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PAPER

Tunable spectral singularities: coherent perfect absorber and laser in an atomic medium

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Abstract

We propose a scheme for a coherent perfect absorber (CPA) and a laser in an atomic medium with gain and loss, obeying simultaneously a spectral singularity and a time-reversed spectral singularity, both occurring at different wavevectors (or frequencies). We term this system a CPA-*and*-laser and investigate its features allowing to obtain asymmetric lasing and absorbing properties, switches, etc. We show that the CPA-*and*-laser can be obtained by modifying characteristics of a CPA-laser of an initial \mathcal{PT} -symmetric configuration, provided there are at least three tunable parameters. The physical mechanism of emergence of a CPA-*and*-laser in this way, is based on splitting of a self-dual spectral singularity of the \mathcal{PT} -symmetric CPA-laser into the spectral singularity and time-reversed spectral singularity. After the discussion of a particular example of a bilayer consisting of one active and one passive slabs, we suggest a realistic physical system for implementing a CPA-*and*-laser. It consists of two adjacent atomic cells filled with isotopes of Λ -type three-level rubidium atoms interacting with probe and control fields, allowing for the required number of tunable parameters.

1. Introduction

Spectral singularity, a concept known in the theory of non-Hermitian operators since the works of Naimark [1] (see [2, 3] for a review), means an isolated point of a continuous spectrum of a Schrödinger operator with a complex-valued potential, at which the associated Jost solutions become linearly dependent and the eigenfunctions lose completeness [3]. Recently, spectral singularities attracted increasing attention in the context of diverse physical applications. They were shown to correspond to zero-width resonances and determine lasing properties of active media [4, 5], to divergence of reflection and transmission coefficients for a specific wavevectors k , what is expressed mathematically as zeros of the diagonal element $M_{22}(k)$ of the transfer matrix $M(k)$ [5, 6], to a secular growth of plane waves diffracted off a parity-time (\mathcal{PT}) symmetric photonic crystals [7], as well as to the regime of a coherent perfect absorber (CPA) viewed as time-reversed laser [8, 9]. In the last case, the spectral singularity occurs for the conjugated potential and is expressed by a zero of $M_{11}(k)$ element of the transfer matrix [3, 10], what is referred to as a time-reversed spectral singularity.

In a generic case, spectral singularity and time-reversed spectral singularity do not coincide, and the existence of one of them does not imply the existence of the other. If, however, the system obeys a \mathcal{PT} -symmetry, the latter imposes strong constraints on the spectrum [11], requiring the coincidence of real zeros of $M_{22}(k)$ and $M_{11}(k)$ [12] (so-called self-dual spectral singularity [13]). In this case the system acts simultaneously as a CPA and as a laser at the *same* frequency, giving origin to CPA-laser introduced in [12].

It is relevant to note here that the paradigm is quite general and can be implemented in systems beyond the conventional optical media with gain and loss [14, 15]. In particular, one can exploit metamaterials where spectral singularities are manipulated by both complex permittivity and complex permeability, as suggested in

[16]. A CPA-laser can also be realized experimentally using a two-port \mathcal{PT} -symmetric electronic dimer [17]. For a recent review of diverse physical systems where \mathcal{PT} -symmetry and spectral singularities can be observed, we mention [15]. Thus, the results of the present study can be extended to these diverse physical systems. Furthermore, we note that \mathcal{PT} symmetry is not a necessary condition for the existence of self-dual spectral singularities; the latter can exist in non- \mathcal{PT} -symmetric systems, as shown in [13].

The effects mentioned above are related to single spectral singularities. Interplay between two spectral singularities can give rise to new interesting phenomena. Unidirectional spectral singularities emergent from coincidence of two independent ones were recently reported in [18]. The respective media were characterized by zero left and infinite right reflection preserving reciprocal in terms of the wave transmission.

In this paper we investigate another possibility for obtaining an optical potential and its conjugate, which allows for spectral singularities occurring simultaneously at *different* wavevectors. Although in the strict mathematical sense these are spectral singularities of different potentials, they describe physical effects of the same optical medium, and for the sake of brevity below they are referred to as *lasing* and *absorbing* singularities. To distinguish such a structure from the CPA-laser [12], here we call it CPA-*and*-laser, thus emphasizing that lasing and absorption occur in the same structure but at *different* wavevectors.

Another goal of the present work is to show that atomic cells filled with isotopes of Λ -type three-level rubidium (Rb) atoms interacting with a probe and a control laser fields, proposed in [19] for creating enhanced real susceptibilities and exploited in [20, 21] for designing linear and nonlinear \mathcal{PT} -symmetric waveguides, is a versatile system allowing for creating optical potentials possessing spectral singularities, and in particular for obtaining lasing and absorbing singularities (in the sense specified above) with tunable characteristics.

The paper is organized as follows. In section 2, we show that the lasing and absorbing singularities of a system can be simultaneously obtained by weakly deviating from an exactly \mathcal{PT} -symmetric configuration, which in its turn obeys a self-dual spectral singularity. For this aim the potential in the system must have several, at least three, adjustable control parameters. In section 3, we discuss some of specific functionalities that can be offered by the two simultaneous singularities. In section 4, we consider a simple example consisting of two separated domains with constant gain and losses (which can be interpreted as a two independent atomic cells similarly to the original proposal in [22]), which allows us to present a detailed study and give quantitative analytical and numerical results. In section 5, we demonstrate that a CPA-*and*-laser can be designed by using two adjacent atomic cells filled with Λ -type three-level Rb atoms. Finally, the main outcomes are summarized in section 6. The details of tedious but straightforward calculations in the text are outlined in appendices.

2. CPA-*and*-laser

2.1. Statement of the problem

Consider incidence of a monochromatic electromagnetic wave on an active layer parallel to the (y, z) plane, whose center is located in the plane $x = 0$ and whose width is $2L$. The dielectric permittivity of the slab is given by $\varepsilon = \varepsilon_0(1 + \chi(x))$ where ε_0 is the vacuum permittivity and the susceptibility $\chi(x)$ is a complex function inside the layer (i.e. at $|x| < L$). Assume that the dielectric permittivity depends on a number, say n , of tunable parameters, which we denote by $\lambda_1, \dots, \lambda_n$. To shorten notations below we use Λ for the whole set of the parameters, i.e. $\Lambda = \{\lambda_1, \dots, \lambda_n\}$. Thus, $\chi(x) \equiv \chi(x; \Lambda)$. Generally speaking, the slab width can also be viewed as a relevant parameter. However, for the sake of convenience, as well as bearing in mind physical realization of describe phenomenon with atomic cells whose geometrical size is usually fixed (see section 5 below), we keep L constant and moreover we set $L = 1$ (i.e. measure all distances in the units of L) below.

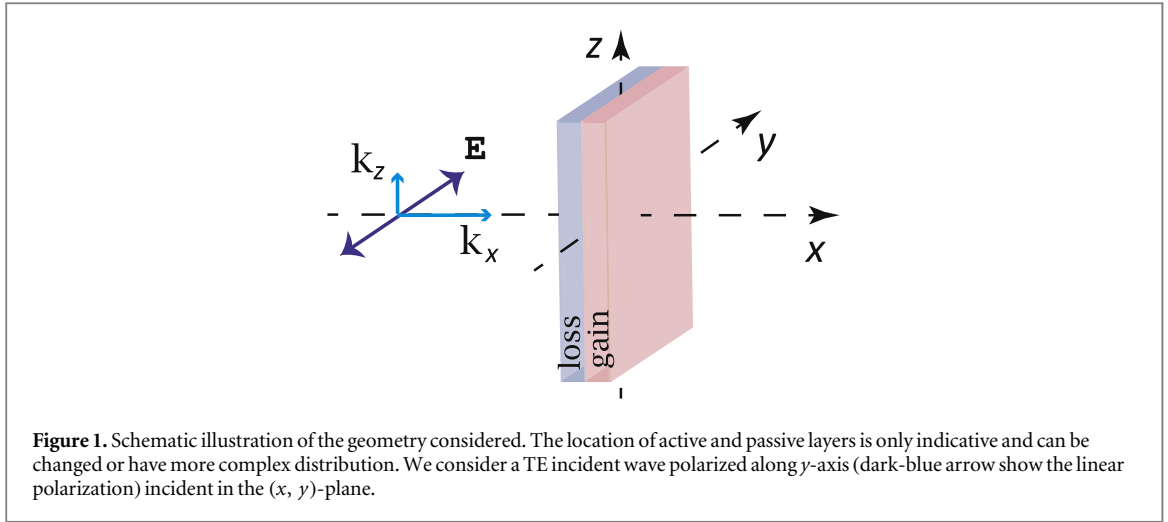
A monochromatic transverse-electric wave $\mathbf{E} = (0, E(x, z, t), 0)$ of the frequency ω is incident on the layer, (x, z) being the plane of incidence (as this is illustrated in figure 1). Using the ansatz $E(x, z, t) = E_0 e^{ik_z z - i\omega t} \psi(x)$, where $\psi(x)$ denotes the spatial modulation of the wave amplitude in the direction orthogonal to the layer. The Helmholtz equation describing the wave scattering by the layer takes the form

$$\frac{d^2 \psi}{dx^2} + U(x; \Lambda) \psi = -k^2 \psi, \quad (1)$$

where $U(x; \Lambda) = k_0^2 \chi(x; \Lambda)$ in the interval $|x| < L$ and is zero for $|x| > L$, $k_0 = \omega/c$, and the spectral parameter is $k^2 = k_0^2 - k_z^2$. Below we concentrate on the scattering problem described by equation (1).

2.2. Spectral singularities

Let us now briefly recall some properties of spectral singularities (for more details see e.g. [2–5, 12]) of the Schrödinger operator (1). To this end we denote $\psi_-(x) = a_1 e^{ikx} + a_2 e^{-ikx}$ and $\psi_+(x) = b_1 e^{ikx} + b_2 e^{-ikx}$ the solutions of (1) at $x < -L$ and $x > L$, respectively. The transfer matrix $M(k)$ (it is unimodular: $\det M = 1$) is defined through the relation among the field amplitudes $a_{1,2}$ and $b_{1,2}$:



$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = M(k) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \quad (2)$$

The left ('l') and right ('r') transmission $t_{l,r}$ and reflection $r_{l,r}$ coefficients are determined by the transfer-matrix elements:

$$t_l = t_r \equiv t = \frac{1}{M_{22}}, \quad r_l = -\frac{M_{21}}{M_{22}}, \quad r_r = \frac{M_{12}}{M_{22}}. \quad (3)$$

Let now a real k belongs to the continuous spectrum of the Schrödinger equation (1). Then, if at some value $k = k_\star^{(l)}$ we have that $M_{22}(k_\star^{(l)}) = 0$, it is said that $(k_\star^{(l)})^2$ is a spectral singularity. In that case the scattering problem (1) has a solution with nonzero outgoing radiation a_2 and b_2 at zero incoming radiation, $a_1 = b_1 = 0$, what means that the system operates as laser. If at some $k = k_\star^{(a)}$ we have that $M_{11}(k_\star^{(a)}) = 0$, then $(k_\star^{(a)})^2$ is a time reversed spectral singularity. In the last case there exists a solution of (1) with only incoming waves (i.e. $a_1, b_2 \neq 0$ and $a_2 = b_1 = 0$) for which the system operates as a CPA. According to the abbreviated terminology introduced above $[k_\star^{(l)}]^2$ and $[k_\star^{(a)}]^2$ are referred to as *lasing* and *absorbing* singularities, respectively.

For a generic complex potential $U(x, \Lambda)$, the roots of $M_{jj}(k)$ are located in the complex plain and thus there are no spectral singularities. If however the complex potential has a tunable parameter λ (say strength of gain or absorption) by varying this parameter one can 'force' the roots of the diagonal elements of the transfer matrix to shift in the complex k -plain and at certain values of λ (usually) one of the roots, of either M_{11} or M_{22} , reaches the real axis and either absorbing or lasing singularity occurs. Examples of such 'dynamics' of roots is described e.g. in [8]. The location and the dynamics of the roots of M_{11} or M_{22} become constraint in the case of a \mathcal{PT} -symmetric potential: now laser and absorber singularities occur simultaneously for the both diagonal elements of the transfer matrix, i.e. $k_\star^{(l)} = k_\star^{(a)} = k_\star$ such that $M_{11}(k_\star) = M_{22}(k_\star) = 0$. This gives origin to the CPA-laser [12] and k_\star is referred to as self-dual spectral singularity [13].

Occurrence of laser and absorbing singularities in the same optical medium, i.e. simultaneous existence of $k_\star^{(l)}$ and $k_\star^{(a)}$, where $k_\star^{(l)} \neq k_\star^{(a)}$, is a less common event for a complex potential of a general type. Such situation however can exist and even can be implemented using algorithmic approach. Indeed, given a \mathcal{PT} -symmetric potential, any spectral singularity occurs simultaneously with time-reversed spectral singularity at the same wavelength. Thus one can pose a problem of perturbing such a potential in a way to split a self-dual spectral-singularity. If impose also a constraint that the zeros of the diagonal elements of the transfer matrix of the perturbed potential remain on the real axis, the deformed potential will act as a laser and as a CPA at different wavelengths. This algorithm is described in details in the next subsection.

2.3. Self-dual spectral singularity split

Consider a complex potential $U(x, \Lambda)$. Suppose that for a given set of the control parameters $\Lambda = \Lambda^{(0)}$ (where $\Lambda^{(0)} = \{\lambda_1^{(0)}, \dots, \lambda_n^{(0)}\}$) the two conditions are satisfied: (i) the potential is \mathcal{PT} -symmetric, i.e. $U(x, \Lambda^{(0)}) = U^*(-x, \Lambda^{(0)})$ at $\Lambda = \Lambda^{(0)}$ and (ii) the potential has a spectral singularity at $k = k_\star$. Generically, at $\Lambda \neq \Lambda^{(0)}$ the \mathcal{PT} -symmetry is broken i.e. $U(x, \Lambda) \neq U^*(-x, \Lambda)$. Thus, for small shifts $\lambda_j = \lambda_j^{(0)} + \lambda_j^{(1)}$ where $|\lambda_j^{(1)}| \ll |\lambda_j^{(0)}|$ the deformed potential losses the \mathcal{PT} -symmetry. Moreover, if no constraints are imposed on $\lambda_j^{(1)}$ we have that $M_{11}(k, \Lambda)M_{22}(k, \Lambda) \neq 0$ for real k in a vicinity of k_\star . In other words, generally speaking, the roots k_1 and k_2 of M_{11} and M_{22} , respectively, are shifted from the real axis. Now $k_{1,2} = k_\star + \kappa_{1,2}$ where $\kappa_{1,2}$ depend on $\{\lambda_1^{(1)}, \dots, \lambda_n^{(1)}\}$, become zero when all $\lambda_j^{(1)} = 0$, and are generally speaking complex.

Now we pose the problem of finding $\lambda_j^{(1)}$ such that the potential $U(x, \Lambda)$ which is not \mathcal{PT} -symmetric any more, and its complex conjugate dual have spectral singularities. Thus we are looking for a lasing singularity $k_*^{(l)} = k_* + \kappa_2 (M_{22}(k_*^{(l)}, \Lambda) = 0)$ and absorbing singularity $k_*^{(a)} = k_* + \kappa_1 (M_{11}(k_*^{(a)}, \Lambda) = 0)$. Obviously, these singularities split out from k_* and are characterized by *real* $\kappa_{1,2}$.

Furthermore, for practical purposes it is of interest to have a possibility to control values of the singularities, i.e. locations of $\kappa_{1,2}$ on the real axis (this would allow one to manipulate the properties of such systems). Thus the problem is reduced to finding $\lambda_j^{(1)}$ such that $\kappa_{1,2} \in \mathbb{R}$ and depend on (at least) one free control parameter.

The formal solution of the problem is straightforward. Indeed, considering $\kappa_{1,2}$ small enough (of order of the small deformations $\lambda_j^{(1)}$) one can expand the elements of the transfer matrix in the Taylor series:

$$M_{jj}(k_j, \Lambda) \approx \frac{\partial M_{jj}^{(0)}}{\partial k} \kappa_j + \sum_n \frac{\partial M_{jj}^{(0)}}{\partial \lambda_n} \lambda_n^{(1)}, \quad j = 1, 2, \quad (4)$$

where the upper index (0) of the transfer matrix elements indicates that the respective elements and their derivatives are computed at $k = k_*$ and at $\Lambda = \Lambda^{(0)}$. In (4) it is taken into account that $M_{jj}^{(0)}(k_*, \Lambda^{(0)}) = 0$ since k_* is a self-dual spectral singularity of the \mathcal{PT} -symmetric limit. Requiring $k_1 = k_*^{(a)}$ and $k_2 = k_*^{(l)}$ to give lasing and absorbing singularities, we obtain from (4)

$$\kappa_j = - \sum_n \frac{\partial M_{jj}^{(0)}}{\partial \lambda_n} \lambda_n^{(1)} \bigg/ \left(\frac{\partial M_{jj}^{(0)}}{\partial k} \right), \quad j = 1, 2. \quad (5)$$

The requirement $\kappa_{1,2} \in \mathbb{R}$ leads to the two constraints on the parameters

$$\sum_n \beta_{jn} \lambda_n^{(1)} = 0, \quad \text{where} \quad \beta_{jn} = \text{Im} \left[\frac{\partial M_{jj}^{(0)}}{\partial \lambda_n} \bigg/ \left(\frac{\partial M_{jj}^{(0)}}{\partial k} \right) \right], \quad (6)$$

which are to be satisfied for $j = 1$ and $j = 2$ simultaneously.

The goal is to obtain the desired singularities $[k_*^{(l)}]^2$ and $[k_*^{(a)}]^2$, with the least possible number of the control parameters. Since in a general situation the functional forms of the elements M_{jj} and of their derivatives are given, i.e., they are not adjustable at will, the system of two algebraic equations (6) has a nontrivial solution if it has at least *three* control parameters. If the described problem of obtaining an optical potential with two (closely located) spectral singularities can be solved, then the obtained system will act simultaneously as a CPA and as a laser at different wavevectors, i.e. represents CPA-and-laser.

3. Functionality features of a CPA-and-laser

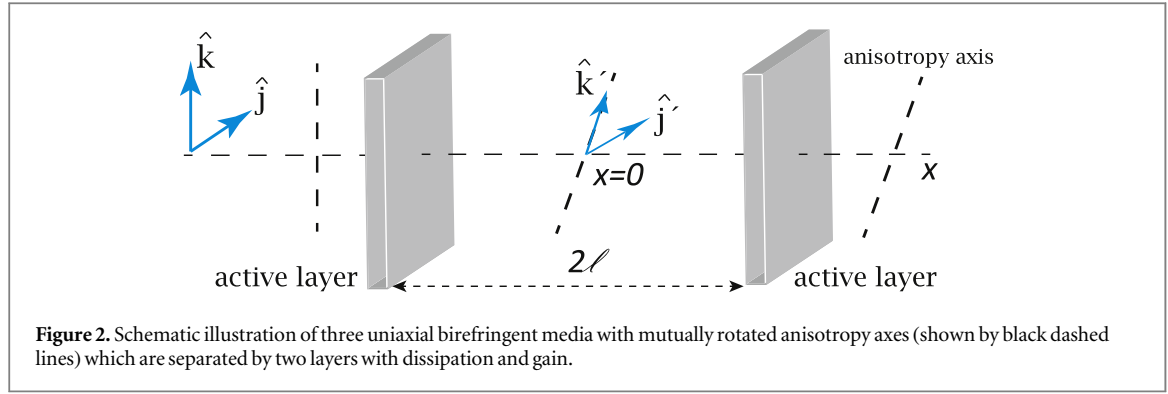
Now we discuss some functionalities of such CPA-and-laser. To this end, we consider a wavevector k located in the interval $k_*^{(a)} < k < k_*^{(l)}$ (for the sake of definiteness in this analysis we set $k_*^{(a)} < k_*^{(l)}$) and define the splitting $\delta = k_*^{(l)} - k_*^{(a)} \ll 1$ (recall that we have set dimensionless units, in which $L = 1$). Then in the leading order the transfer matrix can be written as

$$M = \begin{pmatrix} m_1(k - k_*^{(a)}) + \mathcal{O}(\delta^2) & m + \mathcal{O}(\delta^2) \\ -1/m + \mathcal{O}(\delta^2) & m_2(k_*^{(l)} - k) + \mathcal{O}(\delta^2) \end{pmatrix}, \quad (7)$$

where $m, m_{1,2} = \mathcal{O}(1)$ for any $k \in (k_*^{(a)}, k_*^{(l)})$. We note that particular values of the introduced parameters m and $m_{1,2}$, which characterize the transfer matrix at the CPA-laser limit ($k_*^{(l)} = k_*^{(a)}$), depend on the specific characteristics of the scattering layer. For the analysis of possible applications of the CPA-and-laser, however, these values are not important; a specific example will be given in the next section.

It readily follows from (3) that the reflection and transmission coefficients are now as large as $1/(k_*^{(l)} - k)$ (for $k_*^{(l)} - k \gg \delta^2$). Even if the incident wavevector approaches $k_*^{(a)}$, the amplification remains anomalously strong ($\sim 1/\delta \gg 1$). This amplifying regime is the ‘reminiscence’ of the lasing property of the CPA-laser at $\Lambda = \Lambda^{(0)}$. In order to understand this phenomenon, we notice that it follows from (2) and (7) that $b_2 \sim (mb_1 + a_2)/(m_1(k - k_*^{(a)}))$ and $a_1 \sim (b_1 + a_2/m)/(m_1(k - k_*^{(a)}))$, and hence the outgoing radiation is negligibly small (by the order of $1/\delta$) compared to the incident one from both sides of the slab.

The existence of *two* singularities of a scattering layer offers new possibilities when operating with two waves having wavevectors $k_*^{(a)}$ and $k_*^{(l)}$. In particular, one can implement diverse regimes of asymmetric and unidirectional absorption, amplification and lasing. We illustrate this on a simple example of a monochromatic wave propagating along x -axis in a system composed by three intervals of identical uniaxial birefringent media, which are separated by two identical CPA-and-lasers, each having with $2L$, placed at $x = -\ell - L$ and $x = \ell + L$, i.e. 2ℓ is the distance between the layers (figure 2). The layers are parallel to the (y, z) -plane, i.e.



orthogonal to the direction of propagation, and are characterized by the spectral singularities at $k_{\star}^{(a)}$ and $k_{\star}^{(l)}$. We also assume that the ordinary and extraordinary axes of the medium at $x < -\ell - L$ coincide with the y and z axes, respectively, while the same axes of the media between the layers (i.e. in the interval $-\ell < x < \ell$), as well as to the right of the second layer (i.e. at $x > \ell + 2L$) are rotated by the angle φ with respect to the axes of left medium (shown by dashed lines in figure 2). The rotation causes coupling of the waves. For the sake of further simplification we assume that the wavevectors of ordinary and extraordinary waves in the birefringent media exactly coincide with $k_{\star}^{(a)}$ and $k_{\star}^{(l)}$.

Respectively, the field to the left and to the right of the cavity created by the two layers can be written as

$$\psi_{\text{left}}(x) = [a_1 e^{ik_{\star}^{(a)}x} + a_2 e^{-ik_{\star}^{(a)}x}] \hat{\mathbf{j}} + [a_3 e^{ik_{\star}^{(l)}x} + a_4 e^{-ik_{\star}^{(l)}x}] \hat{\mathbf{k}}, \quad (8)$$

$$\psi_{\text{right}}(x) = [b_1 e^{ik_{\star}^{(a)}x} + b_2 e^{-ik_{\star}^{(a)}x}] \hat{\mathbf{j}}' + [b_3 e^{ik_{\star}^{(l)}x} + b_4 e^{-ik_{\star}^{(l)}x}] \hat{\mathbf{k}}', \quad (9)$$

where $\hat{\mathbf{j}}' = \cos(\varphi) \hat{\mathbf{j}} - \sin(\varphi) \hat{\mathbf{k}}$ and $\hat{\mathbf{k}}' = \sin(\varphi) \hat{\mathbf{j}} + \cos(\varphi) \hat{\mathbf{k}}$ are the unitary vectors obtained by the rotation of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ by the angle φ . Now the transfer matrix of the whole system relates the four component vectors $\mathbf{a} = (a_1, a_2, a_3, a_4)^T$ and $\mathbf{b} = (b_1, b_2, b_3, b_4)^T$ (the upper index T stands for the transpose): $\mathbf{b} = M_{\text{sys}} \mathbf{a}$ and has the form $M_{\text{sys}} = M_1 D M_1$. Here M_1 is the transfer matrix describing each of the active layers. In our case it has the form:

$$M_1 = \begin{pmatrix} 0 & -1/m_1 & 0 & 0 \\ m_1 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & -1/m_2 \\ 0 & 0 & m_2 & 0 \end{pmatrix}, \quad (10)$$

where m_j and μ_j are nonzero constants. This choice of the layer transfer matrix means that the ordinary wave corresponds to the CPA regime while the extraordinary mode corresponds to the lasing regime. The 4×4 transfer matrix D describes free propagation in the cavity between the layers. It has the form

$$D = \begin{pmatrix} D_1(k_{\star}^{(a)}) & -D_2(k_{\star}^{(a)}, k_{\star}^{(l)}) \\ D_2(k_{\star}^{(l)}, k_{\star}^{(a)}) & D_1(k_{\star}^{(l)}) \end{pmatrix}, \quad (11)$$

where the 2×2 matrices $D_1(k)$ and $D(k_1, k_2)$ are given by

$$D_1(k) = \cos(\varphi) \begin{pmatrix} e^{ik\ell} & 0 \\ 0 & e^{-ik\ell} \end{pmatrix}, \quad (12)$$

$$D_2(k_1, k_2) = \frac{\sin \varphi}{2k_1} \begin{pmatrix} (k_1 + k_2) e^{ik_1 \ell} & (k_1 - k_2) e^{ik_1 \ell} \\ (k_1 - k_2) e^{-ik_1 \ell} & (k_1 + k_2) e^{-ik_1 \ell} \end{pmatrix}. \quad (13)$$

Depending on the parameters the described system may allow for diverse regimes. We list some of them for the simplest case, when the rotation angle is $\varphi = \pi/2$, i.e. when from the left and right sides of the left layer the extraordinary and axes in the transverse direction are exchanged.

Switch of an incident extraordinary wave with the wavevector $k_{\star}^{(l)}$ (lasing spectral singularity) and with the amplitude a_3 , from the left, to an outgoing ordinary wave with the wavevector $k_{\star}^{(a)}$ (absorbing spectral singularity) with the amplitude

$$b_1 = \frac{(k_{\star}^{(a)} + k_{\star}^{(l)}) m_2 + (k_{\star}^{(a)} - k_{\star}^{(l)}) \mu_2}{2m_1 k_{\star}^{(a)}} a_3, \quad (14)$$

occurs if the parameters of the system are chosen to satisfy

$$\mu_1 = e^{-2ik_*^{(a)}\ell} m_1 \frac{(k_*^{(a)} - k_*^{(l)})m_2 + (k_*^{(a)} + k_*^{(l)})\mu_2}{(k_*^{(a)} + k_*^{(l)})m_2 + (k_*^{(a)} - k_*^{(l)})\mu_2}. \quad (15)$$

Notice that the polarization vectors of the incident and transmitted waves remain parallel. The amplitudes of the remaining waves in (8) are zero: $a_1 = a_2 = a_4 = b_2 = b_3 = b_4 = 0$.

Lasing of ordinary and extraordinary waves respectively to the right and to the left from the cavity is achieved if

$$\mu_1 = e^{-2ik_*^{(a)}\ell} m_1 \frac{k_*^{(a)} + k_*^{(l)}}{k_*^{(l)} - k_*^{(a)}}. \quad (16)$$

The relation between amplitudes of the modes in this case is given by

$$\frac{b_1}{a_4} = \frac{k_*^{(l)} - k_*^{(a)}}{2m_1 m_2 k_*^{(a)}} e^{-ik_*^{(a)}\ell}. \quad (17)$$

The amplitudes of the remaining waves in (8) are zero: $a_1 = a_2 = a_3 = b_2 = b_3 = b_4 = 0$.

Asymmetric CPA absorbs an incident ordinary wave from the left (the amplitude and the wavevector are a_1 and $k_*^{(a)}$, respectively) and extraordinary wave from the right (the amplitude and the wavevector are b_4 and $k_*^{(l)}$, respectively), if

$$\mu_1 = e^{-2ik_*^{(l)}\ell} \frac{k_*^{(a)} + k_*^{(l)}}{2(k_*^{(l)} - k_*^{(a)})}. \quad (18)$$

The relation between the amplitudes of the absorbed waves is given by

$$\frac{b_4}{a_1} = e^{2ik_*^{(l)}\ell} \frac{m_1 m_2 (k_*^{(a)} + k_*^{(l)})}{2k_*^{(l)}}. \quad (19)$$

The amplitudes of the remaining waves in (8) are zero: $a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = 0$.

By considering arbitrary rotation angle φ and adding rotation between axes of the media in the cavity $x \in (-\ell, \ell)$ and of the right semi-infinite media ($x > \ell + 2L$) one can offer more possibilities for manipulating scattering properties, and in particular for designing unidirectional CPAs and lasers.

4. An example of a CPA-and-laser

4.1. The model and the transfer matrix

Let us consider a simple and meantime experimentally feasible example of a CPA-and-laser. Details of a possible physical realization of such system using atomic cells are discussed in the next section. We consider a bilayer which consists of an absorbing and active layers located in the intervals $-L < x < 0$ and $0 < x < L$, respectively. Thus $2L$ is the total width of the layer. The respective susceptibility is given by

$$\chi(x) = \begin{cases} \chi_{1r} + i\chi_{1i}, & -L < x < 0, \\ \chi_{2r} - i\chi_{2i}, & 0 < x < L, \end{cases} \quad (20)$$

where $\chi_{1r}, \chi_{2r} > 1$ and $\chi_{1i}, \chi_{2i} \geq 0$, are the real and imaginary parts of the dielectric susceptibility, and $\chi(x) = 0$ for $|x| > L$.

As in the preceding section here we set the dimensionless variables where the length is measured in the units of L , i.e. formally we set $L = 1$. The respective dimensionless potential $U(x, \Lambda)$ can be recast in the form $U(x, \Lambda) = U_0(x) + u(x)$ where

$$U_0(x) = \begin{cases} u_0 + i\gamma_0, & x \in (-1, 0) \\ u_0 - i\gamma_0, & x \in (0, 1), \end{cases} \quad (21)$$

with the parameters $u_0 = k_0^2(\chi_{1r} + \chi_{2r})/2$ and γ_0 is the \mathcal{PT} -symmetric part: $U_0(x) = U_0^*(-x)$, and

$$u(x) = \begin{cases} w + ig_1, & x \in [-1, 0) \\ -w - ig_2, & x \in (0, 1], \end{cases} \quad (22)$$

where $w = k_0^2(\chi_{1r} - \chi_{2r})/2$, $g_1 = k_0^2\chi_{1i} - \gamma_0$, and $g_2 = k_0^2\chi_{2i} - \gamma_0$ are small constants describing non- \mathcal{PT} -symmetric deformation. In the new notations u_0 and γ_0 are considered as given, while the set of control parameters is given by $\Lambda = \{w, g_1, g_2\}$ and $\Lambda^{(0)} = \{0, 0, 0\}$. Thus, we have the required three control parameters: the mismatch between local dielectric permittivities in the gain and loss domains w , and deviations of the gain and loss from their values in the \mathcal{PT} -symmetric configuration $g_{1,2}$. We also require u_0 and γ_0 be such that at some $k = k_*$, a spectral singularity takes place.

The potential (21) was introduced in [22] in the framework of the Schrödinger equation simulating \mathcal{PT} -symmetric quantum mechanics and describing transmission of slowly modulated wave-packet through a \mathcal{PT} -symmetric bilayer. Helmholtz equation of a bilayer having refractive index of the form (21) and its spectral singularities were studied in [24, 25].

Denoting solution of (1) in the interval $|x| < 1$ by ψ_0 , from the continuity conditions at $x = \pm 1$ we have obvious relations

$$\begin{aligned} a_1 &= \frac{1}{2} \left[\psi_0(-1) + \frac{1}{ik} \psi'_0(-1) \right] e^{ik}, & a_2 &= \frac{1}{2} \left[\psi_0(-1) - \frac{1}{ik} \psi'_0(-1) \right] e^{-ik}, \\ b_1 &= \frac{1}{2} \left[\psi_0(1) + \frac{1}{ik} \psi'_0(1) \right] e^{-ik}, & b_2 &= \frac{1}{2} \left[\psi_0(1) - \frac{1}{ik} \psi'_0(1) \right] e^{ik}, \end{aligned} \quad (23)$$

where the amplitudes $a_{1,2}$ and $b_{1,2}$ are introduced in (2) and the primes denote derivatives with respect to x . Furthermore, for the potential (21) and (22) the solution can be searched in the form

$$\psi_0(x) = \begin{cases} Ae^{i\nu_-x} + Be^{-i\nu_-x}, & x \in (-1, 0) \\ Ce^{i\nu_+x} + De^{-i\nu_+x}, & x \in (0, 1), \end{cases} \quad (24)$$

where $\nu_{\pm}(k) = \nu'_{\pm}(k) + i\nu''_{\pm}(k)$ ($\nu'_{\pm}, \nu''_{\pm} \in \mathbb{R}$) and satisfy

$$\begin{aligned} \nu_-^2 &= k^2 + u_0 + w + i(\gamma_0 + g_1), \\ \nu_+^2 &= k^2 + u_0 - w - i(\gamma_0 + g_2). \end{aligned} \quad (25)$$

The branches of ν_{\pm} will be chosen to ensure that at $w = g_1 = g_2 = 0$ we have $\nu_-(k_*) = \nu_+^*(k_*) = \nu_0(k_*)$ where

$$\begin{aligned} \nu_0 &= \sqrt{k_*^2 + u_0 + i\gamma_0} = |\nu_0|e^{i\chi}, \\ |\nu_0|^2 &= \sqrt{(k_*^2 + u_0)^2 + \gamma_0^2}, & \chi &= \frac{1}{2} \arctan \frac{\gamma_0}{k_*^2 + u_0}. \end{aligned} \quad (26)$$

The relation among amplitudes A , B , C , and D , are to be determined from the continuity of the field and its derivatives at $x = 0$:

$$\begin{pmatrix} C \\ D \end{pmatrix} = \mathcal{M}_0(k) \begin{pmatrix} A \\ B \end{pmatrix}, \quad \mathcal{M}_0 = \frac{1}{2} \begin{pmatrix} 1 + \frac{\nu_-}{\nu_+} & 1 - \frac{\nu_-}{\nu_+} \\ 1 - \frac{\nu_-}{\nu_+} & 1 + \frac{\nu_-}{\nu_+} \end{pmatrix}. \quad (27)$$

From the relations (23) and the solution (24), one can obtain (the calculations are outlined in appendix A)

$$M = e^{-i\sigma_3 k} \tilde{M} e^{-i\sigma_3 k}, \quad \tilde{M} = M_+ e^{i\sigma_3 \nu_+} \mathcal{M}_0 e^{i\sigma_3 \nu_-} M_-^{-1}, \quad (28)$$

where

$$M_- = \frac{1}{2} \begin{pmatrix} 1 + \frac{\nu_-}{k} & 1 - \frac{\nu_-}{k} \\ 1 - \frac{\nu_-}{k} & 1 + \frac{\nu_-}{k} \end{pmatrix}, \quad M_+ = \frac{1}{2} \begin{pmatrix} 1 + \frac{\nu_+}{k} & 1 - \frac{\nu_+}{k} \\ 1 - \frac{\nu_+}{k} & 1 + \frac{\nu_+}{k} \end{pmatrix}. \quad (29)$$

Obviously the spectral singularities as well as time-reversed spectral singularities of the matrices M and \tilde{M} coincide. Therefore in what follows we concentrate on \tilde{M} .

4.2. \mathcal{PT} -limit: CPA-laser

Starting with the \mathcal{PT} limit, i.e. with $w = g_1 = g_2 = 0$, one can obtain a CPA-laser. To this end we compute

$$\tilde{M}_{22}^{(0)} = |\cos \nu_0|^2 - |\sin \nu_0|^2 \cos(2\chi) - i \operatorname{Re} \left[\sin(\nu_0^*) \cos(\nu_0) \left(\frac{k_*}{\nu_0^*} + \frac{\nu_0^*}{k_*} \right) \right]. \quad (30)$$

Hereafter the upper index ‘(0)’ stands for the matrices computed in the \mathcal{PT} limit, i.e. at $\kappa = w = g_1 = g_2 = 0$.

The value of $k_* \in \mathbb{R}$ is defined as a root of diagonal matrix elements: $\tilde{M}_{22}^{(0)}(k_*) = 0$. The roots k_* cannot be found in the analytical form, but can be readily computed numerically. The result is shown in figure 3. The numerical values of the roots we obtained by solving first the equation $f(|\nu_0|, \chi) = \operatorname{Re} \tilde{M}_{11}^{(0)} = 0$, which does not contain k_* in the explicit form. As the next step from $\operatorname{Im} \tilde{M}_{22}^{(0)}(k_*) = 0$ one can express k_* through ν_0 explicitly

$$k_*^2 = |\nu_0|^2 \frac{\sinh(2|\nu_0| \sin \chi) \sin \chi - \sin(2|\nu_0| \cos \chi) \cos \chi}{\sinh(2|\nu_0| \sin \chi) \sin \chi + \sin(2|\nu_0| \cos \chi) \cos \chi}. \quad (31)$$

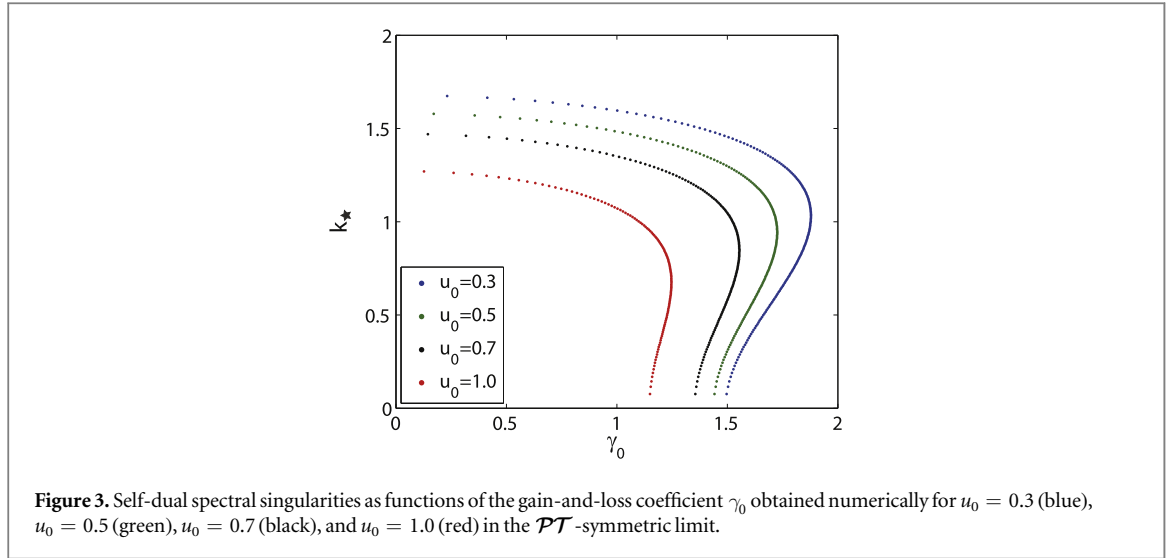


Figure 3. Self-dual spectral singularities as functions of the gain-and-loss coefficient γ_0 obtained numerically for $u_0 = 0.3$ (blue), $u_0 = 0.5$ (green), $u_0 = 0.7$ (black), and $u_0 = 1.0$ (red) in the \mathcal{PT} -symmetric limit.

Notice that the parameters of the \mathcal{PT} -symmetric potential, i.e. u_0 and γ_0 , in these numerical approach enter the solutions implicitly. Moreover, numerical simulations can be limited to reduced interval for the argument χ since the function $f(|\nu_0|, \chi)$ obeys the symmetry $f(|\nu_0|, \chi) = f(|\nu_0|, -\chi) = f(|\nu_0|, \pi - \chi)$. Spectral singularities obviously exist only for nonzero gain-and-loss coefficient. For each given real part of the potential (real part of the symmetric refractive index) u_0 one find an interval of the gain-loss parameter γ_0 at which there exist two self-dual singularities (in the parameter domain scanned numerically).

We also mention that the nondiagonal elements of the scattering matrix can be obtained in the explicit form:

$$\tilde{M}_{12}^{(0)}(k_*) = -i|\sin \nu_0|^2 \sin 2\chi - i\text{Re} \left[\sin(\nu_0) \cos(\nu_0^*) \left(\frac{k_*}{\nu_0} - \frac{\nu_0}{k_*} \right) \right], \quad (32)$$

$$\tilde{M}_{21}^{(0)}(k_*) = -i|\sin \nu_0|^2 \sin 2\chi + i\text{Re} \left[\sin(\nu_0) \cos(\nu_0^*) \left(\frac{k_*}{\nu_0} - \frac{\nu_0}{k_*} \right) \right]. \quad (33)$$

One can verify that $\tilde{M}_{12}^{(0)}(k_*) \tilde{M}_{21}^{(0)}(k_*) = -1$.

4.3. Non- \mathcal{PT} -limit: CPA-and-laser

Now we turn to nonzero deformations κ , w , and $g_{1,2}$. One can use the property $\mathcal{M}_0 = M_+^{-1} M_-$, allowing one to express $\tilde{M} = A_+ A_-$, where $A_{\pm} = M_{\pm} e^{i\sigma_3 \nu_{\pm}} M_{\pm}^{-1}$. Noticing that the dependence on the parameters $g_{1,2}$ and w is only in ν_{\pm} we compute

$$\begin{aligned} \frac{\partial M_{\pm}}{\partial \nu_{\pm}} &= \frac{1}{2k} S, \quad \frac{\partial M_{\pm}^{-1}}{\partial \nu_{\pm}} = -\frac{k}{2\nu_{\pm}^2} S, \quad S = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \\ \left(\frac{\partial M_{\pm}}{\partial k} \right)_{\nu_{\pm}} &= -\frac{\nu_{\pm}}{2k^2} S, \quad \left(\frac{\partial M_{\pm}^{-1}}{\partial k} \right)_{\nu_{\pm}} = \frac{1}{2\nu_{\pm}} S, \end{aligned}$$

where the derivative with respect to k is computed with $\nu_{\pm} = \text{const}$. Subsequently

$$\frac{\partial A_{\pm}}{\partial \nu_{\pm}} = \frac{1}{2\nu_{\pm}} (S A_{\pm} - A_{\pm} S) + \frac{i}{2} \left(\frac{k}{\nu_{\pm}} + \frac{\nu_{\pm}}{k} \right) \sigma_3 A_{\pm} + \frac{i}{2} \left(\frac{k}{\nu_{\pm}} - \frac{\nu_{\pm}}{k} \right) \sigma_2 A_{\pm}.$$

Finally, we compute the first-order terms of the expansion

$$\begin{aligned} \tilde{M} &\approx \tilde{M}^{(0)} + \left(\frac{\partial \tilde{M}^{(0)}}{\partial k} \right)_{\nu_{\pm}} \kappa + \left(\frac{\partial \nu_+}{\partial w} w + \frac{\partial \nu_+}{\partial g_1} g_1 + \frac{\partial \nu_+}{\partial g_2} g_2 + \frac{\partial \nu_+}{\partial k} \kappa \right) \\ &\times \left[\frac{1}{2\nu_+} (S \tilde{M}^{(0)} - A_+^{(0)} S A_-^{(0)}) + iJ_+^* \sigma_3 \tilde{M}^{(0)} + J_-^* \sigma_2 \tilde{M}^{(0)} \right] \\ &+ \left(\frac{\partial \nu_-}{\partial w} w + \frac{\partial \nu_-}{\partial g_1} g_1 + \frac{\partial \nu_-}{\partial g_2} g_2 + \frac{\partial \nu_-}{\partial k} \kappa \right) \\ &\times \left[\frac{1}{2\nu_-} (A_+^{(0)} S A_-^{(0)} - \tilde{M}^{(0)} S) + iJ_+ A_+^{(0)} \sigma_3 A_-^{(0)} + J_- A_+^{(0)} \sigma_2 A_-^{(0)} \right], \end{aligned} \quad (34)$$

where ν_0 is defined in (26) and $J_{\pm} = \frac{1}{2} \left(\frac{k_*}{\nu_0} \pm \frac{\nu_0}{k_*} \right)$.

Further simplification of (34) requires tedious calculations whose main steps are outlined in appendix B. Here we show only the final result. The expansion (4) in our set of parameters $\Lambda = \{w, g_1, g_2\}$ reads

$$\tilde{M}_{11} = \left(T_0 + \frac{k_*}{\nu_0} T_{11} + \frac{k_*}{\nu_0^*} T_{12} \right) \frac{\kappa_1}{2} + \left(\frac{T_{11}}{4\nu_0} - \frac{T_{12}}{4\nu_0^*} \right) w + \frac{iT_{11}}{4\nu_0} g_1 - \frac{iT_{12}}{4\nu_0^*} g_2, \quad (35)$$

$$\tilde{M}_{22} = \left(T_0 + \frac{k_*}{\nu_0} T_{21} + \frac{k_*}{\nu_0^*} T_{22} \right) \frac{\kappa_2}{2} + \left(\frac{T_{21}}{4\nu_0} - \frac{T_{22}}{4\nu_0^*} \right) w + \frac{iT_{21}}{4\nu_0} g_1 - \frac{iT_{22}}{4\nu_0^*} g_2, \quad (36)$$

where

$$T_0 = \frac{i}{k_*} [F_- \sin(2|\nu_0| \cos \chi) \cos \chi + F_+ \sinh(2|\nu_0| \sin \chi) \sin \chi], \quad (37)$$

$$\begin{aligned} T_{11}(\nu_0) = T_{21}^*(\nu_0^*) = & \left(iF_+ + \frac{i}{\nu_0} \sin \chi + \frac{1}{|\nu_0|} \right) \cos(2|\nu_0| \cos \chi) \cos \chi \\ & + \left(F_- \frac{i}{\nu_0} \cos \chi - \frac{i}{|\nu_0|} \right) \cosh(2|\nu_0| \sin \chi) \sin \chi \\ & + \left(\frac{\sin(2\chi)}{2k_*} - \frac{F_+}{2\nu_0} \sin \chi + \left(\frac{i}{k_*} - 2 \right) \cos^2 \chi \right) \sin(2|\nu_0| \cos \chi) \\ & + \left(-\frac{i \sin(2\chi)}{2k_*} + \frac{F_-}{2\nu_0} \cos \chi - \left(2i + \frac{1}{k_*} \right) \sin^2 \chi \right) \sinh(|\nu_0| \sin \chi), \end{aligned} \quad (38)$$

$$\begin{aligned} T_{12}(\nu_0) = T_{22}^*(\nu_0^*) = & -\frac{ik_*}{2} \left(\frac{k_*}{\nu_0^*} - \frac{\nu_0^*}{k_*} \right) T_0 \\ & - \frac{1}{2i\nu_0^*} \left(\frac{k_*}{\nu_0^*} - \frac{\nu_0^*}{k_*} \right) [i \sin(2|\nu_0| \cos \chi) + \sinh(2|\nu_0| \sin \chi)] \\ & + \left[\frac{1}{2} \left(\frac{k_*}{\nu_0^*} - \frac{\nu_0^*}{k_*} \right) - \frac{i}{\nu_0^*} \right] [\cos(2|\nu_0| \cos \chi) - \cosh(2|\nu_0| \sin \chi)] \sin(2\chi), \end{aligned} \quad (39)$$

where we use the abbreviated notation $F_{\pm} = \frac{k_*}{|\nu_0|} \pm \frac{|\nu_0|}{k_*}$ and recall that the change $\nu_0 \rightarrow \nu_0^*$ implies the change $\chi \rightarrow -\chi$.

Below we consider modulation of the real part of the potential w as a tunable control parameter. Then variations of gain and losses ensuring the existence of lasing and absorbing singularities (i.e. obtained from (6)) are given by

$$g_1 = \frac{w}{\Delta} [A_1' (A_2'' - B_2'') - A_2' (A_1'' - B_1'')], \quad (40)$$

$$g_2 = \frac{w}{\Delta} [B_1' (A_2'' - B_2'') - B_2' (A_1'' - B_1'')], \quad (41)$$

where $\Delta = A_1' B_2' - A_2' B_1'$, $A_j' = \text{Re } A_j$, $A_j'' = \text{Im } A_j$, $B_j' = \text{Re } B_j$, and $B_j'' = \text{Im } B_j$, and the abbreviated notations

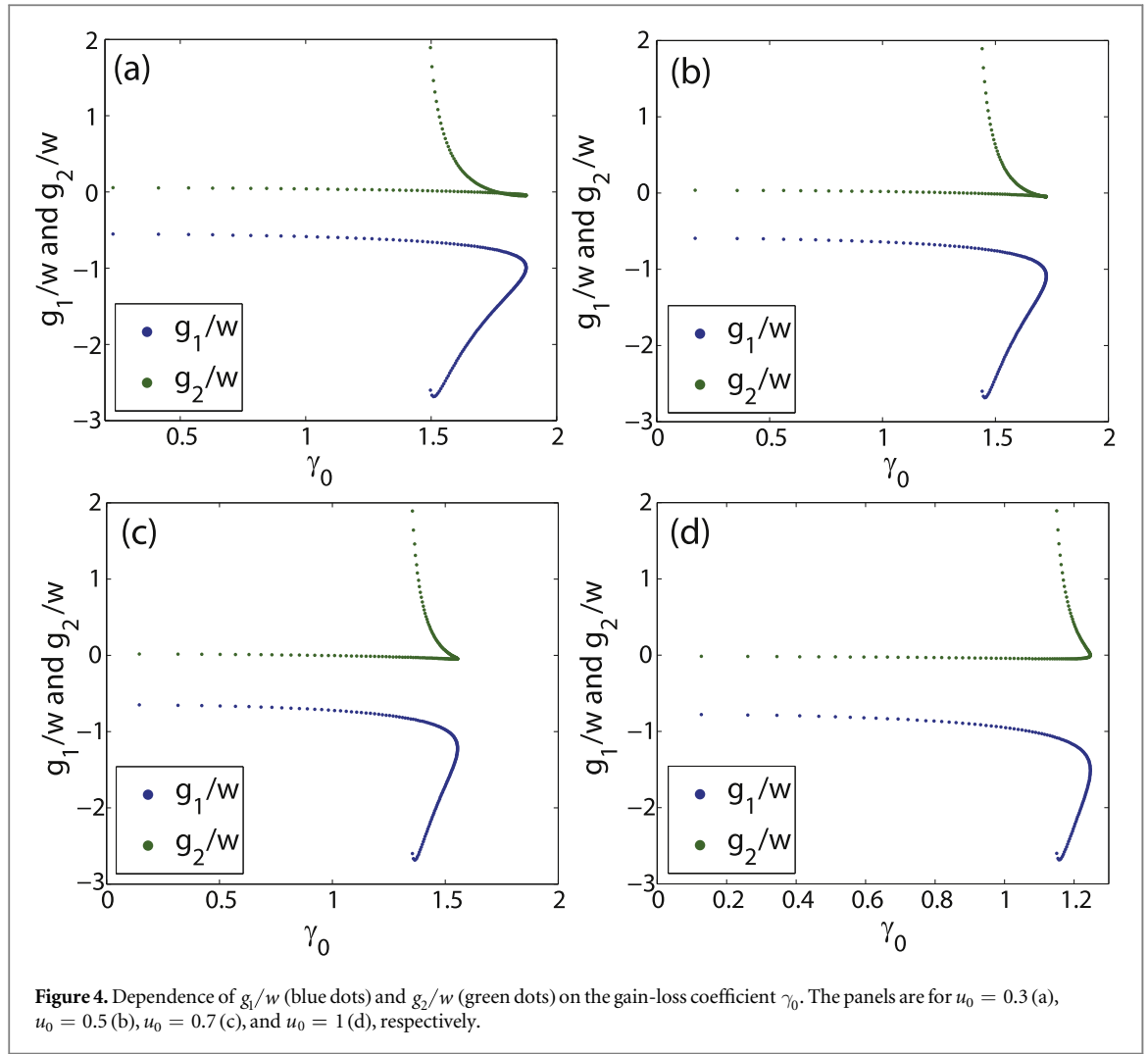
$$A_j = \frac{\alpha_j}{1 + k_*(\alpha_j + \beta_j)}, \quad B_j = \frac{\beta_j}{1 + k_*(\alpha_j + \beta_j)}, \quad \alpha_j = \frac{T_{j2}}{\nu_0^* T_0}, \quad \beta_j = \frac{T_{j1}}{\nu_0 T_0}$$

are introduced.

In panels of figure 4 we show the dependence of g_1/w and g_2/w on the gain-loss coefficient γ_0 for the real parts of the \mathcal{PT} -symmetric potential corresponding to ones shown in figure 3. We notice that in the cases $u_0 = 0.3$, $u_0 = 0.5$, and $u_0 = 0.7$, shown respectively in panels (a)–(c), we have $g_1 < 0$ and $g_2 > 0$ for the whole range of γ_0 . For $u_0 = 1$ this conclusion is valid almost for the whole γ_0 domain, except small vicinity of the maximal γ_0 where both g_1 and g_2 are negative. For the whole explored domain of γ_0 we observed that for split of self-dual spectral singularities into lasing and absorbing singularities the imbalance between gain and losses in active and absorbing layers, i.e. $g_2 - g_1$, must increase.

Finally, substituting these expression in (5) we obtain the shifts of the singular points from k_* along the real axis

$$\begin{aligned} \kappa_1 = & \frac{w}{2\Delta} [B_2' |A_1|^2 + A_2' |B_1|^2 - B_1' (A_1' A_2' + A_1'' A_2'') - A_1' (B_1' B_2' + B_1'' B_2'') \\ & + B_1'' (A_1' A_2'' - A_1'' A_2') + A_1'' (B_1' B_2'' - B_1'' B_2')] \end{aligned} \quad (42)$$



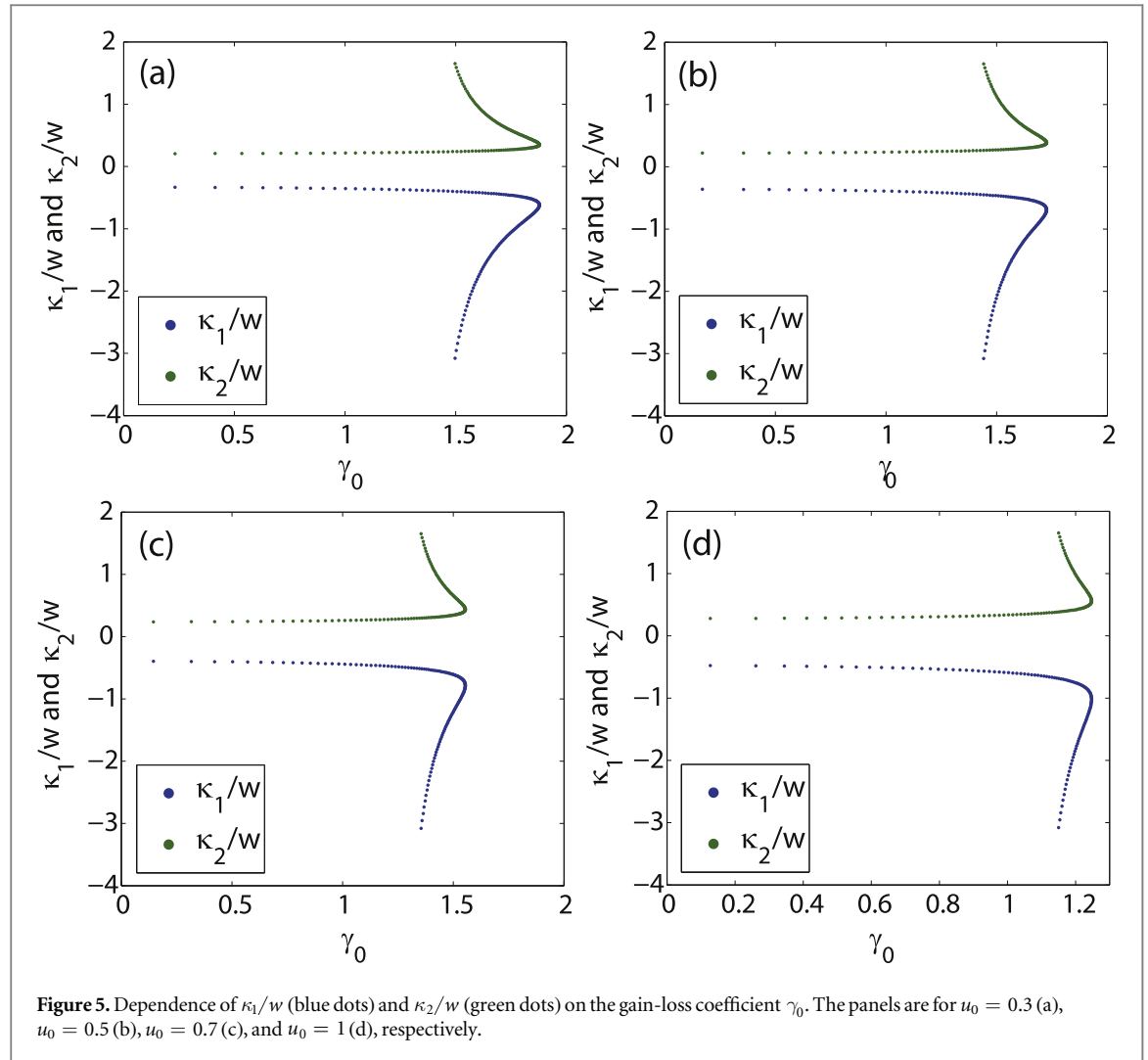
$$\kappa_2 = \frac{w}{2\Delta} [-B_1'|A_2|^2 - A_1'|B_2|^2 + B_2'(A_1'A_2' + A_1''A_2'') + A_2'(B_1'B_2' + B_1''B_2'') + B_2''(A_1'A_2'' - A_1''A_2') + A_2''(B_1'B_2'' - B_1''B_2')]. \quad (43)$$

In figure 5, we show the shifts of the wave-vectors yielding lasing and absorbing singularities as functions of the gain-loss parameter γ_0 for the real refractive indexes explored in figure 3. We observe that for relatively small γ_0 , when only one self-dual singularity exist (see figure 3) the shifts $\kappa_{1,2}$, as well as the required gain and loss changes shown in figure 4, are very weakly dependent on the values γ_0 . On the contrary, the split of the second spectral singularity, i.e. corresponding to the smaller roots k_* in the lower branches shown in figure 3, are strongly γ_0 -dependent.

5. CPA-and-laser in an atomic cell

In the previous section, we have argued that, having at least three adjustable parameters of a medium with gain and losses, one can control the simultaneous appearance of *two* spectral singularities, provided the system features \mathcal{PT} -symmetry for a specific point in the parameter space. As a simple example, we have considered a system consisting of two independent media (possibly of two atomic cells) and found that specific shifts of the spectral singularities are possible. We now present a practically feasible physical system, in which the dielectric susceptibility can be actively designed and manipulated, providing specific parameters at which it can operate as a CPA-and-laser, i.e. the system may have two spectral singularities.

To this end we consider two neighboring atomic cells filled with different cold atomic gases. To be specific, we assume that the left cell is filled with ^{87}Rb atoms, corresponding to an absorbing medium region ($-L < x < 0$), while the right cell is filled with ^{85}Rb atoms, creating an active region ($0 < x < L$), as it is illustrated schematically in figure 6. For two rubidium isotopes we assign $|g, s\rangle = |5S_{1/2}, F = 1\rangle$, $|a, s\rangle = |5S_{1/2}, F = 2\rangle$, and $|e, s\rangle = |5P_{1/2}, F = 0\rangle$, with $s = 1, 2$ indicating ^{87}Rb and ^{85}Rb atoms, respectively.



Each atom has Λ -type configuration, in which a weak probe field \mathbf{E}_p (with the angular frequency ω_p and the wavevector $\mathbf{k}_p = (k, 0, k_z)$; $\sqrt{k^2 + k_z^2} = k_p = \omega_p/c$) drives the ground state $|g, s\rangle$ to the excited level $|e, s\rangle$ and a strong control field \mathbf{E}_c (with the angular frequency ω_c and the wavevector $\mathbf{k}_c = (0, 0, k_c)$; $k_c = \omega_c/c$) drives the lower state $|a, s\rangle$ to the excited level $|e, s\rangle$, respectively.

We notice that in comparison with two-level systems, say described by the Lorentz model as those used in [22], three-level systems offer advantages of being significantly more stable what is particularly relevant for obtaining optical gain and dissipation. The enhanced stability is due to presence of two populated ground states without requiring highly populated excited states (the latter may be subjected to significant spontaneous emission). Additionally, a three-level atomic system is governed by a larger number of tunable parameters, thus offering better controllability (which for the present consideration is crucial).

The probe-field susceptibilities in the absorbing and gain regions are defined by $\chi_1 = p_{eg}^2 N_1 \rho_{eg,1} / (\epsilon_0 \hbar \Omega_p)$ and $\chi_2 = p_{eg}^2 N_2 \rho_{eg,2} / (\epsilon_0 \hbar \Omega_p)$, respectively. Here the half Rabi frequency of the probe field (control field) is given by $\Omega_p = |\mathbf{e}_p \cdot \mathbf{p}_{eg}| E_p / (2\hbar)$ ($\Omega_c = |\mathbf{e}_c \cdot \mathbf{p}_{ea}| E_c / (2\hbar)$), with \mathbf{p}_{eg} (\mathbf{p}_{ea}) the electric dipole matrix element associated with the transition $|e, s\rangle \leftrightarrow |g, s\rangle$ ($|e, s\rangle \leftrightarrow |a, s\rangle$) and being assumed to be approximately equal for both isotopes ($p_{eg} \approx p_{ea} = 10^{-27} \cdot 2.54$ C cm for the selected levels of the Rb atoms [27]); N_s is the density of the s -th isotope, and $\rho_{eg,s}$ is the atomic coherence of the s -th isotope which can be computed from the optical Bloch equations [26].

The atomic coherence $\rho_{eg,s}$ was calculated in [19, 20], by which one can obtain the expressions of the probe-field susceptibilities in the absorbing and gain medium regions of the forms

$$\frac{\chi_1}{\chi_0} = \frac{\delta_1 - i\gamma_{ag}}{(\delta_1 + \Delta_1 - i\gamma_{eg})(\delta_1 - i\gamma_{ag}) - |\Omega_c|^2}, \quad (44)$$

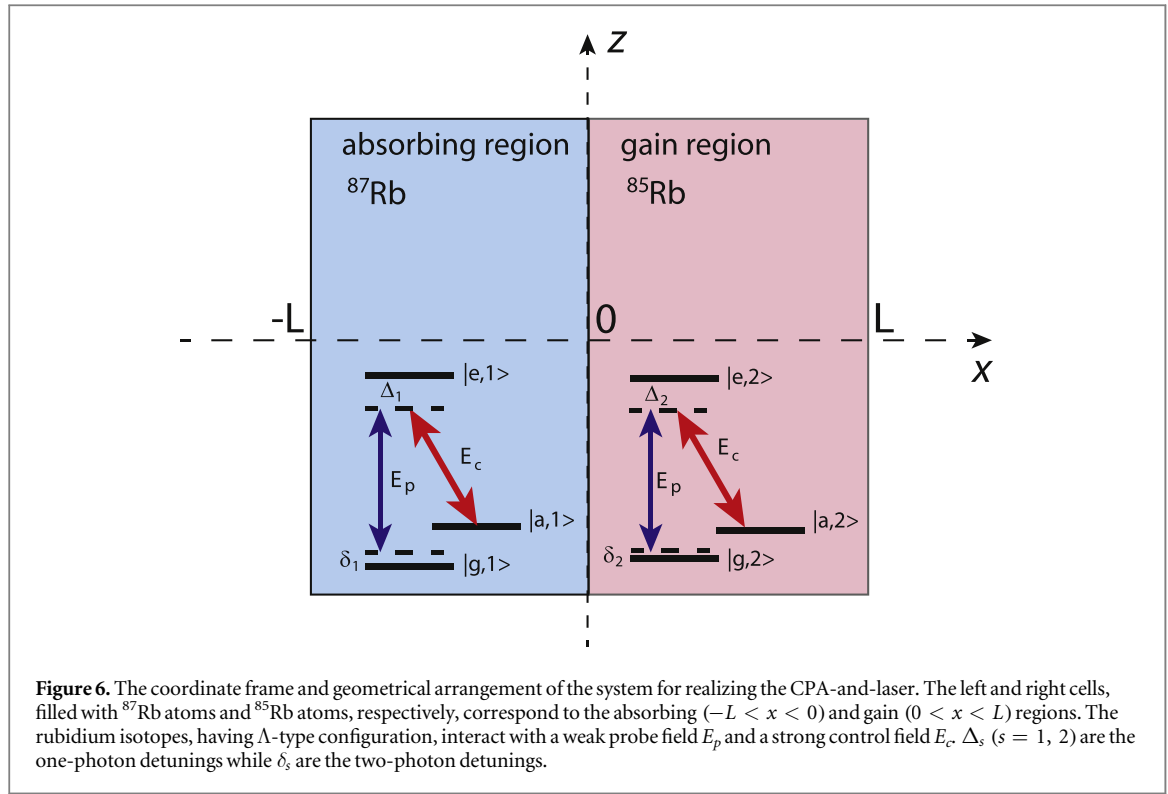


Figure 6. The coordinate frame and geometrical arrangement of the system for realizing the CPA-and-laser. The left and right cells, filled with ^{87}Rb atoms and ^{85}Rb atoms, respectively, correspond to the absorbing ($-L < x < 0$) and gain ($0 < x < L$) regions. The rubidium isotopes, having Λ -type configuration, interact with a weak probe field E_p and a strong control field E_c . Δ_s ($s = 1, 2$) are the one-photon detunings while δ_s are the two-photon detunings.

$$\frac{\chi_2}{\chi_0} = -\eta \frac{|\Omega_c|^2 (\Delta_2 + i\gamma_{ea})^{-1}}{(\delta_2 + \Delta_2 - i\gamma_{eg})(\delta_2 - i\gamma_{ag}) - |\Omega_c|^2}, \quad (45)$$

where $\chi_0 = N_1 p_{eg}^2 / (\epsilon_0 \hbar)$, $\eta = N_2 / N_1$, $\Delta_s = \omega_e^s - \omega_a^s - \omega_c$; $\delta_s = \omega_a^s - \omega_g^s - (\omega_p - \omega_c)$ are respectively one-photon and two-photon detunings, with ω_a^s (ω_g^s) the eigenfrequency of level $|a, s\rangle$ ($|g, s\rangle$). The composite decay rate γ_{jl} is given by $\gamma_{jl} = (\Gamma_j + \Gamma_l)/2 + \gamma_{jl}^{\text{col}}$. Here $\Gamma_j = \sum_{j' < l} \Gamma_{jl}$, with Γ_{jl} being the spontaneous emission decay rate from $|j, s\rangle$ to $|l, s\rangle$ and γ_{jl}^{col} being the dephasing rate reflecting the loss of phase coherence between $|j, s\rangle$ and $|l, s\rangle$ without changing of population [26].

In order to acquire additional adjustable parameters, we assume a far-detuned laser field (called Stark field) of the form $E_S \cos(\omega_S t)$ is applied to the system, which induces the energy shift $\Delta E_{j,s} = -\alpha_j E_S^2 / 4$ for the levels $|j, s\rangle$, with α_j being the scalar polarizability ($\alpha_e - \alpha_g \approx 2\pi \hbar \cdot 0.1223 \text{ Hz}(\text{cm}/\text{V})^2$ and $\alpha_g \approx \alpha_a$ for the selected levels of the Rb atoms [27]). We thus obtain $\Delta_s = \Delta_{s0} - (\alpha_e - \alpha_g) E_S^2 / (4\hbar)$ (with Δ_{s0} corresponding to the detunings for $E_S = 0$) whereas δ_s remains to be constant. Notice that now we have *three* adjustable control parameters, η , Ω_c , and E_S at hand.

The target susceptibility of the medium is given by equation (20). After taking $\xi = k_S x$ and $L = 1/k_S$, the equation describing the wave scattering (1) can be written into the dimensionless form $d^2\psi/d\xi^2 + U_0(\xi)\psi = -(k^2/k_S^2)\psi$, where $U_0(\xi) \equiv U_{01} = (k_p^2/k_S^2)(\chi_{1r} + i\chi_{1i})$ for $\xi \in (-1, 0)$, $U_0(\xi) \equiv U_{02} = (k_p^2/k_S^2)(\chi_{2r} - i\chi_{2i})$ for $\xi \in (0, 1)$, and $U_0(\xi) = 0$ for $|\xi| > 1$. Thus, we obtain the relations $u_0 = (\text{Re}U_{01} + \text{Re}U_{02})/2$, $w = (\text{Re}U_{01} - \text{Re}U_{02})/2$, $g_1 = \text{Im}U_{01} - \gamma_0$, and $g_1 = \text{Im}U_{02} - \gamma_0$.

In order to verify that the atomic system described above can indeed be used to design a CPA-and-laser, we provide a set of realistic physical parameters of the system, i.e. $\Delta_1 = 2\Gamma$, $\Delta_2 = 2.72\Gamma$, $\delta_1 = 1.81\Gamma$, and $\delta_2 = 0$, with $\gamma_{eg} \approx \gamma_{ea} \equiv \Gamma = 2\pi \times 334 \text{ MHz}$ [19]. Taking the wavelength of the Stark field to be $\lambda_S \approx 6.28 \mu\text{m}$ (so that $L \approx 1 \mu\text{m}$) and the atomic densities of the first and the second isotopes $N_1 \approx 4.88 \times 10^{13} \text{ cm}^{-3}$ and $N_2 \approx 9.53 \times 10^{13} \text{ cm}^{-3}$, we obtain $\chi_0 \approx 1$ and $\eta \approx 1.3$; additionally, by assuming $\Omega_c \approx 3.47 \times 10^9 \text{ s}^{-1}$ and $E_S \approx 1.14 \times 10^5 \text{ V cm}^{-1}$, we obtain $U_{01} \approx 0.50 + i0.45$ and $U_{01} \approx 0.90 - i0.59$, and hence $u_0 \approx 0.7$ and $w \approx -0.2$. Thus if $\gamma_0 \approx 0.45$ we obtain $g_1 \approx 0$ and $g_2 \approx 0.14$, i.e. $g_1/w \approx 0$ and $g_2/w \approx -0.66$, which coincide with the data given in figure 4(c). On the other hand, from figure 3 we know that $k_* \approx 1.45$ at $\gamma_0 \approx 0.45$, and from figure 5(c) we see that $\kappa_1/w \approx -0.4$ and $\kappa_2/w \approx 0.24$, i.e. $\kappa_1 \approx 0.08$ and $\kappa_2 \approx -0.05$. Thus, we have two spectral singularities at different wavevectors, i.e. $k_*^{(a)} = k_* + \kappa_1 \approx 1.53$ for absorbing singularity and $k_*^{(l)} = k_* + \kappa_2 \approx 1.40$ for lasing singularity. In physical units, they are $1.53 \times 10^6 \text{ s}^{-1}$ and $1.40 \times 10^6 \text{ s}^{-1}$, respectively.

In order to change the wavevector k continuously, one can change the angle θ between the probe field and the z axis. The relation between k and θ is given by $\tan \theta = k / \sqrt{\omega_p^2/c^2 - k^2}$, where the refractive index in the

atomic cells $n_s = \sqrt{1 + \chi_s} \approx 1$. Hence, lasing and absorbing spectral singularities are obtained at different angles, i.e. $\theta^{(a)} \approx 0.19$ for the absorbing singularity and $\theta^{(l)} \approx 0.18$ for the lasing singularity.

6. Conclusion

In this paper we have explored a possibility of using a medium with a complex refractive index and tunable parameters to design a system that can function as a laser at one wavelength, corresponding to a lasing singularity of the complex potential, and as a CPA at another wavevector, corresponding to an absorbing singularity of the complex conjugate potential. Such a CPA-and-laser can be obtained in a prescribed way if the original system is \mathcal{PT} -symmetric, obeys a spectral singularity, and has at least three available tunable parameters. Then by tuning the system parameters of such a CPA-laser in a given manner, one can make the self-dual spectral singularity of the \mathcal{PT} -symmetric system to split into two desirable absorbing and lasing singularities. We have demonstrated that the system consisting of the atomic cells filled with a cold gas of Λ -type three-level Rb isotopes is an efficient way for creating and manipulating the optical potential leading to two spectral singularities occurring simultaneously at different wavevectors, and thus is promising for the physical realization of the suggested CPA-and-laser experimentally.

While our consideration was based on simple models allowing explicit calculations of the scattering matrix elements through the potential characteristics and focused on the two singularities of different types ('direct' and time-reversed), the analysis presented above imply several extensions. To mention a few of them we notice that, in case of more sophisticated potentials, for which the transfer matrix is not computed analytically but weakly deviates from exactly solvable potentials, the spectral singularities can be obtained by means of a perturbation approach [28]. Similar perturbation approach is expected to be useful for the description of splitting self-dual singularities. It is of practical interest to consider the existence of spectral singularities, of time-reversed spectral singularities and of their splitting for TM waves, as well as to design practical systems for such waves. Another ramification of the reported results is the study of the splitting of self-dual singularities in non- \mathcal{PT} -symmetric systems, similar to ones reported in [13]. Furthermore, we have mentioned only a few functionalities of the spectral singularities, which obviously do not exhaust the whole diversity of the phenomena reachable in the suggested medium. In particular, since the effect is based on two different specific wavevectors, the polarization dynamics of light beams in birefringent media with active layers obeying both singularities may be a suitable framework for such studies. Finally, it is of practical relevance to design a CPA-and-laser in a single cell using spatially dependent control and Stark fields.

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Appendix A. Derivation of (28)

To describe the field inside the slab we introduce the transfer matrices $\mathcal{M}_{\pm}(k)$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \mathcal{M}_-(k) \begin{pmatrix} A \\ B \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathcal{M}_+(k) \begin{pmatrix} C \\ D \end{pmatrix}, \quad (\text{A.1})$$

The total transfer matrix is given by $M = \mathcal{M}_+ \mathcal{M}_0 \mathcal{M}_-^{-1}$ where

$$\mathcal{M}_- = \frac{1}{2} \begin{pmatrix} \left(1 + \frac{\nu_-}{k}\right) e^{-i(\nu_- - k)} & \left(1 - \frac{\nu_-}{k}\right) e^{i(\nu_- + k)} \\ \left(1 - \frac{\nu_-}{k}\right) e^{-i(\nu_- + k)} & \left(1 + \frac{\nu_-}{k}\right) e^{i(\nu_- - k)} \end{pmatrix}, \quad (\text{A.2})$$

$$\mathcal{M}_+ = \frac{1}{2} \begin{pmatrix} \left(1 + \frac{\nu_+}{k}\right) e^{i(\nu_+ - k)} & \left(1 - \frac{\nu_+}{k}\right) e^{-i(\nu_+ + k)} \\ \left(1 - \frac{\nu_+}{k}\right) e^{i(\nu_+ + k)} & \left(1 + \frac{\nu_+}{k}\right) e^{-i(\nu_+ - k)} \end{pmatrix}. \quad (\text{A.3})$$

Rewriting

$$\mathcal{M}_- = e^{i\sigma_3 k} M_- e^{-i\sigma_3 \nu_-}, \quad \mathcal{M}_+ = e^{-i\sigma_3 k} M_+ e^{i\sigma_3 \nu_+}, \quad (\text{A.4})$$

where M_{\pm} is given by (29), we obtain (28).

Appendix B. Expansion of the matrix \tilde{M}

From (25) we obtain in the leading order:

$$\begin{aligned} \frac{\partial \nu_+}{\partial g_1} = \frac{\partial \nu_-}{\partial g_2} = 0, \quad \frac{\partial \nu_+}{\partial g_2} g_2 = -i \frac{g_2}{2\nu_0^*}, \quad \frac{\partial \nu_-}{\partial g_1} g_1 = i \frac{g_1}{2\nu_0} \\ \frac{\partial \nu_+}{\partial k} \kappa = \frac{k_*}{\nu_0^*} \kappa, \quad \frac{\partial \nu_-}{\partial k} \kappa = \frac{k_*}{\nu_0} \kappa, \quad \frac{\partial \nu_+}{\partial w} w = -\frac{w}{2\nu_0^*}, \quad \frac{\partial \nu_-}{\partial w} w = \frac{w}{2\nu_0}. \end{aligned} \quad (\text{B.1})$$

For the next calculations we use that

$$\tilde{M}_{11} = \frac{1}{2} \text{Tr}[(1 + \sigma_3)\tilde{M}], \quad \tilde{M}_{22} = \frac{1}{2} \text{Tr}[(1 - \sigma_3)\tilde{M}]. \quad (\text{B.2})$$

Since in the self-dual singularity $\tilde{M}_{11}^{(0)} = \tilde{M}_{22}^{(0)} = 0$, we can use (B.1) to obtain from (34) the following expression:

$$\begin{aligned} \text{Tr}[(1 \pm \sigma_3)\tilde{M}] \approx \text{Tr} \left[(1 \pm \sigma_3) \frac{\partial \tilde{M}^{(0)}}{\partial k} \right] \kappa \\ + \text{Tr} \left\{ (1 \pm \sigma_3) \left[\frac{1}{2\nu_0^*} (S\tilde{M}^{(0)} - A_+^{(0)} S A_-^{(0)}) + iJ_+^* \sigma_3 \tilde{M}^{(0)} + J_-^* \sigma_2 \tilde{M}^{(0)} \right] \right\} \\ \times \left(-\frac{w}{2\nu_0^*} - i \frac{g_2}{2\nu_0^*} + \frac{k_*}{\nu_0^*} \kappa \right) \\ + \text{Tr} \left\{ (1 \pm \sigma_3) \left[\frac{1}{2\nu_0} (A_+^{(0)} S A_-^{(0)} - \tilde{M}^{(0)} S) + iJ_+ A_+^{(0)} \sigma_3 A_-^{(0)} + J_- A_+^{(0)} \sigma_2 A_-^{(0)} \right] \right\} \\ \times \left(\frac{w}{2\nu_0} + i \frac{g_1}{2\nu_0} + \frac{k_*}{\nu_0} \kappa \right). \end{aligned} \quad (\text{B.3})$$

It is straightforward to check that at $k = k_*$

$$\text{Tr}[(1 + \sigma_3)S\tilde{M}^{(0)}] = \text{Tr}[(1 - \sigma_3)\tilde{M}^{(0)}S] = -2\tilde{M}_{21}^{(0)}, \quad (\text{B.4})$$

$$\text{Tr}[(1 - \sigma_3)S\tilde{M}^{(0)}] = \text{Tr}[(1 + \sigma_3)\tilde{M}^{(0)}S] = -2\tilde{M}_{12}^{(0)} \quad (\text{B.5})$$

and

$$A_-^{(0)} = iJ_+ \sigma_3 \sin \nu_0 + J_- \sigma_2 \sin \nu_0 + \cos \nu_0, \quad (\text{B.6})$$

$$A_+^{(0)} = iJ_+^* \sigma_3 \sin \nu_0^* + J_-^* \sigma_2 \sin \nu_0^* + \cos \nu_0^*, \quad (\text{B.7})$$

Thus we compute

$$\left(\frac{\partial A_-^{(0)}}{\partial k} \right)_{\nu_{\pm}} = \frac{\sin \nu_0}{k_*} (iJ_- \sigma_3 + J_+ \sigma_2), \quad (\text{B.8})$$

$$\left(\frac{\partial A_+^{(0)}}{\partial k} \right)_{\nu_{\pm}} = \frac{\sin \nu_0^*}{k_*} (iJ_-^* \sigma_3 + J_+^* \sigma_2). \quad (\text{B.9})$$

and subsequently

$$\begin{aligned} \left(\frac{\partial \tilde{M}^{(0)}}{\partial k} \right)_{\nu_{\pm}} = \frac{i\sigma_3}{2k_*} [\sin(\nu_0 + \nu_0^*)(J_- + J_-^*) + \sin(\nu_0 - \nu_0^*)(J_- - J_-^*)] \\ + \frac{\sigma_2}{2k_*} [\sin(\nu_0 + \nu_0^*)(J_+ + J_+^*) + \sin(\nu_0 - \nu_0^*)(J_+ - J_+^*)]. \end{aligned} \quad (\text{B.10})$$

Further we have to compute the following traces

$$\begin{aligned} \text{Tr}[(1 \pm \sigma_3)A_+^{(0)} S A_-^{(0)}] = 2[|\cos \nu_0|^2 - (J_+ - J_-)(J_+^* + J_-^*)|\sin \nu_0|^2] \\ \pm 2i[(J_+ - J_-)\sin(\nu_0)\cos(\nu_0^*) + (J_+^* + J_-^*)\sin(\nu_0^*)\cos(\nu_0)], \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} \text{Tr}[(1 \pm \sigma_3)A_+^{(0)} \sigma_2 A_-^{(0)}] = \sin(\nu_0 + \nu_0^*)(J_- + J_-^*) + \sin(\nu_0 - \nu_0^*)(J_- - J_-^*) \\ \pm 2i|\sin \nu_0|^2 (J_+ J_-^* + J_+^* J_-), \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} \text{Tr}[(1 \pm \sigma_3)A_{\pm}^{(0)}\sigma_3A_{\mp}^{(0)}] &= i[\sin(\nu_0 + \nu_0^*)(J_+ + J_+^*) + \sin(\nu_0 - \nu_0^*)(J_+ - J_+^*)] \\ &\quad \pm 2[|\cos \nu_0|^2 - |\sin \nu_0|^2(|J_+|^2 + |J_-|^2)]. \end{aligned} \quad (\text{B.13})$$

Gathering all derivatives in the trace formulas we obtain the coefficients T_0 , T_{11} , T_{12} , T_{12} , and T_{22} :

$$\begin{aligned} T_0 &= \text{Tr} \left[(1 \pm \sigma_3) \left(\frac{\partial \tilde{M}^{(0)}}{\partial k} \right)_{\nu_0} \right] = \frac{i}{k_*} [\sin(\nu_0 + \nu_0^*)(J_- + J_-^*) \\ &\quad + \sin(\nu_0 - \nu_0^*)(J_- - J_-^*)], \end{aligned} \quad (\text{B.14})$$

$$\begin{aligned} T_{11} &= \text{Tr} \left\{ (1 + \sigma_3) \left[\frac{1}{2\nu_0} (A_{\pm}^{(0)}SA_{\mp}^{(0)} - \tilde{M}^{(0)}S) + A_{\pm}^{(0)}(iJ_+\sigma_3 + J_-\sigma_2)A_{\mp}^{(0)} \right] \right\} \\ &= \frac{1}{\nu_0} \tilde{M}_{12}^{(0)} + \left(\frac{1}{\nu_0} + 2iJ_+ \right) |\cos \nu_0|^2 - \left(\frac{1}{\nu_0^*} + i2J_+^*(J_+^2 - J_-^2) \right) |\sin \nu_0|^2 \\ &\quad + \frac{i}{\nu_0} [(J_+ - J_-)\sin(\nu_0)\cos(\nu_0^*) + (J_+^* + J_-^*)\sin(\nu_0^*)\cos(\nu_0)] - (|J_+|^2 + J_+^2 \\ &\quad - |J_-|^2 - J_-^2)\sin(\nu_0 + \nu_0^*) + (|J_+|^2 - J_+^2 - |J_-|^2 + J_-^2)\sin(\nu_0 - \nu_0^*), \end{aligned} \quad (\text{B.15})$$

$$\begin{aligned} T_{12} &= \text{Tr} \left\{ (1 + \sigma_3) \left[\frac{1}{2\nu_0^*} (S\tilde{M}^{(0)} - A_{\pm}^{(0)}SA_{\mp}^{(0)}) + (iJ_+^*\sigma_3 + J_-^*\sigma_2)\tilde{M}^{(0)} \right] \right\} \\ &= - \left(\frac{1}{\nu_0^*} + 2iJ_-^* \right) \tilde{M}_{21}^{(0)}(k_*) - \frac{1}{\nu_0^*} [|\cos \nu_0|^2 - (J_+ - J_-)(J_+^* + J_-^*)|\sin \nu_0|^2] \\ &\quad - \frac{i}{\nu_0^*} [(J_+ - J_-)\sin(\nu_0)\cos(\nu_0^*) + (J_+^* + J_-^*)\sin(\nu_0^*)\cos(\nu_0)], \end{aligned} \quad (\text{B.16})$$

The final simplifications involve the relations: $J_+^2 - J_-^2 = 1$,

$$\begin{aligned} J_+ &= 2(E \cos \chi - iE_+ \sin \chi), \\ J_- &= 2(E_+ \cos \chi - iE \sin \chi), \\ |J_+|^2 - |J_-|^2 &= \cos(2\chi), \\ J_+J_-^* - J_+^*J_- &= i\sin(2\chi), \end{aligned}$$

as well as

$$\cosh(2|\nu_0|\sin \chi) \sin^2 \chi = -\cos(2|\nu_0|\cos \chi) \cos^2 \chi, \quad (\text{B.17})$$

$$E_- \sinh(2|\nu_0|\sin \chi) \sin \chi = -E_+ \sin(2|\nu_0|\cos \chi) \cos \chi, \quad (\text{B.18})$$

following from the condition $\tilde{M}_{22}^{(0)}(k_*) = 0$. This leads to the final expressions (37)–(39).

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