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PAPER

Resilience of networks formed of interdependent modular networks

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**Abstract**

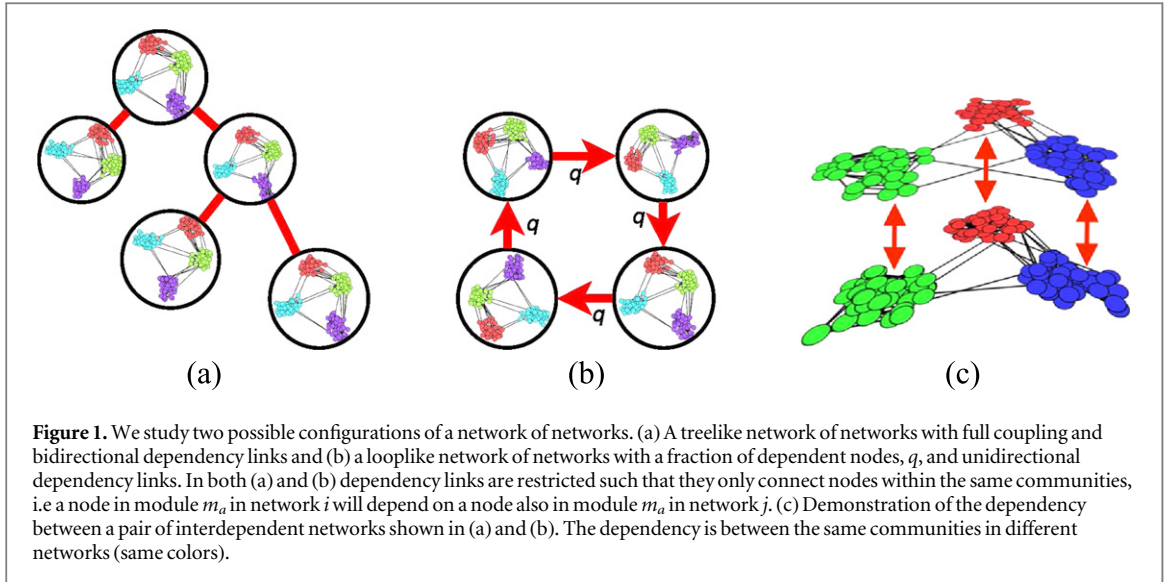
Many infrastructure networks have a modular structure and are also interdependent with other infrastructures. While significant research has explored the resilience of interdependent networks, there has been no analysis of the effects of modularity. Here we develop a theoretical framework for attacks on interdependent modular networks and support our results through simulations. We focus, for simplicity, on the case where each network has the same number of communities and the dependency links are restricted to be between pairs of communities of different networks. This is particularly realistic for modeling infrastructure across cities. Each city has its own infrastructures and different infrastructures are dependent only within the city. However, each infrastructure is connected within and between cities. For example, a power grid will connect many cities as will a communication network, yet a power station and communication tower that are interdependent will likely be in the same city. It has previously been shown that single networks are very susceptible to the failure of the interconnected nodes (between communities) (Shai *et al* 2014 arXiv:1404.4748) and that attacks on these nodes are even more crippling than attacks based on betweenness (da Cunha *et al* 2015 arXiv:1502.00353). In our example of cities these nodes have long range links which are more likely to fail. For both treelike and looplike interdependent modular networks we find distinct regimes depending on the number of modules, m . (i) In the case where there are fewer modules with strong intraconnections, the system first separates into modules in an abrupt first-order transition and then each module undergoes a second percolation transition. (ii) When there are more modules with many interconnections between them, the system undergoes a single transition. Overall, we find that modular structure can significantly influence the type of transitions observed in interdependent networks and should be considered in attempts to make interdependent networks more resilient.

1. Introduction

Many recent studies have explored interdependent and multilayer networks [3–34]. Further studies have also explored more realistic structures such as spatially embedded networks [35–40], and different types of realistic failures such as response under degree-based attacks [41, 42] and localized attacks [26, 39]. Nevertheless, all previous studies on interdependent networks ignored the realistic effect of modularity on the resilience of interdependent networks.

Many real world networks have a modular structure including biological networks [43], infrastructure such as the power grid, internet [44] and airport networks [45], and financial networks [46]. Several studies have explored the robustness of individual modular systems (i.e. single networks) [47, 48] yet no study has considered the effect of interdependence in modular networks. It is well known that many of these systems are also interdependent and thus it is crucial to understand how the modular structure affects the resilience of interdependent networks.

There has also been considerable work on understanding various types of attacks on networks [26, 39, 49–51]. Recent work by Shai *et al* [1] developed an analytical method where the attack is carried out on



interconnected nodes, i.e. nodes that connect communities. Further work by da Cunha *et al* [2] showed that in real networks, attacks on interconnected nodes are even more damaging than attacks based on betweenness. It is particularly important to consider the effectiveness of this type of attack in interdependent networks since the researchers in [2] showed that the US power grid is among the most susceptible networks to this type of attack and it is well known that power grids are interdependent with other infrastructures.

In our model we assume modular networks composed of m modules, and a fixed ratio, α , between the probability for an intramodule link and intermodule link. We further fix the total average degree, k_{tot} , of the network. Using these three parameters we can determine the average intramodule degree, k_{intra} and the average intermodule degree k_{inter} . We obtain

$$\frac{k_{\text{intra}}}{k_{\text{inter}}} = \frac{\alpha}{m - 1}, \quad (1)$$

$$k_{\text{tot}} = k_{\text{intra}} + k_{\text{inter}}. \quad (2)$$

Note that k_{inter} increases with m , since networks with more modules have more interlinks [1]. We generate n of these modular networks, and for simplicity we assume that each network has the same m , α and k . We then create dependency links between the nodes in the different networks. The fraction of nodes in network i which depend on nodes in network j is defined as q_{ij} . The dependency links are either bidirectional (in our analysis of treelike dependencies, section 3, figure 1(a)) or unidirectional (in our analysis of loopleftike dependencies, section 4, figure 1(b)). In both cases we further restrict the dependency links such that a node in community m_a in network i will depend on a node in community m_a in network j , i.e. the dependency links are within the same community. This restriction is particularly reasonable in our example of cities where each city has its own tightly connected, interdependent infrastructure with relatively few connectivity links between the cities.

We focus on the case of attack on the interconnected nodes, i.e. nodes with at least one connectivity link to a different module. Many studies have shown that these links are particularly susceptible to failure in biological networks and serve as efficient targets in attacking infrastructure [2]. Further in our example of cities, these nodes have the longer distance links that are more likely to fail [52].

Our attack randomly removes a fraction $1 - r$, of the interconnected nodes in the network until there are no remaining interconnected nodes and then continues to remove nodes randomly. The fraction of unremoved interconnected nodes, r , is related to the overall fraction of unremoved nodes, p , by

$$r = \frac{p - \bar{p}_{\text{inter}}}{1 - \bar{p}_{\text{inter}}}, \quad (3)$$

where $\bar{p}_{\text{inter}}$ is the probability that a node is *not* interconnected.

We study this model on both treelike networks of networks (NoNs) and loopleftike NoNs (figure 1) and find that there are two distinct regimes depending on the number of modules, m . In the first regime, for small m , the network first separates into isolated, interdependent, yet functional modules, i.e. there is a transition at $r = 0$. However, for larger m the network collapses due to the removal of a finite nonzero fraction of interconnected nodes. We provide an analytic solution that predicts the critical point, m^* , where the system has a transition

from one behavior to the other, as well as solutions for the size of the giant component as a function of the fraction of removed nodes. We support our theory by simulations.

2. Failure and attack on interdependent modular networks

We now extend the method of Callaway *et al* [50] to the case of interdependent networks or network of networks (NoN). We will then specifically apply this framework to the case of interdependent modular networks.

We begin by recalling the derivation in [50] for a single network i with a given degree distribution $P_i(k)$ described by the generating function

$$G(x) = \sum_{k=0}^{\infty} P_i(k) x^k. \quad (4)$$

For degree based attacks where nodes with degree k are removed with probability $1 - r_k$, the generating function is [50]

$$F_0(x) = \sum_{k=0}^{\infty} r_k P_i(k) x^k. \quad (5)$$

The generating function of the branching process is then

$$F_1(x) = F'_0(x)/G'(x). \quad (6)$$

Following Callaway *et al* [50] we obtain the probability that a randomly chosen edge leads to a cluster of given size, $H_1(x)$, as

$$H_1(x) = 1 - F_1(x) + xF_1[H_1(x)] \quad (7)$$

and the probability that a randomly chosen node leads to a cluster of a given size, $H_0(x)$

$$H_0(x) = 1 - F_0(1) + xF_0[H_1(x)]. \quad (8)$$

The fraction of nodes in the giant component is

$$P_{\infty}(x) = 1 - H_0(1) = F_0(1) - F_0(u), \quad (9)$$

with u being the smallest non-negative real solution of the self consistency equation

$$u = 1 - F_1(1) + F_1(u). \quad (10)$$

In order to generalize this framework for the case of attack on a NoN we must include the fact that nodes have an additional uniform likelihood to fail. Since this likelihood to fail is random we can include it in the equations by defining a probability, p_{dep} , of surviving failure from the dependency links. We note that p_{dep} will depend on the specific topology of the NoN. Essentially, nodes now must survive the initial failure and failure from dependency links, which lead equation (5) to become $F_0(x) = \sum_{k=0}^{\infty} p_{\text{dep}} r_k P_i(k) x^k$. The overall likelihood of failure for each node is now $r_k p_{\text{dep}}$. However, since p_{dep} is constant with respect to the degree of the node, we can extract it from the sum to obtain

$$\tilde{F}_0(x) = p_{\text{dep}} \sum_{k=0}^{\infty} r_k P_i(k) x^k, \quad (11)$$

where $\tilde{F}_0(x)$ is the analogue of equation (5) for interdependent networks and p_{dep} can be extracted from the sum since it does not depend on the degree of the node. The rest of the derivation continues the same for equations (5)–(9).

We note that equation (10) is the same as before only now with the extra factor of p_{dep} , giving

$$1 = p_{\text{dep}} \frac{F_1(u) - F_1(1)}{u - 1} \quad (12)$$

and equation (9) also will have the same factor p_{dep} . This pair of equations, equations (9) and (12), can be used to find the giant component of a NoN under any sort of degree based attack and can be used as an alternative method to that of Huang *et al* [42] and Dong *et al* [41].

We will now generalize this framework to the case where each network in the NoN is a modular network with the parameters m , α , and k , defined above. For the case of a modular network where the modules are made up of Erdős–Rényi structures, equation (3) becomes

$$r = \frac{p - e^{-k_{\text{inter}}}}{1 - e^{-k_{\text{inter}}}}. \quad (13)$$

We recall the results from Shai *et al* [1] for the generating functions

$$G(x) = e^{-(k_{\text{intra}}+k_{\text{inter}})(x-1)}, \quad (14)$$

$$F_0(x) = e^{k_{\text{intra}}(x-1)-k_{\text{inter}}}(1-r) + rG(x), \quad (15)$$

$$F_1(x) = F_0'(x)/G'(x). \quad (16)$$

We now combine the results from Shai *et al* [1] for a single network with equations (9) and (12) for a NoN. We note that these equations not only describe the case of an interdependent modular NoN but also the case where we have both random failure with probability $1 - p_{\text{rand}}$ and targeted attack. For our purposes we will assume that $p_{\text{rand}} = p_{\text{dep}}$, i.e. the random damage is solely the result of the dependencies. Our new generating functions are thus

$$\begin{aligned} \tilde{F}_0(x) &= p_{\text{dep}} F_0(x), \\ \tilde{F}_1(x) &= p_{\text{dep}} F_1(x). \end{aligned} \quad (17)$$

The average connected component size is given by Leicht and D'Souza [6]

$$\langle s \rangle = p_{\text{dep}} F_0(1) + p_{\text{dep}} r k_{\text{intra}} F_0(1) j_0 + p_{\text{dep}} r k_{\text{inter}} j_1, \quad (18)$$

where

$$j_0 = p_{\text{dep}} F_0(1) + p_{\text{dep}} r k_{\text{intra}} F_0(1) j_0 + p_{\text{dep}} r k_{\text{inter}} j_1, \quad (19)$$

$$j_1 = p_{\text{dep}} r + p_{\text{dep}} r k_{\text{intra}} j_0 + p_{\text{dep}} r k_{\text{inter}} j_1. \quad (20)$$

Combining the above equations, we find that there is a giant component that spans the system when

$$\begin{aligned} & \left(1 - p_{\text{rand}} k_{\text{intra}} \left(e^{-k_{\text{inter}}} + r_c - r_c e^{-k_{\text{inter}}}\right)\right) \left(1 - p_{\text{rand}} r_c k_{\text{inter}}\right) - p_{\text{rand}}^2 r_c^2 k_{\text{intra}} k_{\text{inter}} = 0 \\ & \Rightarrow r_c^2 \left(p_{\text{rand}}^2 k_{\text{intra}} k_{\text{inter}} e^{-k_{\text{inter}}} + r_c \left(p_{\text{rand}} k_{\text{inter}} + p_{\text{rand}} k_{\text{intra}} - p_{\text{rand}} k_{\text{intra}} e^{-k_{\text{inter}}} - p_{\text{rand}}^2 k_{\text{intra}} k_{\text{inter}} e^{-k_{\text{inter}}}\right)\right) \\ & + \left(p_{\text{rand}} k_{\text{intra}} e^{-k_{\text{inter}}} - 1\right) = 0 \end{aligned} \quad (21)$$

which can be solved using the quadratic formula for r_c .

3. Treelike network of interdependent modular networks

We now consider the case of a treelike NoN (see figure 1(a)) with full dependency and no-feedback, i.e. if network i depends on network j then each node in i depends on a single node in j and vice versa. As stated earlier, this can serve as a good model of infrastructure across cities. We remove a fraction $1 - r$ of the interconnected nodes from one of the networks and aim to obtain the mutual giant connected component. The fraction of nodes which fail due to the effect of the dependencies is given by

$$p_{\text{dep}} = \left(1 - e^{-(k_{\text{intra}}+k_{\text{inter}})P_{\infty}}\right)^{n-1}. \quad (22)$$

This can be understood by noting that each node is a part of a n -tuple of interdependent nodes and we require that all these n nodes be in their networks respective giant components. Therefore, besides the node itself, there are $n - 1$ dependent nodes that must all be in their networks' respective giant components after the initial attack.

If we combine equations (11), (12), and (22) we obtain the mutual giant connected component in the NoN,

$$P_{\infty} = \begin{cases} \left(e^{-k_{\text{inter}}}(1-r) \left(1 - e^{-k_{\text{intra}}P_{\infty}}\right) + r \left(1 - e^{-(k_{\text{intra}}+k_{\text{inter}})P_{\infty}}\right) \right) \left(1 - e^{-(k_{\text{intra}}+k_{\text{inter}})P_{\infty}}\right)^{n-1} & 0 < r < 1, \\ \frac{P}{m} \left(1 - e^{-mk_{\text{intra}}P_{\infty}}\right)^n & r < 0. \end{cases} \quad (23)$$

Since the dependency links are within modules, equation (23) for $r < 0$ is simply the equation for a treelike NoN with $k = k_{\text{intra}}$ and a giant component that is smaller by a factor of m . Nonetheless the value of p_c will be the same as for a regular treelike NoN with $k = k_{\text{intra}}$. We show analytic solutions of these equations along with simulations in figure 2 for varying m and varying n . By varying either of these parameters we may have either one or two abrupt percolation transitions. Note that for the case of varying n the point of the first abrupt drop is the same for all curves. This is because the point where the modules separate depends only on m , α and k_{tot} , but not on n (see figure 3). It is worth noting though that for larger n , the second regime, where modules exist

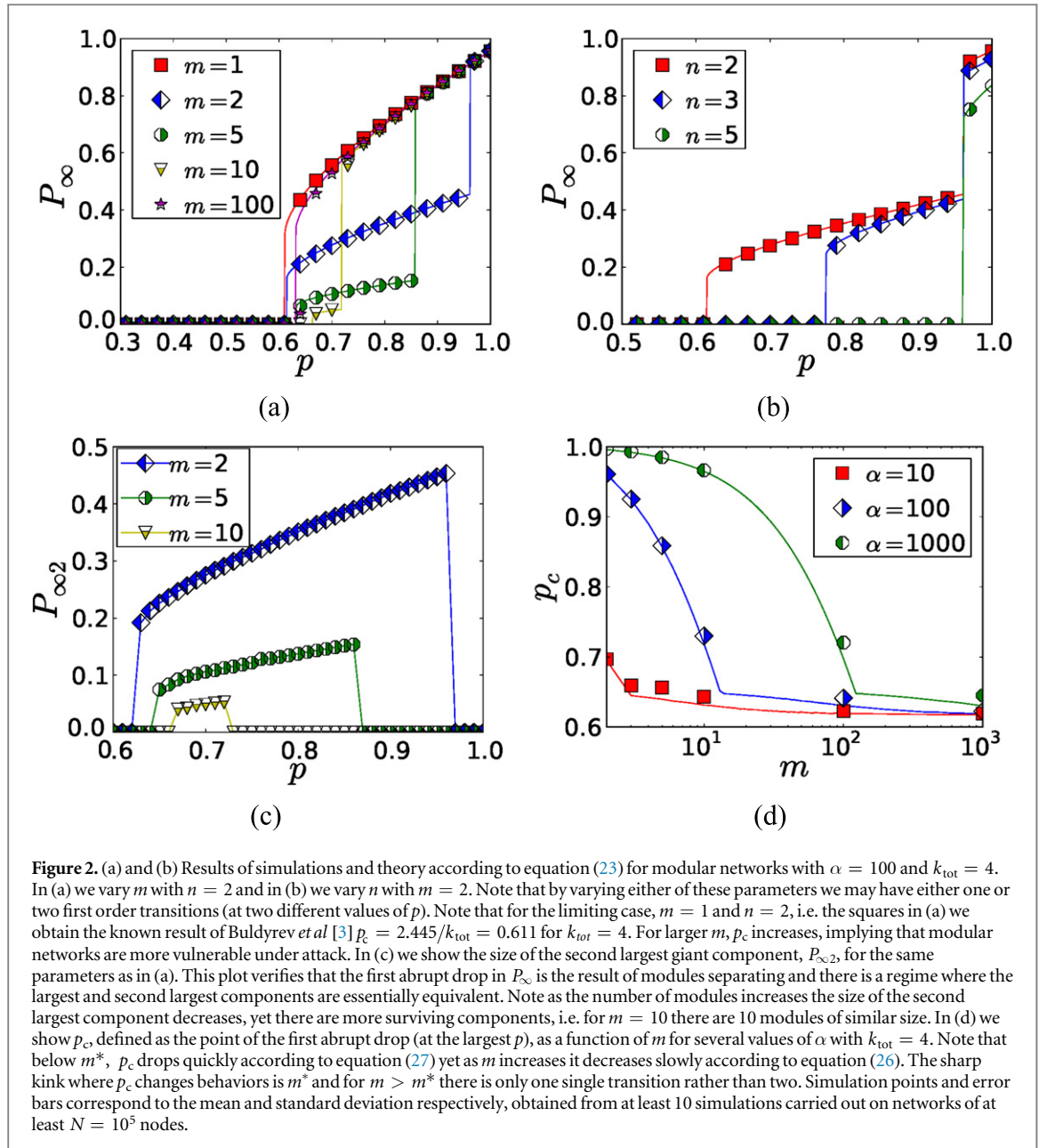


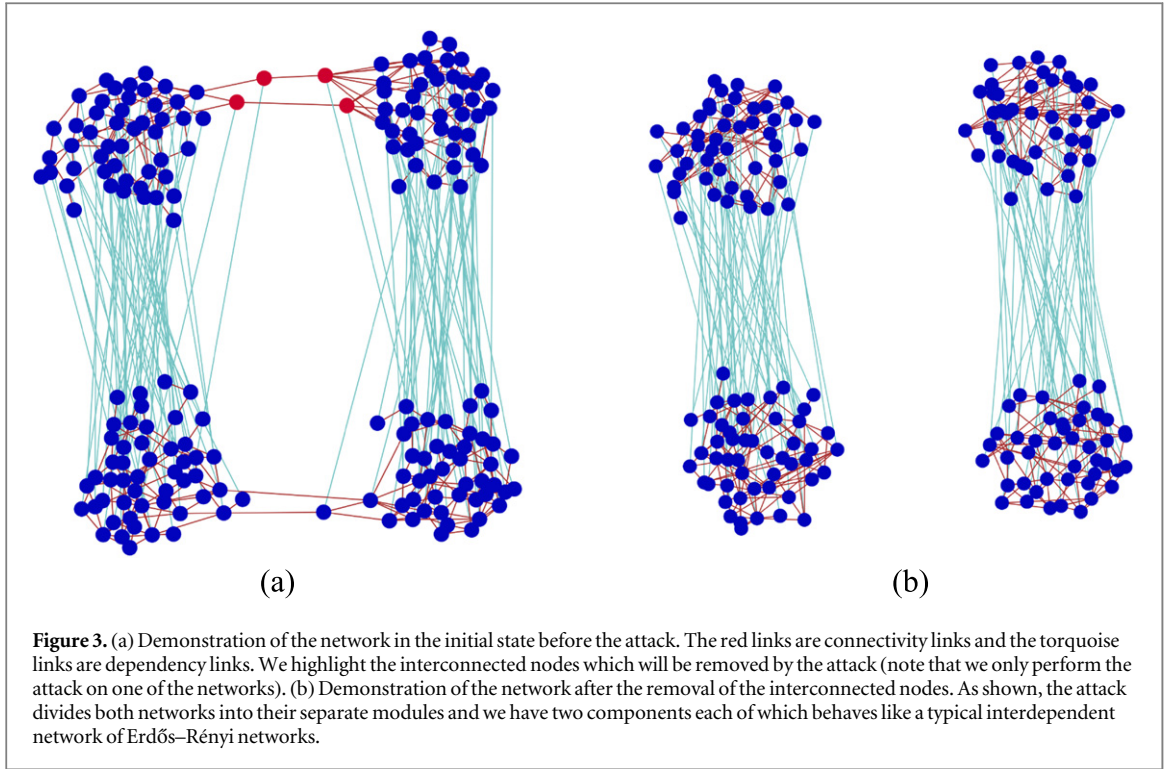
Figure 2. (a) and (b) Results of simulations and theory according to equation (23) for modular networks with $\alpha = 100$ and $k_{\text{tot}} = 4$. In (a) we vary m with $n = 2$ and in (b) we vary n with $m = 2$. Note that by varying either of these parameters we may have either one or two first order transitions (at two different values of p). Note that for the limiting case, $m = 1$ and $n = 2$, i.e. the squares in (a) we obtain the known result of Buldyrev *et al* [3] $p_c = 2.445/k_{\text{tot}} = 0.611$ for $k_{\text{tot}} = 4$. For larger m , p_c increases, implying that modular networks are more vulnerable under attack. In (c) we show the size of the second largest giant component, $P_{\infty 2}$, for the same parameters as in (a). This plot verifies that the first abrupt drop in P_∞ is the result of modules separating and there is a regime where the largest and second largest components are essentially equivalent. Note as the number of modules increases the size of the second largest component decreases, yet there are more surviving components, i.e. for $m = 10$ there are 10 modules of similar size. In (d) we show p_c , defined as the point of the first abrupt drop (at the largest p), as a function of m for several values of α with $k_{\text{tot}} = 4$. Note that below m^* , p_c drops quickly according to equation (27) yet as m increases it decreases slowly according to equation (26). The sharp kink where p_c changes behaviors is m^* and for $m > m^*$ there is only one single transition rather than two. Simulation points and error bars correspond to the mean and standard deviation respectively, obtained from at least 10 simulations carried out on networks of at least $N = 10^5$ nodes.

independently (see figure 3(b)), may not be present if the individual modules do not have enough intralinks to survive the damage from the dependencies.

We demonstrate that in the case where there are two abrupt transitions, the first transition (at higher p) represents the separation of the modules and we therefore have m individual modules functioning separately. This is indicated by the spike in the size of the second largest component, $P_{\infty 2}$, in figure 2(c) and is demonstrated visually in figure 3. In general for a network composed of m modules, there are m equal components functioning independently. After the separation into modules, we observe a second abrupt transition due to the interdependence.

As m increases, the network begins to act more like a regular network of n interdependent Erdős–Rényi networks. It is clear from the framework developed until now that there exists a critical value of m , m^* above which the system undergoes a single transition whereas below m^* the system first separates into separate modules before collapsing entirely.

We will now determine p_c and m^* for this system using the framework developed in chapter II. For certain regimes there exist two first-order, discontinuous jumps in the size of the giant component (see figure 2(a)). We define p_c as the point where the first (higher p) jump occurs, regardless of whether there is a second jump afterwards. If desired, the point of the second jump can be found by solving for the critical threshold for a typical Erdős–Rényi NoN with $k = k_{\text{intra}}$. We will now solve for p_c by first finding r_c using equation (21) and then



converting it to p_c using equation (13). We recall that for the case of trees we use $p_{\text{dep}} = (1 - e^{-(k_{\text{intra}} + k_{\text{inter}})P_{\infty}})^{n-1}$ in equation (21) as was done in equation (23). This gives us the following system of equations

$$p_{\text{dep}} = \left(1 - e^{-(k_{\text{intra}} + k_{\text{inter}})P_{\infty}}\right)^{n-1}, \quad (24)$$

$$P_{\infty} = \left(e^{-k_{\text{inter}}(1 - r_c)}(1 - e^{-k_{\text{intra}}P_{\infty}}) + r_c(1 - e^{-(k_{\text{intra}} + k_{\text{inter}})P_{\infty}})\right)\left(1 - e^{-(k_{\text{intra}} + k_{\text{inter}})P_{\infty}}\right)^{n-1}, \quad (25)$$

$$r_c = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (26)$$

with the values of a , b and c being the coefficients from equation (21).

We see in figure 2(d) that the simulations fit the theory well except near the point where we switch to having a single abrupt transition. This small discrepancy is because there are large fluctuations in the fraction of individual modules that are part of the giant component, which are not considered in the mean-field approach. Once we move away from this critical point the theory again fits the simulations.

In order to find m^* we compare the value of p when $r = 0$ to the p_c for a treelike network of networks with $k = k_{\text{intra}}$. Essentially this means comparing the fraction of nodes needed to remove all interconnected nodes and the value of p_c where the modules themselves break apart.

We can trivially find the value of p when $r = 0$ by noting that $r = \frac{p - e^{-k_{\text{inter}}}}{1 - e^{-k_{\text{inter}}}}$. We obtain in terms of m

$$p(r = 0) = e^{-k_{\text{tot}} \frac{m-1}{\alpha+m-1}}. \quad (27)$$

We note that the value of p_c for a treelike network of networks (without communities) was determined by Gao *et al* [10]. In order to find p_c we first find the value of $\bar{k}_{\text{min}}$, the degree at which the system undergoes a transition. This is done using the system of equations

$$f_c = -\left[nW\left(-\frac{1}{n}e^{-1/n}\right)\right]^{-1}, \quad (28)$$

$$\bar{k}_{\text{min}} = \left[nf_c(1 - f_c)^{(n-1)}\right]^{-1}, \quad (29)$$

where $W(x)$ is the Lambert function.

Then $p_c = \bar{k}_{\text{min}} / \langle k \rangle$, where $\langle k \rangle$ is the average degree of each network, in our case $\langle k \rangle = k_{\text{intra}}$. We obtain

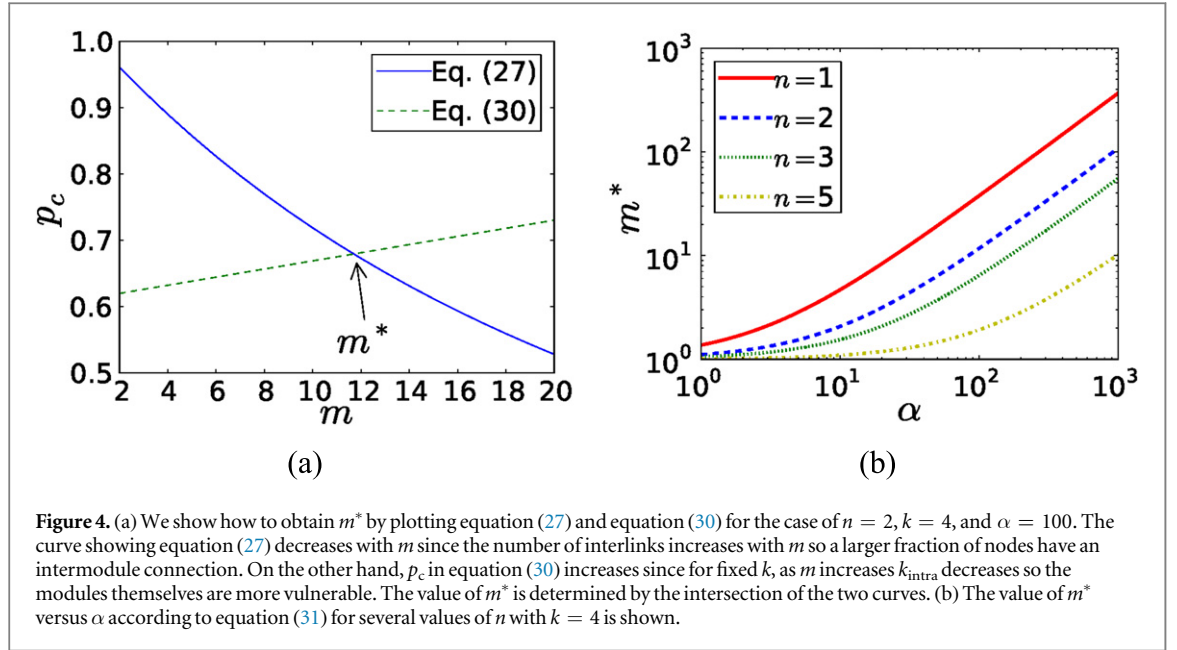


Figure 4. (a) We show how to obtain m^* by plotting equation (27) and equation (30) for the case of $n = 2$, $k = 4$, and $\alpha = 100$. The curve showing equation (27) decreases with m since the number of interlinks increases with m so a larger fraction of nodes have an intermodule connection. On the other hand, p_c in equation (30) increases since for fixed k , as m increases k_{intra} decreases so the modules themselves are more vulnerable. The value of m^* is determined by the intersection of the two curves. (b) The value of m^* versus α according to equation (31) for several values of n with $k = 4$ is shown.

$$p_c = \frac{\bar{k}_{\min}(\alpha + m - 1)}{k_{\text{tot}} \alpha}. \quad (30)$$

The value of m^* is found by setting equation (27) equal to equation (30) to obtain

$$\frac{\bar{k}_{\min}(\alpha + m^* - 1)}{k_{\text{tot}} \alpha} = e^{-k_{\text{tot}} \frac{m^* - 1}{\alpha + m^* - 1}}. \quad (31)$$

This equation can be solved numerically for any value of n and α . Essentially, there are two competing effects in this equation. The left side represents the value of p_c for an interdependent network of networks with $k = k_{\text{intra}}$ and the right side represents the point where we have removed all nodes with an interlink. As m increases, k_{intra} decreases, meaning that the modules become more vulnerable and p_c from the left side of the equation increases. On the other hand, as m increases, k_{inter} increases so more nodes have an interlink and p_c on the right side of the equation decreases (more resilient). We demonstrate these two effects and show the point where they intersect in figure 4(a). Further we show the value of m^* for various values of α and n in figure 4(b). Also, as n increases, $\bar{k}_{\min}$ increases, meaning the affect of the interdependence is stronger. Essentially it causes the dashed curve in figure 4(a) to increase, thus decreasing m^* , the point where the curves intersect. Thus for larger n we have a lower m^* , as seen in figure 4(b).

4. Loop of interdependent modular networks

We will now demonstrate our approach on another example where we consider the specific case of a directed loop of modular networks. In this case, for simplicity, we perform the initial attack on all of the networks rather than just one of them. Also, whereas the dependency links in chapter III. were bidirectional here we use unidirectional links as part of a loop (see figure 1(b)). The fraction of nodes which are interdependent is defined as q , i.e. $1 - q$ is the fraction of autonomous nodes, and is assumed here, for simplicity, to be the same for all pairs of networks. Here again, we will only consider dependency links which are restricted to being between two networks but within the same module. It has been shown that for looplike NoNs the number of networks in the loop is not relevant to the calculation [4, 38, 53].

The equation governing this system is once again equation (12) with $p_{\text{dep}} = (1 - q + qP_{\infty})$. Substituting this into equation (12) gives

$$P_{\infty} = \begin{cases} \left(e^{-k_{\text{inter}}(1-r)}(1 - e^{-k_{\text{intra}}P_{\infty}}) + r(1 - e^{-(k_{\text{intra}}+k_{\text{inter}})P_{\infty}}) \right) (1 - q + qP_{\infty}) & 0 < r < 1, \\ \frac{P}{m}(1 - e^{-k_{\text{intra}}mP_{\infty}})(1 - q + qmP_{\infty}) & r < 0. \end{cases} \quad (32)$$

We compare simulations and theory according to equation (32) in figure 5. We note that here if the system undergoes two transitions, the first one (at a higher p) is abrupt and the second one is either continuous or abrupt. For the parameters we used, looplike networks of networks undergo a second order transition [10, 53], yet for different values of k_{tot} and q they can undergo an abrupt transition. For looplike NoNs, the number of

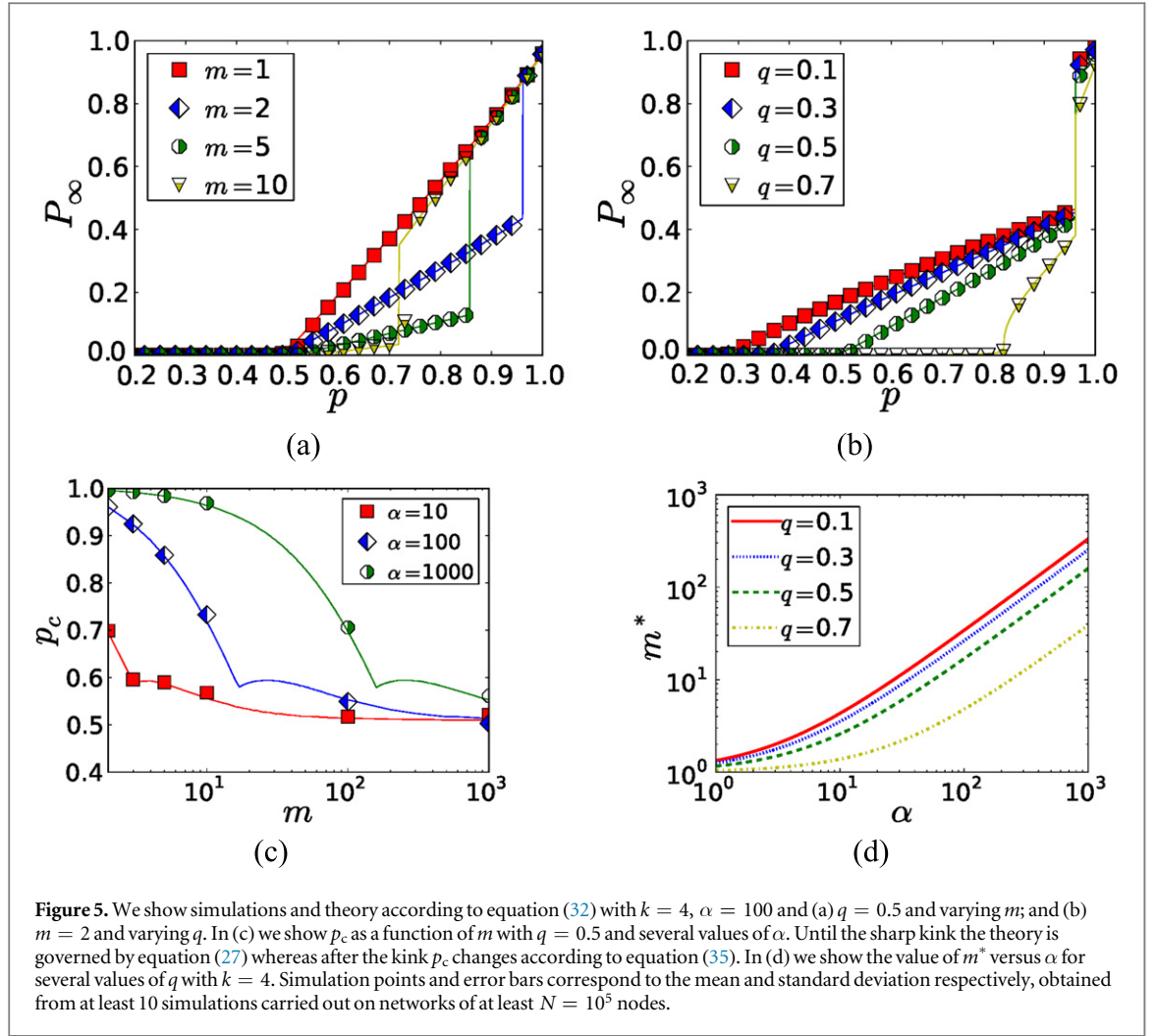


Figure 5. We show simulations and theory according to equation (32) with $k = 4$, $\alpha = 100$ and (a) $q = 0.5$ and varying m ; and (b) $m = 2$ and varying q . In (c) we show p_c as a function of m with $q = 0.5$ and several values of α . Until the sharp kink the theory is governed by equation (27) whereas after the kink p_c changes according to equation (35). In (d) we show the value of m^* versus α for several values of q with $k = 4$. Simulation points and error bars correspond to the mean and standard deviation respectively, obtained from at least 10 simulations carried out on networks of at least $N = 10^5$ nodes.

networks, n , does not play a role but increasing q the fraction of interdependent nodes, weakens the resilience of the system (see figure 5(b)). We further note that for looplike NoNs there is a maximum coupling, $q_{\max}$ above which the entire network collapses even for $p \rightarrow 1$. Our value for $q_{\max}$ will be the same as that of Gao *et al* [53], namely $q_{\max} = 1 - 1/k_{\text{tot}}$ since for $p = 1$ our networks respond to random percolation like ER networks. Indeed, in figure 5(b) we do not show higher values of q since the system does not survive even for $p = 1$ if $q > 1 - 1/k_{\text{tot}}$. For example, with $k_{\text{tot}} = 4$ (the value in our plots) we obtain a maximum coupling of $q = 0.75$, above which the system collapses even for $p \rightarrow 1$.

Following the analysis in chapter II we can find p_c using equation (21). In this case $p_{\text{dep}} = (1 - q + qP_\infty)$, which we will substitute into equation (21). Our system corresponding to equation (26) becomes

$$p_{\text{dep}} = (1 - q + qP_\infty), \quad (33)$$

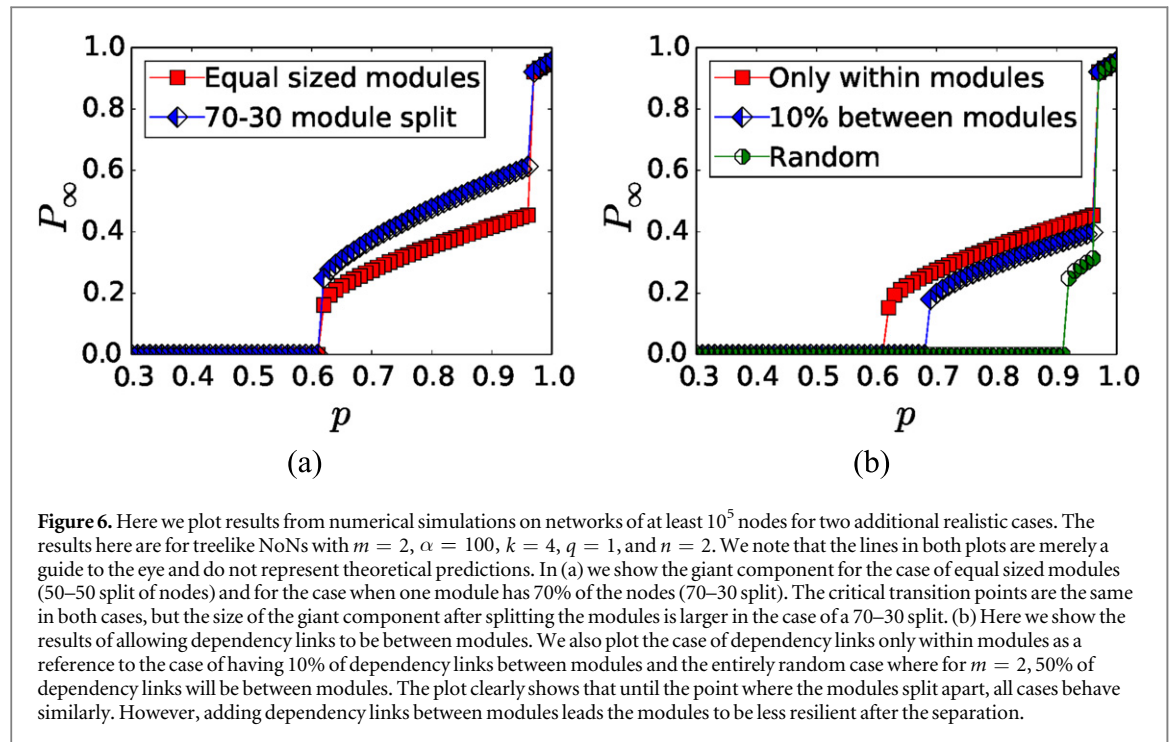
$$P_\infty = \left(e^{-k_{\text{inter}}(1-r)} (1 - e^{-k_{\text{intra}}P_\infty}) + r (1 - e^{-(k_{\text{intra}}+k_{\text{inter}})P_\infty}) \right), \quad (34)$$

$$r_c = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (35)$$

with the values of a , b and c being the coefficients from equation (21). We plot the p_c obtained based on the theory and compare with simulations in figure 5(c).

To find m^* we use the same method as was done in chapter III for treelike networks of networks. Specifically we compare the p_c at which the modules become separated and the p_c for a network of networks with $k = k_{\text{intra}}$. We have from Gao *et al* [53] that p_c for a looplike NoN is $p_c = \frac{1}{k(1-q)}$. If we solve this and use the representation with α and m rather than k_{intra} , we obtain

$$\frac{\alpha + m^* - 1}{k_{\text{tot}}\alpha(1-q)} = e^{-\frac{k_{\text{tot}}(m^*-1)}{\alpha+m-1}}. \quad (36)$$



The values of m^* , as a function of α , based on equation (36) can be seen in figure 5(d). Increasing q decreases the robustness of the individual modules and thus decreases the value of m^* since the system is more likely to collapse before the modules are separated.

5. More realistic features of interdependent modular networks

The previous cases focused on a simple model of equal sized modules with dependency links being restricted to be within the same module. Here we will consider more realistic variations of this model and discuss what effect they have on our results.

First, it is noteworthy that real communities, e.g. cities, are not all of equal size. In fact the distribution of city sizes is actually rather heavy-tailed meaning that some cities are significantly larger than others [54]. Nonetheless, we assume that the *level of connectivity* within the various cities is likely similar. In other words, power stations in different cities tend to have approximately the same number of connections to other power stations in the same city and to power stations in other cities. So long as k_{inter} and k_{intra} remain the same, our results for r_c and p_c are robust to varying sizes of modules. This is essentially due to the fact that once the system size becomes large enough its behavior approaches the infinite limit (with non-equal sized modules the relative size of each module changes, yet all modules have the same critical properties). The sole difference in the case of non-equal sized modules occurs in the case of $m < m^*$ and $r < 0$, in other words after the modules have separated. In this case P_∞ will be slightly larger since one of the surviving modules will contain a larger fraction of the remaining nodes. For example, for the case shown in figure 6(b) one module has 70% of the nodes, thus when the modules split apart, the giant component is larger since one of the remaining modules is larger. Nonetheless the value of p_c is the same as for equal-sized modules.

A second question is what happens if dependency links are not strictly required to be within the same modules. Presumably there could be some overlap of dependency links between different cities, even if these links are relatively scarce. We note that for the case $0 < r < 1$, the system has identical behavior because the damage is the same in each module regardless of whether its source is from dependent nodes in the same module or other modules. Thus, it can be modeled using the corresponding parts of equations (23) and (32).

Now, due to dependencies between different modules the modules themselves become less robust as seen in figure 6(a). When the modules split apart in one layer, the damage from the split in one layer encompasses parts of different modules in the other layers (as opposed to figure 3). Thus an individual module ends up being part of different components, reducing the number of intraconnections between the nodes in the module.

One can see this most clearly in the limiting case of totally random dependency links (i.e. no bias towards dependency links being within the same module) where while one network will separate according to its modules, the components in the other networks will include parts of several modules and thus will have a lower overall level of connectivity. Essentially the individual module becomes split up with parts of it in different giant components corresponding to the separation of the modules in another network. In the case of random dependency links for treelike networks (the circles in figure 6(a)) the equations governing the system are

$$P_{\infty} = \begin{cases} \left(e^{-k_{\text{inter}}(1-r)}(1 - e^{-k_{\text{intra}}P_{\infty}}) + r(1 - e^{-(k_{\text{intra}}+k_{\text{inter}})P_{\infty}}) \right) (1 - e^{-(k_{\text{intra}}+k_{\text{inter}})P_{\infty}})^{n-1} & 0 < r < 1, \\ \frac{p}{m}(1 - e^{-mk_{\text{intra}}P_{\infty}})(1 - e^{-k_{\text{intra}}P_{\infty}})^{n-1} & r < 0. \end{cases} \quad (37)$$

It is also worth noting that the decreased robustness of the modules themselves decreases m^* since when the network breaks apart it is now less likely that individual modules have enough remaining connectivity to continue functioning.

6. Discussion

In summary, we have developed a framework for studying attacks on interdependent modular networks. Our results show that modular NoNs can behave significantly differently from random NoNs or spatially embedded NoNs in that they may undergo two separate percolation phase transitions. One transition occurs (at a higher p) where the modules become separated, and a second transition occurs when the individual modules collapse. For the case of a fully interdependent treelike NoN with bidirectional dependency links, both of these transitions are first order, whereas for a looplike NoN with unidirectional dependency links the first transition (higher p) is abrupt while the second is either abrupt or continuous depending on the parameters. These results might be relevant for many interdependent systems such as financial networks, biological networks and are particularly relevant for models of city infrastructure where most of the interdependence presumably occurs within a single city (regarded here as a community) even though there are connections to other cities. Another reason our attack is realistic for city infrastructure is because the interconnected nodes contain the interconnected links that are longer and therefore more likely to fail [52].

We also note that the theoretical approach developed here can be used for many other types of targeted attack and for other network of network structures as well.

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