^{d} \langle i | \rho | i \rangle \sum_{k \in \{1, \dots, d\} \setminus K} W_k^{\dagger} | i \rangle \langle i | W_k.$$

$$\tag{41}$$

Example 7 (Coarse graining). For $\mathbb{C}^d = \mathbb{C}^n \otimes \mathbb{C}^D$, consider

$$M(\rho) = \int_{U(D)} \mathrm{d}U\Big(\bigoplus_{i=1}^{n} U\Big)\rho^{T}\Big(\bigoplus_{i=1}^{n} U\Big)^{\dagger}, \quad m = D,$$
(42)

where the integration is with respect to the Haar measure.

The averaging operation in M may physically be interpreted as a coarse graining of an operation, which is only capable of resolving n blocks of size D within a $d = n \cdot D$ dimensional system. In order to prove that the above M leads to an admissible and for n > 1 not entanglement-breaking channel, let us first note that we may, after a suitable reshuffle, equivalently write

$$M(\rho) = \int dU(\mathbb{1}_n \otimes U)\rho^T(\mathbb{1}_n \otimes U)^{\dagger} = \rho_n^T \otimes \frac{\mathbb{1}_D}{D}, \qquad (43)$$

where the tensor product is that of $\mathbb{C}^d = \mathbb{C}^n \otimes \mathbb{C}^D$ and ρ_n^T is the reduction of ρ^T with respect to the first tensor factor \mathbb{C}^n . Obviously, M is positive, trace-preserving and for ρ_0 with $\langle i|\rho_0|j\rangle = 1/d$, we obtain a normalized projection of rank D. Complete positivity of T is equivalent to

$$(T \otimes \mathrm{id})(\Omega) \ge 0,\tag{44}$$

where $|\Omega\rangle = (1/\sqrt{d}) \sum_{i=1}^{d} |i, i\rangle$ is again the state vector of a maximally entangled state Ω . Exploiting again that the latter is related to the flip operator $\mathbb{F}|i, j\rangle = |j, i\rangle$ via partial transposition, i.e. $\Omega^{T_2} = \mathbb{F}/d$, we obtain

$$(T \otimes \mathrm{id})(\Omega) = \left(\frac{\mathbb{1}_{d^2}}{d} - \frac{1}{d} \mathbb{F}_n \otimes \mathbb{1}_{D^2}\right) / (d - D),$$
(45)

where \mathbb{F}_n is the flip operator on $\mathbb{C}^n \otimes \mathbb{C}^n$. Since the latter has eigenvalues ± 1 , the channel defined as above is indeed completely positive. In order to prove that *T* is not entanglement breaking, it is sufficient to show that the partial transpose of equation (45) is no longer positive, which is true since the negative term picks up an additional factor *n*.

Finally, additivity of the minimal output entropy holds for any channel for which there exists a pure output state, leading to a vanishing output entropy. In this case additivity of the

minimal output entropy in the form of equation (2) is evident. However, strong additivity within the considered class of channels is still a non-trivial result. This applies in particular to instances of the 3 and 4-state channels of [29] and the class of so-called diagonal channels, for which strong additivity was proven recently in [30]:

Example 8 (Diagonal channels). Consider $T : S(\mathbb{C}^d) \to S(\mathbb{C}^d)$ with

$$T(\rho) = \sum_{k=1}^{K} A_k \rho A_k^{\dagger}, \tag{46}$$

where A_k , k = 1, ..., K, are all diagonal in a distinguished basis.

5. Classical information capacity

So far we have considered the minimal output entropy of quantum channels and their additivity properties. It turns out that for a large subset of the considered channels, including all the discussed examples 3–8, one can indeed infer the additivity of the Holevo capacity as well. On the one hand, for each covariant instance of a quantum channel from which we know that the minimal output entropy is additive, we can conclude that the Holevo capacity is also additive [26]. For example, this argument applies to the Werner–Holevo channel itself. One the other hand, a quantum channel does not necessarily have to be covariant for a very similar argument to be valid. Subsequently, we will restate the result of [26] using weaker assumptions. The main difference is that for a given channel, one may exploit properties of the state for which the output entropy is minimal. This is particularly useful in our case at hand, where these optimal input states can always be identified in a straightforward manner. We will first state the modified proposition in a general way, and then apply it to the channels at hand of the form as in theorem 1.

Theorem 3 (Strong additivity for the classical information capacity). Let $T : S(\mathbb{C}^d) \to S(\mathbb{C}^d)$ be a quantum channel for which the minimal output von-Neumann entropy is additive, and let $\{\rho_i\}$ be a set of input states for which the minimal output entropy is achieved. If for any probability distribution $\{p_i\}$ and $\overline{\rho} := \sum_i p_i \rho_i$, we have that

$$(S \circ T)(\overline{\rho}) = \sup(S \circ T)(\rho) \tag{47}$$

holds, then the Holevo capacity C(T) is additive and the classical information capacity is given by

$$C_{\rm Cl}(T) = (S \circ T)(\overline{\rho}) - \nu_1(T). \tag{48}$$

Moreover, if the assumptions are satisfied by an arbitrary number of different channels $\{T_k\}$ among which we have strong additivity of the minimal output entropy, then $C(\bigotimes_k T_k) = \sum_k C(T_k)$.

Proof. Let us first consider the Holevo capacity of a single channel. Obviously, C(T) is always upper bounded by the maximal minus the minimal output entropy. Due to the assumed properties of the set $\{\rho_i\}$ this bound is, however, saturated and we have

$$C(T) = \sup \left[S\left(\sum_{j} p_{j} T(\rho_{j})\right) - \sum_{j} p_{j} (S \circ T)(\rho_{j}) \right] = (S \circ T)(\overline{\rho}) - \nu_{1}(T).$$
(49)

In other words, the supremum in C(T) can be calculated separately for the positive and the negative part. Now consider the expression $C(\bigotimes_k T_k)$. If we again separate the two suprema, then by the assumed strong additivity the maximum of the negative part is attained for product inputs. The same is true for the positive part, since the entropy satisfies the sub-additivity inequality $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$. Hence, by evaluating the suprema separately, we obtain an upper bound which coincides with the sum of the achievable upper bounds for the single channels. \Box

In practice, one is often in the position to have a channel which is weakly covariant on an input state ρ_0 which minimizes the output entropy. That is, there are unitary (not necessarily irreducible) representations π and Π of a compact Lie group or a finite group *G*, such that for all $g \in G$

$$T(\pi(g)\rho_0\pi(g)^{\dagger}) = \Pi(g)T(\rho_0)\Pi(g)^{\dagger};$$
(50)

in addition the image of the group average of ρ_0 under T is the maximally mixed state. That is, in the case of a finite group

$$\frac{1}{|G|} \sum_{g \in G} \Pi(g) T(\rho_0) \Pi(g)^{\dagger} = \frac{\mathbb{1}_d}{d},$$
(51)

where we have to replace the sum by an integral with respect to the Haar measure if G is a compact Lie group. The optimal set of states $\{\rho_j\}$ in theorem 3 is then taken to be the set of equally distributed states $\{\pi(g)\rho_0\pi(g)^{\dagger}\}$ (i.e. $p_g = |G|^{-1}$ for all $g \in G$ for a finite group). In fact, the discussed examples 3–8 are of this weakly covariant form.

Obviously, quantum channels which are covariant with respect to an irreducible representation of a compact Lie group always have the required properties. For instance, for the *d*-dimensional Werner–Holevo channel, one may take for the group G = SU(d), the defining representation π , and the conjugate representation Π . Note, however, that the property of the channel required by theorem 3 is significantly weaker than covariance.

To construct new instances of quantum channels for which the additivity of the classical information capacity is found, let us consider the above-mentioned examples. To start with example 3, we know that the state ρ_0 with elements $\langle i | \rho_0 | j \rangle = 1/d$ for i, j = 1, ..., d is an optimal input. To construct an appropriate group *G*, consider the set of unitaries,

$$U_{j} := \sum_{l=0}^{d-1} e^{\frac{2\pi i l j}{d}} |l\rangle \langle l|, \quad j = 1, \dots, d.$$
(52)

It is straightforward to show that

$$T(U_j\rho_0 U_j^{\dagger}) = U_j T(\rho_0) U_j^{\dagger}, \tag{53}$$

$$\frac{1}{d} \sum_{j=1}^{d} U_j T(\rho_0) U_j^{\dagger} = \frac{1}{d}.$$
(54)

That is, by virtue of theorem 3 the channel in example 3 has a classical information capacity of

$$C_{\rm Cl}(T) = \log(d) - \log(d-1).$$
(55)

Example 4 can be treated in a similar fashion. Let us choose the basis in which the projections are diagonal, and take $\rho_0 = |1\rangle\langle 1|$. Obviously, we have that

$$T(W_i \rho_0 W_i^{\dagger}) = W_i T(\rho_0) W_i^{\dagger}, \quad i = 1, ..., d,$$
(56)

$$\frac{1}{d} \sum_{i=1}^{d} W_i T(\rho_0) W_i^{\dagger} = \frac{\mathbb{1}_d}{d},$$
(57)

where the W_i are again the unitary shift operators, again forming an appropriate finite group G. The classical information capacity is given by $C_{Cl}(T) = \log (d) - \log (d - 1)$. Note that the same argument using shift operators, leading to a classical information capacity of $C_{Cl}(T) = \log (d)$, can be applied to the class of diagonal channels of example 8. This result of a maximal classical information capacity is no surprise, however, as one can encode classical information in such a way that information transmission through the channel is entirely lossless.

Then, example 5 is another example of a channel with additive Holevo capacity. This becomes manifest as a consequence of the fact that every Casimir channel [31] based on some representation of SU(2), is covariant under the respective representation. Such Casimir channels are convenient building blocks to construct a large number of channels with additive Holevo capacity. So let us consider for G = SU(2), a *d*-dimensional representation π of *G* [32]. The generators of the associated Lie algebra are denoted with J_k , k = 1, 2, 3. In a mild abuse of notation, we will denote with $\pi(J_k)$ the generators of the Lie algebra of the group SU(2) in the representation π . The respective Casimir channel is given by

$$T(\rho) = \frac{1}{\lambda_{\pi}} \sum_{k=1}^{3} \pi(J_k) \rho \pi(J_k),$$
(58)

where normalization follows from the Casimir operator

$$\sum_{k=1}^{3} \pi (J_k)^2 = \lambda_{\pi} \mathbb{1}_d.$$
(59)

For irreducible representations π of SU(2), we have that $\lambda_{\pi} = (d - 1)(d + 1)/4$. The covariance of the resulting quantum channels can be immediately deduced from the structural constants of the Lie algebra specified as

$$[J_i, J_j] = i\varepsilon_{i,j,k}J_k, \quad i, j, k \in \{1, 2, 3\}.$$
(60)

by making use of the exponential mapping into the group SU(2). Casimir channels $T : S(\mathbb{C}^d) \to S(\mathbb{C}^d)$ with respect to a *d*-dimensional representation π as in equation (58) are covariant in the sense that

$$T(\pi(g)\rho\pi(g)^{\dagger}) = \Pi(g)T(\rho)\Pi(g)^{\dagger}$$
(61)

for all states ρ , where Π is either the defining or the conjugate representation of SU(2).

For d = 3, for example, we reobtain the Werner–Holevo channel. Then, in example 5 as an example of a Casimir channel with respect to a reducible representation, we find that the channel is covariant with respect to this reducible representation. This channel is covariant with respect to

the chosen representation π of SU(2). Moreover, we may start from the optimal input state ρ_0 as specified in the example, leading to an output $T(\rho_0) = (|3\rangle\langle 3| + |4\rangle\langle 4|)/2$. We can generate then an ensemble of states that averages to the maximally mixed state, assuming the Haar measure. That is, we have that

$$\int_{g \in SU(2)} \mathrm{d}g \,\pi(g) T(\rho_0) \pi(g)^{\dagger} = \frac{1}{4}.$$
(62)

To be very specific, with $U_x := \exp(ix_2\pi(J_2))\exp(ix_1\pi(J_1))\exp(ix_3\pi(J_3))$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, this average amounts to

$$\int_{0}^{4\pi} \mathrm{d}x_1 \int_{0}^{\pi} \mathrm{d}x_2 \int_{0}^{2\pi} \mathrm{d}x_3 \,\frac{\sin(x_2)}{16\pi 2} U_x T(\rho_0) U_x^{\dagger} = \frac{1}{4}.$$
(63)

Therefore, we again conclude that the classical information capacity is given by $C_{Cl}(T) = \log (4) - \log (2) = 1$.

In a similar way, the above coarse graining channel can be shown to exhibit an additive Holevo capacity. Here, $M(\rho)$ can be written as in equation (42). Therefore, the reducible representation of SU(n) corresponding to

$$V \otimes \mathbb{1}_D, \qquad V \in SU(n) \tag{64}$$

can be taken as the group appropriately twirling the output resulting from the optimal input. This argument leads to an additive Holevo capacity such that the classical information capacity becomes

$$C_{\rm Cl}(T) = \log(d) - \log(d - D).$$
 (65)

These examples give substance to the observation that quite many channels of the above type can be identified for which the classical information capacity can be evaluated. At this point, indeed, one may be tempted to think that all of the above channels have an additive Holevo capacity. While we cannot ultimately exclude this option, it is not true that theorem 3 can be applied to all channels of the form as in theorem 1. A simple counterexample is provided by example 2, where only a single optimal input state exists, namely $\rho = \omega^T$, such that theorem 3 cannot be applied.

6. Note on the entanglement cost of concominant bipartite states

Finally, we remark on the implications of the results for the additivity of the entanglement of formation. In [5], the additivity of weakly covariant channels has been directly related to the additivity of the entanglement of formation [33]

$$E_F(\rho) = \inf \sum_{i=1}^n p_i (S \circ \operatorname{tr}_B)(\rho_i),$$
(66)

where the infimum is taken over all ensembles such that $\sum_{i=1}^{n} p_i \rho_i = \rho$. The entanglement cost, in turn, is the asymptotic version,

$$E_C(\rho) = \lim_{N \to \infty} \frac{1}{N} E_F(\rho^{\otimes N}).$$
(67)

This entanglement cost quantifies the required maximally entangled resources to prepare an entangled state: it is the rate at which maximally entangled states are asymptotically necessary in order to prepare a bipartite state using only local operations and classical communication. In contrast to the asymptotic version of the relative entropy of entanglement [34], which is known to be different from the relative entropy of entanglement, for the entanglement of formation no counterexample for additivity is known. Moreover, additivity of the entanglement of formation for all bipartite states has been shown to be equivalent to the strong additivity of the minimal output entropy and that of the Holevo capacity [4].

For the channels considered above, the construction in [5] can readily be applied, yielding further examples of states for which the entanglement cost is known, beyond the examples in [5, 8, 35]. The construction is as follows: from the quantum channel $T : S(\mathbb{C}^d) \to S(\mathbb{C}^d)$, one constructs a Stinespring dilation, via an isometry $U : \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^K$ for appropriate $K \in \mathbb{N}$. For any bipartite state $\rho \in S(\mathbb{C}^d \otimes \mathbb{C}^K)$ with carrier on $\mathcal{K} := U\mathbb{C}^d$ which achieves

$$C(T) = (S \circ \operatorname{tr}_1)(\rho) - E_F(\rho), \tag{68}$$

we know that

$$E_C(\rho) = E_F(\rho) = \nu_1(T).$$
 (69)

The following state is an example of a state with known entanglement cost constructed in this manner.

Example 9 (state with additive entanglement of formation). Let the state vectors from $\mathcal{K} \subset \mathbb{C}^4 \otimes \mathbb{C}^4$ be defined as $\mathcal{K} = \text{span}(|\psi_1\rangle, \dots, |\psi_4\rangle)$, with

$$|\psi_1\rangle = (i(|1,4\rangle + |2,3\rangle - |3,2\rangle) + |4,1\rangle)/2$$
(70)

$$|\psi_2\rangle = (i(-|1,3\rangle + |2,4\rangle + |3,1\rangle) + |4,2\rangle)/2$$
(71)

$$|\psi_3\rangle = (\mathbf{i}(|1,2\rangle - |2,1\rangle + |3,4\rangle) + |4,3\rangle)/2$$
(72)

$$|\psi_4\rangle = (\mathbf{i}(-|1,1\rangle - |2,2\rangle - |3,3\rangle) + |4,4\rangle)/2.$$
(73)

Then $E_C(\rho) = E_F(\rho) = 1$, where

$$\rho = (|\psi_1\rangle\langle\psi_1| + \dots + |\psi_4\rangle\langle\psi_4|)/4. \tag{74}$$

In just the same fashion, a large number of examples with known entanglement cost can be constructed from the above quantum channels.

7. Summary and conclusions

In this paper, we investigated a class of quantum channels for which the norm of the output state is maximized for an output being a normalized projection, with respect to their additivity properties. We introduced three equivalent characterizations of this class of quantum channels. For all channels of this type, which satisfy an additional (presumably weak) positivity condition,

one can infer the additivity of the minimal output von Neumann entropy from the respective additivity in the case of the 2-entropy. Several examples of channels of this type were discussed in quite some detail, showing that a surprisingly large number of quantum channels is included in the considered class. Finally, we investigated instances of this class of quantum channels with a weak covariance property, relating the minimal output entropy to both the classical information capacity. This construction indeed gives rise to a large class of channels with a known classical information capacity.

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