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Uncertainty principle for angular position and angular momentum

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Abstract. The uncertainty principle places fundamental limits on the accuracy with which we are able to measure the values of different physical quantities (Heisenberg 1949 The Physical Principles of the Quantum Theory (New York: Dover); Robertson 1929 Phys. Rev. 34 127). This has profound effects not only on the microscopic but also on the macroscopic level of physical systems. The most familiar form of the uncertainty principle relates the uncertainties in position and linear momentum. Other manifestations include those relating uncertainty in energy to uncertainty in time duration, phase of an electromagnetic field to photon number and angular position to angular momentum (Vaccaro and Pegg 1990 J. Mod. Opt. 37 17; Barnett and Pegg 1990 Phys. Rev. A 41 3427). In this paper, we report the first observation of the last of these uncertainty relations and derive the associated states that satisfy the equality in the uncertainty relation. We confirm the form of these states by detailed measurement of the angular momentum of a light beam after passage through an appropriate angular aperture. The angular uncertainty principle applies to all physical systems and is particularly important for systems with cylindrical symmetry.

It is a fundamental principle of quantum theory that we cannot establish, with arbitrary precision, all the physical properties of any system. This idea has its most concise quantitative statement in the uncertainty principle, which places a lower bound on the product of the underlying uncertainties associated with a chosen pair of observable quantities. The original uncertainty relation, due to Heisenberg, states that the product of the uncertainties in position and momentum for a particle is bounded by Planck's constant [1]: $\Delta x \Delta p \ge \hbar/2$. This inequality has played

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Figure 1. Different forms of the uncertainty principle. The uncertainty principle relates the precision to which various physical quantities of a system can be known. The most familiar form of the principle relates the uncertainty in linear position (a) to the uncertainty in linear momentum (b). Similarly, the uncertainty in angular position (c) is related to the uncertainty in angular momentum (d). For the angular case, the distribution for the intelligent states has a Gaussian form symmetrically truncated within the $-\pi$ to π range. The corresponding distribution of the orbital angular momentum state is the convolution of a Gaussian with a sinc function.

an important role in the study of quantum-limited measurements and in exploring questions concerning the nature of quantum theory [2]. States satisfying the equality in an uncertainty relation are called the intelligent states [3]. If the uncertainty product is bounded by a constant, these intelligent states coincide with the minimum uncertainty-product or critical states [4] that realize the smallest uncertainty product. However, in more complicated relations such as the one between angle and angular momentum, the intelligent states are not necessarily identical with the minimum uncertainty-product states. We will discuss this distinction in more detail elsewhere.

A natural consequence of the continuous Fourier-transform relationship between the linear position and momentum observables is that the intelligent states, which are simultaneously the minimum uncertainty-product states, are Gaussians. Whereas linear position is linked to linear momentum, angular position is linked to angular momentum, and such pairs are called conjugate variables. However, since the angular position is a periodic variable, its relationship with the angular momentum is by a discrete Fourier transform and is more complicated compared with the linear case (see figure 1). For states of precisely defined angular momentum $\Delta L = 0$, there is no restriction on angular position; it can take any value between 0 and 2π with equal probability.



Figure 2. Scheme of the experiment used to observe uncertainties in angular position and angular momentum. Passing a light beam through a 'cake-slice' aperture restricts the angular position, leading to a corresponding broadening of the light's angular momentum states. Experimentally, a spatial light modulator is used to produce both the aperture function and the angular-momentum-analysing hologram. The probability of each angular momentum component is deduced from the fraction of the resulting light transmitted through a pinhole.

Although now unrestricted, the statistical uncertainty in angular position remains finite and readily calculable [5]: $\Delta \phi = \pi/\sqrt{3}$. It is clear, for these states, that the uncertainty product is $\Delta \phi \Delta L = 0$. For other states, where the range of angular position is restricted, it is impossible to know precisely the angular momentum; for small angular uncertainties, the uncertainty principle tends to $\Delta \phi \Delta L = \hbar/2$. Between these extremes, there is a minimum uncertainty-product that varies monotonically from 0 to $\hbar/2$ as the angular uncertainty increases [6]. The general form of the uncertainty principle is a consequence of Robertson's generalization to any pair of observables of Heisenberg's uncertainty principle [1].

In the present paper, by precise measurements on an apertured light beam, we demonstrate for the first time a manifestation of this angular position, angular momentum uncertainty principle. If we place any restriction on the angular position, such as by passing the light beam through an angular aperture (a 'cake-slice'), a distribution of angular momentum states results (see figure 2). We have identified and experimentally verified the form of the aperture that corresponds to the intelligent states for angle and angular momentum.

Light beams carry both linear and angular momenta, which manifest at the macroscopic and single-photon levels. Angular momentum comprises spin and orbital components that are associated with circular polarization and helical-phase fronts respectively. At the photon level, the spin angular momentum can take one of two values, $\pm\hbar$ per photon, corresponding to leftand right-handed circularly polarized light. In contrast, the orbital angular momentum can take one of an unbounded range of values, $\ell\hbar$ per photon [7], where the integer ℓ relates to the azimuthal phase structure of the helically phased beam, $\exp(i\ell\phi)$. At the macroscopic level, both forms of angular momentum can be transferred to particles, causing them to spin about their own axes and orbit about the beam axis, respectively [8]. For single photons, both spin and orbital angular momenta have been shown to be well behaved and to have interesting quantum properties [9, 10].

All physical properties of cylindrical systems are periodic functions of an angular position. For this reason, we must restrict the values of the angle observable to lie within a 2π radian range, and the corresponding angular momentum component L_z can only take on discrete values $\ell\hbar$. The angle operator, $\hat{\phi}_{\theta}$, will have eigenvalues, ϕ , lying in the range θ to $\theta + 2\pi$, with a common choice being $-\pi \leq \phi < \pi$. This dependence on the choice of angular range is denoted by subscript θ in the angle operator. For states with finite uncertainty in angular momentum, the relation between uncertainty in angular position $\Delta \phi_{\theta}$ and uncertainty in angular momentum ΔL_z has the form [6]

$$\Delta \phi_{\theta} \Delta L_z \ge \frac{1}{2} \hbar |1 - 2\pi P(\theta)|, \tag{1}$$

where $\Delta L_z = \Delta \ell \hbar$ and $P(\theta)$ is the angular probability density at the boundary of the chosen angular range. The intelligent states, $|\psi\rangle$, obey the equation

$$[\hat{L}_z - \langle \hat{L}_z \rangle - i\hbar\lambda(\hat{\phi}_\theta - \langle \hat{\phi}_\theta \rangle)]|\psi\rangle = 0, \qquad (2)$$

where $\theta \leq \phi < \theta + 2\pi$ and λ is a real constant [11]. By taking its overlap with the state $|\Phi\rangle = (2\pi)^{-1/2} \sum_{\ell} e^{-i\ell\phi} |\ell\rangle$, we can convert equation (2) into a differential equation,

$$\left[i\frac{\partial}{\partial\phi} + \bar{\ell} + i\lambda(\phi - \bar{\phi}_{\theta})\right]\Psi(\phi) = 0,$$
(3)

where $\langle \hat{L}_z \rangle = \bar{\ell}\hbar$ and $\langle \hat{\phi}_{\theta} \rangle = \bar{\phi}$ denote the mean values of angular momentum and angular position, respectively. The solution of this equation is given by the wavefunction

$$\Psi(\phi) = \frac{(\lambda/\pi)^{1/4}}{\sqrt{\operatorname{erf}(\pi\sqrt{\lambda})}} e^{i\bar{\ell}\phi} \exp\left(-\frac{\lambda}{2}(\phi-\bar{\phi})^2\right),\tag{4}$$

where $\theta \leq \phi < \theta + 2\pi$. This function is normalized so that $\int_{\theta}^{\theta+2\pi} d\phi |\Psi(\phi)|^2 = 1$. Since we are seeking to identify the states giving a finite uncertainty in angular momentum, we require that the wavefunction has no discontinuities, i.e. $\Psi(\theta) = \Psi(\theta + 2\pi)$. Consequently, $\bar{\ell}$ needs to be an integer and the mean value of ϕ must lie at the centre of its allowed range of values, so that $\bar{\phi} = \theta + \pi$. With these restrictions, the truncated Gaussian described by equation (4) is the only realizable intelligent state for angular position (see figure 1(c)).

From equation (4), the corresponding uncertainty in angular position is

$$\Delta \phi = \frac{1}{\sqrt{2\lambda}} \sqrt{1 - \frac{2\sqrt{\pi\lambda} \exp(-\pi^2 \lambda)}{\operatorname{erf}(\pi \sqrt{\lambda})}}.$$
(5)

Here, we have dropped the subscript in ϕ and selected 0 as the mean value of the angle, so that $-\pi \leq \phi < \pi$. We can calculate the angular momentum amplitudes for our intelligent states by

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taking the Fourier transform of equation (4):

$$\psi(\ell) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} d\phi \, \mathrm{e}^{-\mathrm{i}\ell\phi} \psi(\phi)$$
$$= \frac{(\lambda\pi)^{-1/4}}{\mathrm{erf}(\pi\sqrt{\lambda})} \int_{-\infty}^{+\infty} \mathrm{d}k \operatorname{sinc}(k\pi) \, \mathrm{e}^{-(\bar{\ell}-\ell-k)^2/(2\lambda)},\tag{6}$$

which is the convolution of a Gaussian with a sinc function (see figure 1(d)). The corresponding uncertainty in the measured value of the angular momentum quantum number ℓ is

$$\Delta \ell = \lambda \Delta \phi. \tag{7}$$

Taking the product of equations (5) and (7), we confirm that these states do satisfy the equality, i.e.

$$\Delta\phi\Delta\ell = \frac{1}{2}|1 - 2\pi P(\pi)|. \tag{8}$$

It is interesting to note that the intelligent states for angular position are truncated Gaussians of equation (4) with a discontinuity in the gradient at $\pm \pi$. The presence of discontinuity is perhaps surprising, but it does not violate the condition for inequality. The minimum uncertainty-product states, in contrast, exhibit no such discontinuity.

The basis of our experiment is the understanding that if a light beam with a single value of orbital angular momentum is passed through an angular aperture, the angular momentum of the transmitted beam will take on a range of values (see figure 2).

Helically phased light beams are readily generated using diffractive optical components, often termed computer-generated holograms. A frequently used design is similar to a diffraction grating, but one that features an ℓ_{holo} pronged fork dislocation at its centre [12]. After illumination with a plane wave, the first-order diffracted beam has an $\exp(i\ell_{holo}\phi)$ phase structure and an orbital angular momentum of $\ell_{holo}\hbar$ per photon. Such holograms also work more generally, such that when the incident beam is already helically phased, e.g. $\exp(i\ell_{beam}\phi)$, the diffracted beam has a phase structure $\exp(i(\ell_{beam} + \ell_{holo})\phi)$. When $\ell_{beam} + \ell_{holo} = 0$, the diffracted beam has planar phasefronts. Such a system can be used to test for a particular orbital angular momentum state since only a planar diffracted beam can be made to pass through a subsequent pinhole [10].

For accurate measurements, it is essential that both the apex of the aperture and the dislocation within the hologram be centred with respect to the illuminating beam. Consequently, rather than aligning mechanical components, both the aperture and analysing hologram are produced on a spatial light modulator. In principle, separate modulators could be used for the aperture and the analysing holograms; however, it is convenient, in our case, to combine these elements onto a single plane and thereby display them on a single device. The input beam is derived from a 100 mW, 532 nm laser, which is modulated with a mechanical chopper to allow phase-sensitive detection of the light transmitted through the pinhole and, hence, a large dynamic range in the measured angular momentum components. The angular amplitude of the illuminating beam is defined in (4). An angular width for the aperture is picked at random and orientated with a random azimuthal position. The number of dislocations within the analysing hologram is then changed over a wide range of ℓ values to deduce the normalized distribution



Figure 3. Examples of experimental observation. The observed distribution of the angular momenta (left panel) after transmission through the angular apertures (right panel). The solid curve shows the best fit to the theoretically predicted distribution.

of the orbital angular momentum components of the beam for that particular aperture width. Over several hours, many hundreds of aperture widths can be evaluated and the corresponding distribution of orbital angular momentum states measured.

In our experiment, the incident beam is a Gaussian laser mode with zero angular momentum. Figure 3 shows three angular apertures, the measured distribution of angular momentum components and the statistical best fit to the distribution function $|\psi(\ell)|^2$, obtained from (6). Figure 4 shows the observed product $\Delta\phi\Delta\ell$ plotted against $\Delta\phi$, compared with the theoretical prediction of (8).

Each of the individual measurements is subject to error both in $\Delta \phi$, due to the non-linearity of the spatial light modulator, and in $\Delta \ell$, due to noise in the detection system. For small values of $\Delta \phi$, the total power transmitted through the aperture is low, giving a poor signal-to-noise ratio in the measured ℓ components and resulting in an overestimate of $\Delta \ell$. For large values of $\Delta \phi$, the



Figure 4. Experimental measurement and theoretical prediction of the angular uncertainty relation. The observed (diamonds) and predicted (line) relationship between the product of the uncertainties in angular position and angular momentum in units of \hbar plotted against the uncertainty in angular position. For small angles, the uncertainty product is constant and equal to 0.5. In the absence of any restricting aperture, there is no uncertainty in angular momentum and the uncertainty product decreases to zero.

angular momentum distribution is dominated by a single value; however, noise for other values again leads to a similar overestimation of $\Delta \ell$.

In addition to supporting the predictions of our theoretical analysis, figure 4 illustrates the detailed behaviour of the angular form of the uncertainty relationship. For no aperture, $\Delta \phi = \pi/\sqrt{3}$ and, as expected, we find that the orbital angular momentum is precisely defined, i.e. $\Delta \ell = 0$. For very small apertures, $P(\pi)$ tends to zero and $\Delta \phi \Delta \ell \approx 1/2$. The most interesting behaviour occurs for intermediate aperture widths when the wings of the Gaussian do not decrease to zero within the 2π range of the distribution. Under these conditions, we see that $\Delta \phi \Delta \ell$ reduces from 1/2 as $\Delta \phi$ increases.

We note that a similar uncertainty relationship exists between photon number and phase of an electromagnetic field [13], which has been explored experimentally [14]. In this case, however, the fact that the photon number has a minimum value (i.e. zero) means that exact intelligent states, other than the pure photon-number states, do not exist [15]. In this paper, we have observed for the first time the uncertainty relationship governing angular position and angular momentum and have both derived and demonstrated the corresponding intelligent states.

Clearly, experimental results were obtained in the present study from light beams comprising many photons in the same orbital angular momentum state and, hence, are strictly applicable to classical beams only. However, the experiment is fundamentally one in which the measured intensity values arise through interference effects in linear optics. In agreement with other such interference experiments, we can infer that the intensity distribution measured for the many-photon result is proportional to the probability that would be measured if the experiment was repeated with single photons. Consequently, the angular uncertainty relationship between optical orbital angular momentum and the angular aperture applies both in the quantum and classical regimes. This is as anticipated, since the theoretical description relies on linear optics and, thus, is identical for single photons and classical beams.

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