



OPEN ACCESS

What is the relativistic spin operator?

To cite this article: Heiko Bauke *et al* 2014 *New J. Phys.* **16** 043012

View the [article online](#) for updates and enhancements.

You may also like

- [The Relativistic Spin Precession in the Compact Double Neutron Star System PSR J1946+2052](#)
Lingqi Meng, Weiwei Zhu, Michael Kramer et al.
- [Spin precession of Dirac particles in Kerr geometry](#)
Anusar Farooqui
- [Bundle theoretic descriptions of massive single-particle state spaces: with a view toward relativistic quantum information theory](#)
Heon Lee

What is the relativistic spin operator?

Heiko Bauke¹, Sven Ahrens¹, Christoph H Keitel¹ and Rainer Grobe^{1,2}

¹Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany

²Intense Laser Physics Theory Unit and Department of Physics, Illinois State University, Normal, IL 61790-4560, USA

E-mail: heiko.bauke@mpi-hd.mpg.de

Received 6 February 2014

Accepted for publication 3 March 2014

Published 11 April 2014

New Journal of Physics **16** (2014) 043012

doi:[10.1088/1367-2630/16/4/043012](https://doi.org/10.1088/1367-2630/16/4/043012)

Abstract

Although the spin is regarded as a fundamental property of the electron, there is no universally accepted spin operator within the framework of relativistic quantum mechanics. We investigate the properties of different proposals for a relativistic spin operator. It is shown that most candidates are lacking essential features of proper angular momentum operators, leading to spurious zitterbewegung (quivering motion) or violation of the angular momentum algebra. Only the Foldy–Wouthuysen operator and the Pryce operator qualify as proper relativistic spin operators. We demonstrate that ground states of highly charged hydrogen-like ions can be utilized to identify a legitimate relativistic spin operator experimentally.

Keywords: spin, relativistic quantum mechanics, hydrogen-like ions

1. Introduction

Quantum mechanics forms the universally accepted theory for the description of physical processes on the atomic scale. It has been validated by countless experiments and is used in many technical applications. However, even today quantum mechanics presents physicists with some conceptual difficulties. In particular, the concept of spin is related to such difficulties and myths [1, 2]. Although there is consensus that elementary particles have a quantum



Content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](https://creativecommons.org/licenses/by/3.0/). Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

mechanical property called spin, the understanding of the physical nature of the spin is still incomplete [3].

Historically, the concept of spin was introduced in order to explain some experimental findings such as the emission spectra of alkali metals and the Stern–Gerlach experiment. A direct measure of the spin (or more precisely the electron’s magnetic moment) was, however, missing until the pioneering work by Dehmelt [4]. Nevertheless, spin measurement experiments [5–10] still require sophisticated methods. Pauli and Bohr even claimed that the spin of free electrons was impossible to measure for fundamental reasons [11]. Recent renewed interest in the fundamental aspects of the spin arose, for example, from the growing field of (relativistic) quantum information [12–17], quantum spintronics [18], spin effects in graphene [19–21] and in light-matter interaction at relativistic intensities [22–24].

According to the formalism of quantum mechanics, each measurable quantity is represented by a Hermitian operator. Taking the experiments that aim to measure bare electron spins seriously, we have to ask the question: what is the correct (relativistic) spin operator? Although the spin is regarded as a fundamental property of the electron, a universally accepted spin operator for the Dirac theory is still missing. The pivotal question we try to tackle is: which mathematical operator corresponds to an experimental spin measurement? This question may be answered by comparing the experimental results with the theoretical predictions originating from different spin operators, and testing which operator is compatible with the experimental data.

A relativistic spin operator may be introduced by splitting the undisputed total angular momentum operator $\hat{\mathbf{J}}$ into an external part $\hat{\mathbf{L}}$ and an internal part $\hat{\mathbf{S}}$, commonly referred to as the orbital angular momentum and the spin, viz. $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$. The question for the right splitting of the total angular momentum into an orbital part and a spin part is closely related to the quest for the right relativistic position operator [25–27]. This becomes evident by writing $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$ with the position operator $\hat{\mathbf{r}}$ and the kinematic momentum operator $\hat{\mathbf{p}}$, which is in the atomic units as used in this paper, $\hat{\mathbf{p}} = -i\nabla$. Thus, different definitions of the spin operator $\hat{\mathbf{S}}$ induce different relativistic position operators, $\hat{\mathbf{r}}$.

Introducing the position vector $\mathbf{r}$ and the operator $\hat{\mathbf{\Sigma}} = (\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3)^T$ via

$$\hat{\Sigma}_i = -i\alpha_j\alpha_k \quad (1)$$

with (i, j, k) being a cyclic permutation of $(1, 2, 3)$ and the matrices $(\alpha_1, \alpha_2, \alpha_3)^T = \boldsymbol{\alpha}$ obeying the algebra

$$\alpha_i^2 = 1, \quad \alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{i,k}, \quad (2)$$

the operator of the relativistic total angular momentum is given by $\hat{\mathbf{J}} = \mathbf{r} \times \hat{\mathbf{p}} + \hat{\mathbf{\Sigma}}/2$. Thus, the most obvious way of splitting $\hat{\mathbf{J}}$ is to define the orbital angular momentum operator $\hat{\mathbf{L}}_p = \mathbf{r} \times \hat{\mathbf{p}}$ and the spin operator $\hat{\mathbf{S}}_p = \hat{\mathbf{\Sigma}}/2$, which is a direct generalization of the orbital angular momentum operator and the spin operator of the nonrelativistic Pauli theory. This naive splitting, however, suffers from several problems, e.g. $\hat{\mathbf{L}}_p$ and $\hat{\mathbf{S}}_p$ do not commute with the free Dirac Hamiltonian nor with the Dirac Hamiltonian for central potentials. Thus, in contrast to classical and nonrelativistic quantum theory, the angular momenta $\hat{\mathbf{L}}_p$ and $\hat{\mathbf{S}}_p$ are not conserved. This has consequences, e.g. for the labeling of the eigenstates of the hydrogen atom. In

nonrelativistic theory, bound hydrogen states may be constructed as simultaneous eigenstates of the Pauli–Coulomb Hamiltonian, the squared orbital angular momentum, the z -components of the orbital angular momentum and the spin. In the Dirac theory, however, the squared total angular momentum $\hat{\mathbf{J}}^2$, the total angular momentum in the z -direction $\hat{J}_3$, and the so-called spin-orbit operator $\hat{K}$ (or the parity) are utilized [28, 29]. In particular, it is not possible to construct simultaneous eigenstates of the Dirac–Coulomb Hamiltonian and some component of $\hat{S}_p$.

2. Relativistic spin operators

To overcome conceptual problems with the naive splitting of $\hat{\mathbf{J}}$ into $\hat{\mathbf{L}}_p$ and $\hat{\mathbf{S}}_p$, several alternatives for a relativistic spin operator have been proposed. However, there is no single commonly accepted relativistic spin operator, leading to the unsatisfactory situation that the relativistic spin operator is not unambiguously defined. We will investigate the properties of different popular definitions of the spin operator that result from different splittings of $\hat{\mathbf{J}}$ with the aim of finding means that allow us to identify the legitimate relativistic spin operator by experimental methods.

Table 1 summarizes various proposals for a relativistic spin operator $\hat{\mathbf{S}}$. These operators are often motivated by abstract group theoretical considerations rather than by experimental evidence. For example, Wigner showed in his seminal work [54–56] that the spin degree of freedom can be associated with irreducible representations of the sub-group of the inhomogeneous Lorentz group that leaves the four-momentum invariant. We will denote individual components of $\hat{\mathbf{S}}$ by $\hat{S}_i$ with index $i \in \{1, 2, 3\}$. The spin operators are defined in terms of the particle’s rest mass m_0 , the speed of light c , the matrix β such that

$$\beta^2 = 1, \quad \alpha_i \beta + \beta \alpha_i = 0, \quad (3)$$

the free particle Dirac Hamiltonian

$$\hat{H}_0 = c\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + m_0 c^2 \beta, \quad (4)$$

and the operator

$$\hat{p}_0 = \left(m_0^2 c^2 + \hat{\mathbf{p}}^2 \right)^{1/2}. \quad (5)$$

In the nonrelativistic limit, i.e. when the plane wave expansion of a wave packet has only components with momenta that are small compared to $m_0 c$, the expectation values for all operators in table 1 converge to the same value. Note that the nomenclature in table 1 is not universally adopted in the literature and other authors may utilize different operator names. Furthermore, the spin operators can be formulated by various different but algebraically equivalent expressions. For example, the so-called Gürsey–Ryder operator in [46, 47] is equivalent to the Chakrabarti operator of table 1.

One may conclude that an operator can not be considered as a relativistic spin operator if it does not inherit the key properties of the nonrelativistic Pauli spin operator. In particular, we demand the following features from a proper relativistic spin operator.

- (i) It is required to commute with the free Dirac Hamiltonian.

Table 1. Definitions and commutation properties of various relativistic spin operators.

Operator name	Definition	$[\hat{H}_0, \hat{S}] = 0?$	$[\hat{S}_i, \hat{S}_j] = i\varepsilon_{i,j,k}\hat{S}_k?$	Eigenvalues $= \pm 1/2?$
Pauli [30–32]	$\hat{S}_P = \frac{1}{2}\hat{\Sigma}$	no	yes	yes
Foldy– Wouthuysen [33–37]	$\hat{S}_{FW} = \frac{1}{2}\hat{\Sigma} + \frac{i\beta}{2\hat{p}_0}\hat{\mathbf{p}} \times \boldsymbol{\alpha} - \frac{\hat{\mathbf{p}} \times (\hat{\Sigma} \times \hat{\mathbf{p}})}{2\hat{p}_0(\hat{p}_0 + m_0c)}$	yes	yes	yes
Czachor [38]	$\hat{S}_{Cz} = \frac{m_0^2c^2}{2\hat{p}_0^2}\hat{\Sigma} + \frac{im_0c\beta}{2\hat{p}_0^2}\hat{\mathbf{p}} \times \boldsymbol{\alpha} + \frac{\hat{\mathbf{p}} \cdot \hat{\Sigma}}{2\hat{p}_0^2}\hat{\mathbf{p}}$	yes	no	no
Frenkel [39–41]	$\hat{S}_F = \frac{1}{2}\hat{\Sigma} + \frac{i\beta}{2m_0c}\hat{\mathbf{p}} \times \boldsymbol{\alpha}$	yes	no	no
Chakrabarti [42–48]	$\hat{S}_{Ch} = \frac{1}{2}\hat{\Sigma} + \frac{i}{2m_0c}\boldsymbol{\alpha} \times \hat{\mathbf{p}} + \frac{\hat{\mathbf{p}} \times (\hat{\Sigma} \times \hat{\mathbf{p}})}{2m_0c(m_0c + \hat{p}_0)}$	no	yes	yes
Pryce [47, 49–52]	$\hat{S}_{Pr} = \frac{1}{2}\beta\hat{\Sigma} + \frac{1}{2}\hat{\Sigma} \cdot \hat{\mathbf{p}}(1 - \beta)\frac{\hat{\mathbf{p}}}{\hat{p}^2}$	yes	yes	yes
Fradkin–Good [46, 53]	$\hat{S}_{FG} = \frac{1}{2}\beta\hat{\Sigma} + \frac{1}{2}\hat{\Sigma} \cdot \hat{\mathbf{p}}\left(\frac{\hat{H}_0}{c\hat{p}_0} - \beta\right)\frac{\hat{\mathbf{p}}}{\hat{p}^2}$	yes	no	yes

- (ii) A spin operator must feature the two eigenvalues $\pm 1/2$ and it has to obey the angular momentum algebra

$$[\hat{S}_i, \hat{S}_j] = i\varepsilon_{i,j,k}\hat{S}_k \quad (6)$$

with $\varepsilon_{i,j,k}$ denoting the Levi–Civita symbol.

The first property is required to ensure that the relativistic spin operator is a constant of motion if forces are absent, such that spurious Zitterbewegung of the spin is prevented. The second requirement is commonly regarded as *the* fundamental property of angular momentum operators of spin-half particles [57]. The physical quantity that is represented by the operator $\hat{S}$ should not depend on the orientation of the chosen coordinate system. This can be ensured by fulfilling [57]

$$[\hat{J}_i, \hat{S}_j] = i\varepsilon_{i,j,k}\hat{S}_k. \quad (7)$$

The angular momentum algebra (6) and the relation (7) determine the properties of the spin and the orbital angular momentum as well as the relationship between them. As a consequence of (7), the orbital angular momentum $\hat{L} = \hat{J} - \hat{S}$ that is induced by a particular choice of the spin obeys $[\hat{J}_i, \hat{L}_j] = i\varepsilon_{i,j,k}\hat{L}_k$. Thus, $\hat{L}$ is a physical vector operator, too. As $\hat{L}$ represents an angular momentum operator, it must obey the angular momentum algebra. Furthermore, we may say that the total angular momentum $\hat{J}$ is split into an internal part $\hat{S}$ and an external part $\hat{L}$ only if internal and external angular momenta can be measured independently, i.e. $\hat{S}$ and $\hat{L}$ commute.

Both conditions are fulfilled if, and only if, the spin operator $\hat{\mathbf{S}}$ satisfies the angular momentum algebra (6) because the commutator relations

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{i,j,k}\hat{L}_k + [\hat{S}_i, \hat{S}_j] - i\epsilon_{i,j,k}\hat{S}_k, \quad (8)$$

$$[\hat{L}_i, \hat{S}_j] = i\epsilon_{i,j,k}\hat{S}_k - [\hat{S}_i, \hat{S}_j] \quad (9)$$

follow from (7). All spin operators in table 1 fulfill (7). The Czachor spin operator $\hat{\mathbf{S}}_{\text{Cz}}$, the Frenkel spin operator $\hat{\mathbf{S}}_{\text{F}}$, and the Fradkin–Good operator $\hat{\mathbf{S}}_{\text{FG}}$, are however, disqualified as relativistic spin operators by violating the angular momentum algebra (6). Furthermore, the Pauli spin operator $\hat{\mathbf{S}}_{\text{p}}$ and the Chakrabarti spin operator $\hat{\mathbf{S}}_{\text{Ch}}$ do not commute with the free Dirac Hamiltonian, ruling them out as meaningful relativistic spin operators. According to our criteria, only the Foldy–Wouthuysen spin operator $\hat{\mathbf{S}}_{\text{FW}}$ and the Pryce spin operator $\hat{\mathbf{S}}_{\text{Pr}}$ remain as possible relativistic spin operators.

3. Electron spin of hydrogen-like ions

The question of which of the proposed relativistic spin operators (if any) in table 1 provides the correct mathematical description of spin can be answered definitely only by comparing theoretical predictions with experimental results. For this purpose, one needs a physical setup that shows strong relativistic effects and is as simple as possible. Such a setup is provided by the bound eigenstates of highly charged hydrogen-like ions, i.e. atomic systems with an atomic core of Z protons and a single electronic charge. These ions can be produced at storage rings [58] or by utilizing electron beam ion traps [59, 60] up to $Z = 92$ (hydrogen-like uranium). The degenerate bound eigenstates of the corresponding Coulomb–Dirac Hamiltonian

$$\hat{H}_C = \hat{H}_0 - \frac{Z}{|\mathbf{r}|} \quad (10)$$

are commonly expressed as simultaneous eigenstates $\psi_{n,j,m,\kappa}$ of $\hat{H}_C$, $\hat{\mathbf{J}}^2$, $\hat{J}_3$, and the so-called spin-orbit operator $\hat{K} = \beta\{\hat{\mathbf{S}} \cdot [\mathbf{r} \times (-i\nabla) + 1]\}$ fulfilling the eigenequations [28, 29]

$$\hat{H}_C \psi_{n,j,m,\kappa} = \mathcal{E}(n, j) \psi_{n,j,m,\kappa} \quad n = 1, 2, \dots, \quad (11)$$

$$\hat{\mathbf{J}}^2 \psi_{n,j,m,\kappa} = j(j+1) \psi_{n,j,m,\kappa} \quad j = \frac{1}{2}, \frac{3}{2}, \dots, n - \frac{1}{2}, \quad (12)$$

$$\hat{J}_3 \psi_{n,j,m,\kappa} = m \psi_{n,j,m,\kappa} \quad m = -j, (j-1), \dots, j, \quad (13)$$

$$\hat{K} \psi_{n,j,m,\kappa} = \kappa \psi_{n,j,m,\kappa} \quad \kappa = -j - \frac{1}{2}, j + \frac{1}{2}. \quad (14)$$

The eigenenergies are given with α_{el} denoting the fine structure constant by

$$\mathcal{E}(n, j) = m_0 c^2 \left[1 + \left(\frac{\alpha_{\text{el}}^2 Z^2}{n - j - 1/2 + \sqrt{(j - 1/2)^2 - \alpha_{\text{el}}^2 Z^2}} \right) \right]^{-1/2}. \quad (15)$$

In order to establish a close correspondence between the nonrelativistic Schrödinger–Pauli theory and the relativistic Dirac theory, one may desire to find a splitting of $\hat{\mathbf{J}}$ into a sum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ of commuting operators such that both $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ (i) fulfill the angular momentum algebra, and (ii) form a complete set of commuting operators that contains $\hat{H}_C$ as well as $\hat{S}_3$ and/or $\hat{L}_3$.

The latter property would ensure that *all* hydrogenic energy eigenstates are spin eigenstates and/or orbital angular momentum eigenstates, too. Such hypothetical eigenstates would be superpositions of $\psi_{n,j,m,\kappa}$ of the same energy. Consequently, these superpositions are eigenstates of $\hat{\mathbf{J}}^2$, too, because the energy (15) depends on the principal quantum number n as well as the quantum number j . Thus, any complete set of commuting operators for specifying hydrogenic quantum states necessarily includes $\hat{\mathbf{J}}^2$. As a consequence of the postulated angular momentum algebra for $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$, the operator $\hat{\mathbf{J}}^2$ commutes with $\hat{\mathbf{L}}^2$ as well as with $\hat{\mathbf{S}}^2$, but with neither $\hat{S}_3$ nor $\hat{L}_3$ [61], excluding $\hat{S}_3$ and $\hat{L}_3$ from any complete set of commuting operators for specifying relativistic hydrogenic eigenstates. In conclusion, hydrogenic energy eigenstates are generally not eigenstates of any spin operator that fulfills the angular momentum algebra.

In momentum space, the relativistic spin operators introduced in table 1 are simple matrices. Thus, by employing the momentum space representation of $\psi_{n,j,m,\kappa}$, spin expectation values of the degenerate hydrogenic ground states $\psi_\uparrow = \psi_{1,1/2,1/2,1}$ and $\psi_\downarrow = \psi_{1,1/2,-1/2,-1}$ can be evaluated [62]. For simplicity, we measure spin along the z -direction for the remainder of this section. The spin expectation value of a general superposition $\psi = \cos(\eta/2)\psi_\uparrow + \sin(\eta/2)e^{i\zeta}\psi_\downarrow$ of the hydrogenic ground states $\psi_\uparrow$ and $\psi_\downarrow$ is given by

$$\begin{aligned} \langle \psi | \hat{S}_3 | \psi \rangle &= \cos^2 \frac{\eta}{2} \langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle + \sin^2 \frac{\eta}{2} \langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle \\ &\quad + 2 \cos \frac{\eta}{2} \sin \frac{\eta}{2} \cos \zeta \operatorname{Re} \langle \psi_\uparrow | \hat{S}_3 | \psi_\downarrow \rangle. \end{aligned} \quad (16)$$

For all spin operators introduced in table 1, the mixing term $\operatorname{Re} \langle \psi_\uparrow | \hat{S}_3 | \psi_\downarrow \rangle$ vanishes and, furthermore, $\langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle = - \langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle > 0$. Thus, the expectation value (16) is maximal for $\eta = 0$ and minimal for $\eta = \pi$, and the inequality

$$\langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle \leq \langle \psi | \hat{S}_3 | \psi \rangle \leq \langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle \quad (17)$$

holds for all hydrogenic ground states ψ .

For every proposed spin operator in table 1, we get different values for the upper and lower bounds in (17). The spin expectation values $\langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle$ and implicitly $\langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle$ for the operators of table 1 are displayed as a function of the atomic number Z in figure 1. None of the spin operators in table 1 commute with $\hat{H}_C$. Thus, the expectation values $\langle \psi_\uparrow | \hat{S}_3 | \psi_\uparrow \rangle$ and $\langle \psi_\downarrow | \hat{S}_3 | \psi_\downarrow \rangle$ generally do not equal one of the eigenvalues of $\hat{S}_3$. For small atomic numbers ($Z < 20$), all spin operators yield about $\pm 1/2$. For larger Z , however, the expectation values differ significantly from each other. In particular, spin expectation values differ from $\pm 1/2$ even for spin operators

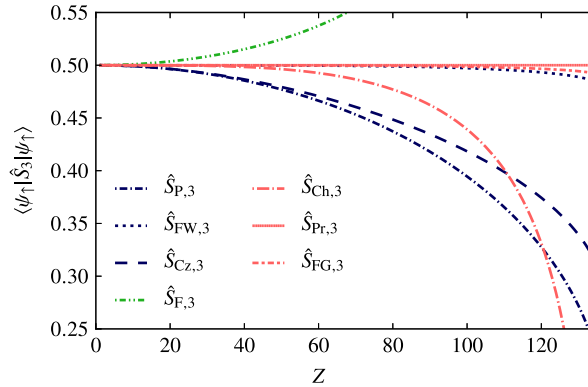


Figure 1. The spin expectation values $\langle \psi_{\uparrow} | \hat{S}_3 | \psi_{\uparrow} \rangle$ of various relativistic spin operators for the hydrogenic ground state $\psi_{\uparrow}$ as a function of the atomic number Z measured in the z -direction. The spin expectation values for $\psi_{\downarrow}$ follow by symmetry via $\langle \psi_{\downarrow} | \hat{S}_3 | \psi_{\downarrow} \rangle = -\langle \psi_{\uparrow} | \hat{S}_3 | \psi_{\uparrow} \rangle$.

with eigenvalues $\pm 1/2$. This means that it is possible to discriminate between different relativistic spin operator candidates. The magnitude of the spin expectation value decreases with growing Z when the Pauli, the Fouldy–Wouthuysen, the Czachor, the Chakrabarti, or the Fradkin–Good spin operator is applied. The Frenkel spin operator yields spin expectation values with the modulus exceeding $1/2$, which is due to the violation of the angular momentum algebra. Only the Pryce operator yields a spin expectation of $\pm 1/2$ for all values of Z . In fact, calculations show that all hydrogenic states $\psi_{n,j,m,\kappa}$ with $m = \pm j$ are eigenstates of the Pryce spin operator, but not those states with $m \neq \pm j$.

4. An experimental test for relativistic spin operators

Theoretical considerations have led to several proposals for a relativistic spin operator, as illustrated in table 1. The identification of the correct relativistic spin operator, however, demands an experimental test. The inequality (17) may serve as a basis for such an experimental test. More precisely, the inequality (17) allows falsification of the hypothesis that the spin measurement procedure is an experimental realization of some operator $\hat{S}$, where $\hat{S}$ is one of the operators in table 1. In this test, the electron of a highly charged hydrogen-like ion is prepared in its ground state $\psi_{\uparrow}$ first, e.g. by exposing the ion to a strong magnetic field in the z -direction and turning it off adiabatically. (Preparing a superposition of $\psi_{\uparrow}$ and $\psi_{\downarrow}$ will reduce the sensitivity of the experimental test.) Afterwards, the spin will be measured along the z -direction, e.g. by a Stern–Gerlach-like experiment, yielding the experimental expectation value s . Comparing this experimental value to each of the seven bounds shown in figure 1 will allow exclusion of some of the proposed spin operators. The hypothesis that the spin measurement procedure is an experimental realization of the operator $\hat{S}$ is compatible with the experimental result s if, and only if, the inequality $\langle \psi_{\downarrow} | \hat{S}_3 | \psi_{\downarrow} \rangle \leq s \leq \langle \psi_{\uparrow} | \hat{S}_3 | \psi_{\uparrow} \rangle$ is fulfilled. Otherwise, this operator is

excluded as a relativistic spin operator by experimental evidence. In particular, realizing full spin-polarization, i.e. $s = \pm 1/2$, eliminates all operators in table 1 except the Pryce operator.

5. Conclusions

We investigated the properties of various proposals for a relativistic spin operator. Only the Fouldy–Wouthuysen operator and the Pryce operator fulfill the angular momentum algebra, and are constants of motion in the absence of forces. While different theoretical considerations lead to different spin operators, the definite relativistic spin operator has to be justified by experimental evidence. The energy eigenstates of highly charged hydrogen-like ions, in particular the ground states, can be utilized to exclude candidates for a relativistic spin operator experimentally. The proposed spin operators predict different maximal degrees of spin polarization. Only the Pryce spin operator allows for a complete polarization of spin in the hydrogenic ground state.

Acknowledgments

We have enjoyed helpful discussions with Prof. C Müller, S Meuren, Prof. Q Su and E Yakaboylu. RG acknowledges the warm hospitality during his sabbatical leave in Heidelberg. This work was supported by the NSF.

References

- [1] Nikolić H 2007 *Found. Phys.* **37** 1563–611
- [2] Giulini D 2008 *Stud. Hist. Philos. Sci.: B Stud. Hist. Philos. Modern Phys.* **39** 557–78
- [3] Morrison M 2007 *Stud. Hist. Philos. Sci. B Stud. Hist. Philos. Modern Phys.* **38** 529–57
- [4] Dehmelt H 1990 *Science* **247** 539–45
- [5] Hanneke D, Fogwell S and Gabrielse G 2008 *Phys. Rev. Lett.* **100** 120801
- [6] Neumann P, Beck J, Steiner M, Rempp F, Fedder H, Hemmer P R, Wrachtrup J and Jelezko F 2010 *Science* **329** 542–4
- [7] Buckley B B, Fuchs G D, Bassett L C and Awschalom D D 2010 *Science* **330** 1212–5
- [8] Close T, Fadugba F, Benjamin S C, Fitzsimons J and Lovett B W 2011 *Phys. Rev. Lett.* **106** 167204
- [9] Sturm S, Wagner A, Schabinger B, Zatorski J, Harman Z, Quint W, Werth G, Keitel C H and Blaum K 2011 *Phys. Rev. Lett.* **107** 023002
- [10] DiSciaccia J and Gabrielse G 2012 *Phys. Rev. Lett.* **108** 153001
- [11] Mott N F 1929 *Proc. R. Soc. Lond. A* **124** 425–42
- [12] Peres A and Terno D R 2004 *Rev. Mod. Phys.* **76** 93–123
- [13] Kim W T and Son E J 2005 *Phys. Rev. A* **71** 014102
- [14] Wiesendanger R 2009 *Rev. Mod. Phys.* **81** 1495–550
- [15] Friis N, Bertlmann R A, Huber M and Hiesmayr B C 2010 *Phys. Rev. A* **81** 042114
- [16] Petersson K D, McFaul L W, Schroer M D, Jung M, Taylor J M, Houck A A and Petta J R 2012 *Nature* **490** 380–3
- [17] Saldanha P L and Vedral V 2012 *New J. Phys.* **14** 023041
- [18] Awschalom D D, Bassett L C, Dzurak A S, Hu E L and Petta J R 2013 *Science* **339** 1174–9
- [19] Abanin D A, Gorbachev R V, Novoselov K S, Geim A K and Levitov L S 2011 *Phys. Rev. Lett.* **107** 096601

- [20] Mecklenburg M and Regan B C 2011 *Phys. Rev. Lett.* **106** 116803
- [21] Güttinger J, Frey T, Stampfer C, Ihn T and Ensslin K 2010 *Phys. Rev. Lett.* **105** 116801
- [22] Di Piazza A, Müller C, Hatsagortsyan K Z and Keitel C H 2012 *Rev. Mod. Phys.* **84** 1177–228
- [23] Ahrens S, Bauke H, Keitel C H and Müller C 2012 *Phys. Rev. Lett.* **109** 043601
- [24] Ahrens S, Bauke H, Keitel C H and Grobe R 2014 Electron-spin dynamics induced by photon spins arXiv:1401.5976
- [25] Newton T D and Wigner E P 1949 *Rev. Mod. Phys.* **21** 400–6
- [26] Jordan T F and Mukunda N 1963 *Phys. Rev.* **132** 1842–8
- [27] O’Connell R and Wigner E 1978 *Phys. Lett. A* **67** 319–21
- [28] Bethe H A and Salpeter E E 2008 *Quantum Mechanics of One- and Two-Electron Atoms* (Mineola: Dover)
- [29] Thaller B 2000 *Advanced Visual Quantum Mechanics* (Heidelberg, New York: Springer)
- [30] Hill E L and Landshoff R 1938 *Rev. Mod. Phys.* **10** 87–132
- [31] Dirac P A M 1971 *Proc. R. Soc. Lond. A* **322** 435–45
- [32] Ohanian H C 1986 *Am. J. Phys.* **54** 500–5
- [33] Foldy L L and Wouthuysen S A 1950 *Phys. Rev.* **78** 29–36
- [34] de Vries E 1970 *Fortschr. Phys.* **18** 149–82
- [35] Costella J P and McKellar B H J 1995 *Am. J. Phys.* **63** 1119–22
- [36] Schweber S S 2005 *An Introduction to Relativistic Quantum Field Theory* (Mineola: Dover)
- [37] Caban P, Rembieliński J and Włodarczyk M 2013 *Ann. Phys.* **330** 263–72
- [38] Czachor M 1997 *Phys. Rev. A* **55** 72–77
- [39] Hilgevoord J and Wouthuysen S A 1963 *Nucl. Phys.* **40** 1–12
- [40] Wightman A S 1960 L’invariance dans la mecanique quantique relativiste *Relations de Dispersion et Particules Élémentaires* (Paris: Hermann) chapter 5
- [41] Bargmann V, Michel L and Telegdi V L 1935 *Phys. Rev. Lett.* **2** 435–6
- [42] Chakrabarti A 1963 *J. Math. Phys.* **4** 1215–22
- [43] Gürsey F 1965 *Phys. Lett.* **14** 330–1
- [44] Gürsey F 1965 Group combining internal symmetries and spin *High Energy Physics* (London: Gordon and Breach) pp 53–88
- [45] Gürsey F 1965 *Phys. Lett.* **14** 330–1
- [46] Kirsch I, Ryder L H and Hehl F W 2001 The Gordon decompositions of the inertial currents of the Dirac electron correspond to a Foldy–Wouthuysen transformation arXiv:hep-th/0102102
- [47] Ryder L H 1999 *Gen. Rel. Grav.* **31** 775–80
- [48] Choi T 2013 *J. Korean Phys. Soc.* **62** 1085–92
- [49] Pryce M H L 1935 *Proc. R. Soc. Lond. Ser. A* **150** 166–72
- [50] Pryce M H L 1948 *Proc. R. Soc. Lond. Ser. A* **195** 62–81
- [51] Macfarlane A J 1963 *J. Math. Phys.* **4** 490–500
- [52] Berg R A 1965 *J. Math. Phys.* **6** 34–39
- [53] Fradkin D M and Good R H Jr 1961 *Rev. Mod. Phys.* **33** 343–52
- [54] Wigner E 1939 *Ann. Math.* **40** 149–204
- [55] Bargmann V and Wigner E P 1948 *Proc. Natl Acad. Sci. USA* **34** 211–23
- [56] Wigner E P 1957 *Rev. Mod. Phys.* **29** 255–68
- [57] Sakurai J J and Napolitano J 2010 *Modern Quantum Mechanics* 2nd edn (Upper Saddle River: Pearson Education)
- [58] Stöhlker T *et al* 1993 *Phys. Rev. Lett.* **71** 2184–7
- [59] Robbins D *et al* 2006 *Phys. Rev. A* **74** 022713
- [60] Kluge H J *et al* 2008 HITRAP: A facility at GSI for highly charged ions ed S Salomonson and E Lindroth *Current Trends in Atomic Physics (Advances in Quantum Chemistry vol 53)* (Academic Press)
- [61] Cohen-Tannoudji C, Diu B and Laloë F 1977 *Quantum Mechanics* vol 2 (John Wiley & Sons)
- [62] Rubinowicz A 1948 *Phys. Rev.* **73** 1330–3