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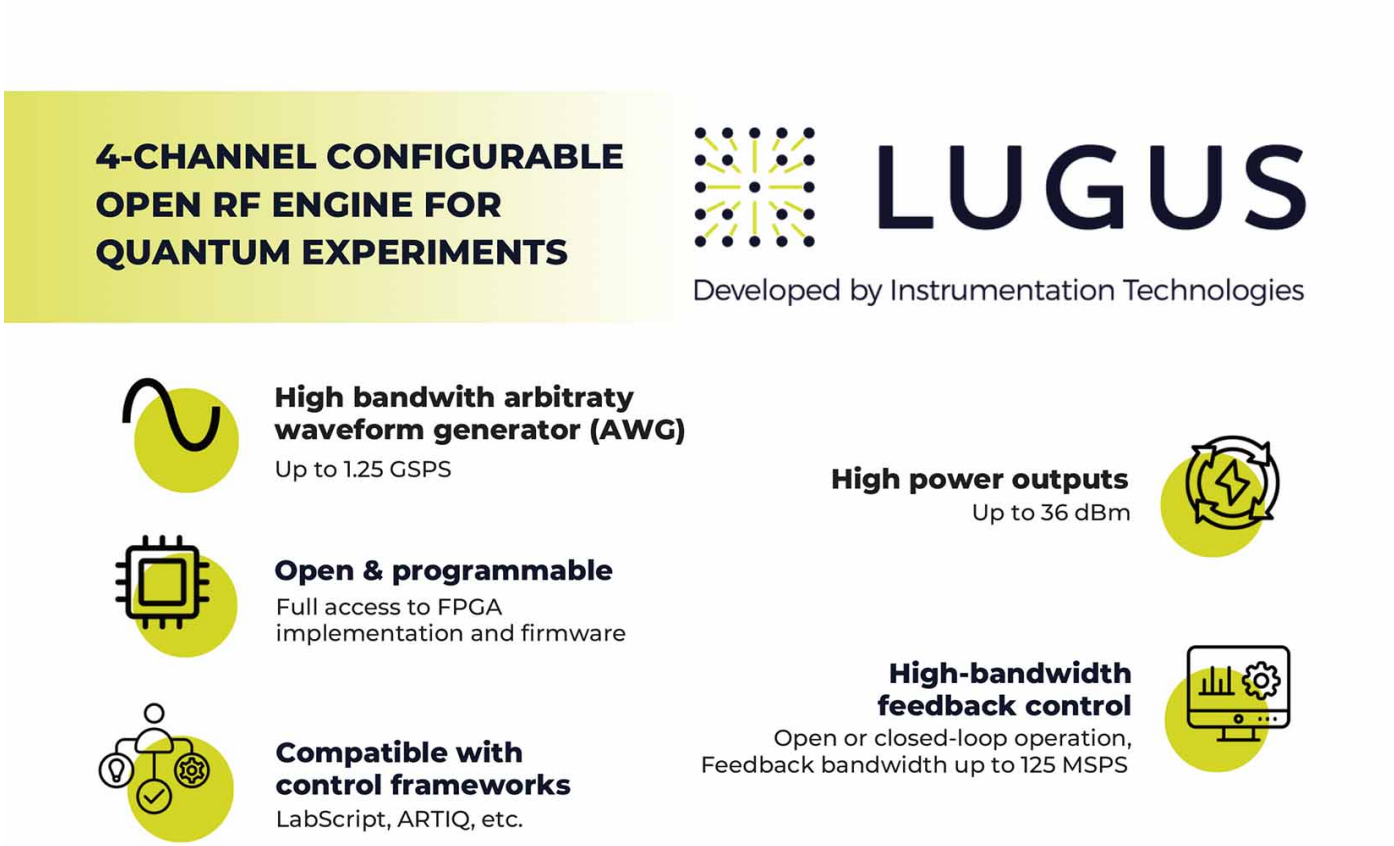
On three-dimensional spherical acoustic cloaking

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On three-dimensional spherical acoustic cloaking

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Abstract. Transformation acoustics opens a new avenue towards the design of acoustic metamaterials, which are materials engineered at the subwavelength scale in order to mimic the parameters in wave equations. The design of the acoustic cloaking is based on the property of equations being invariant under a coordinate transformation, i.e. a specific spatial compression is equivalent to a variation of the material parameters in the original space. In this paper, the sound invisibility performance is discussed for spherical cloaks. The original domain consists of alternating concentric layers made from piezoelectric ceramics and epoxy resin, following a triadic Cantor sequence. The spatial compression, obtained by applying the concave-down transformation, leads to an equivalent domain with an inhomogeneous and anisotropic distribution of the material parameters.

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1. Introduction

The idea of invisibility has fascinated people for years and has been an inspiration for myths, novels and films, from the Greek legend of Perseus versus Medusa to Well's Invisible Man. Newton viewed color as a physical problem, involving light striking objects and entering our

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eyes. For Goethe (1810), the sensations of color reaching our brain are also shaped by our perception, the mechanics of human vision and the way our brains process information.

It was recently found that enhanced control of acoustic waves can be achieved through coordinate transformations, which bring the material parameters into their governing equations. The coordinate transformation is a general method for designing not only the electromagnetic cloaks, but also acoustic cloaks with arbitrary sizes and shapes. The fundamental idea is that the acoustic equation is invariant under a coordinate transformation if the material properties are altered appropriately, i.e. a specific spatial compression is equivalent to a variation of the material parameters in the original space. The required material properties for a three-dimensional (3D) elastic cloak involve an elasticity tensor of order four with up to 34 spatially varying nonvanishing Cartesian entries. A metamaterial variant was constructed by Milton (2007) for which, at a fixed frequency, the momentum density is independent of the local rotation (but still depends on the strain) and the stress is symmetric (but still depends on the acceleration).

Transformation acoustics is key to the design of acoustic metamaterials, which are materials engineered at the subwavelength scale (Alu and Engheta 2003, 2005, Milton and Nicorovici 2006, Nicorovici *et al* 1994, Wolf and Habashy 1993).

Recent works show that acoustic metamaterials could cloak regions of space, making them invisible to sound (Guenneau *et al* 2011, Leonhardt 2006, Pendry *et al* 2006). We refer to acoustic cloaking that occurs when a medium contains a region in which noisy objects can be acoustically hidden. It is easy to imagine an object invisible to sound by building a box around it to prevent the wave from reaching the object.

The principle of how to cloak a region of space to make its contents invisible or transparent to waves was discussed by Miller (2006) and Leonhardt (2006). Recent papers by Pendry *et al* (2006) and Greenleaf *et al* (2003) used the coordinate invariance of Maxwell's equations to show how a region of the space can be made inaccessible to electromagnetic waves by surrounding it with a suitable dielectric shield. Kohn *et al* (2008) analyzed two shortcomings of the aforementioned papers: (i) the cloaks they consider are rather singular and (ii) the analysis by Greenleaf *et al* (2003) does not apply in the space dimension $n = 2$, and they provided a treatment that remedies these shortcomings. They have shown how a regular near-cloak can be obtained using a nonsingular change of variables and proved that the change-of-variable-based scheme achieves perfect cloaking in any dimension $n \geqslant 2$.

As an alternative to a box made from a metamaterial, sonic composites (or sonic crystals) exhibit the full band gaps, where the sound is not allowed to propagate due to complete reflections (Hirsekorn *et al* 2004; Munteanu and Chiroiu 2010). Alternatively, the acoustic field can be mimicked and canceled out by using secondary sources. Such an approach is well known in acoustics as the *antinoise* (Ffowcs 1984, Friot and Bordier 2004, Friot *et al* 2006, Nelson and Elliott 1992). An interesting review lecture in this direction has been given by Ffowcs (1984).

Cummer *et al* (2008) derived the mass density and bulk modulus of a spherical shell that can eliminate scattering from an arbitrary object in the interior of the acoustic shell. Calculations confirmed that the pressure and velocity fields were smoothly bent and excluded from the central region as for previously reported electromagnetic cloaking shells. It is also interesting to note that the ideal 3D acoustic cloaking parameters are similar in structure to the 2D electromagnetic and 2D acoustic parameters in that they contain singularities on the interior edge of the cloak. Cummer and Schurig (2007) have demonstrated that in a 2D geometry, the acoustic equations in a fluid are identical in form to the single polarization Maxwell equations via a variable

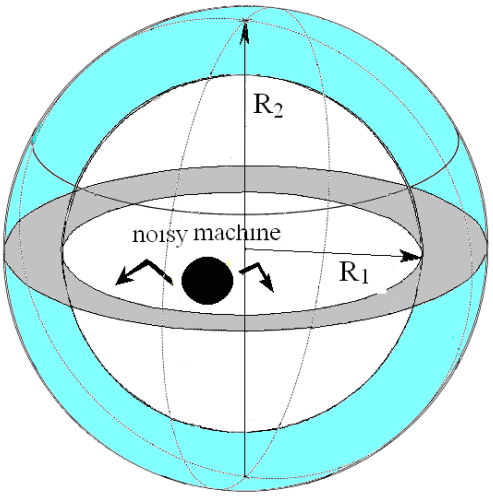


Figure 1. Sketch of the spherical cloak surrounding a noisy machine.

exchange that also preserves boundary conditions. This is in contrast to the 3D electromagnetic cloak (Pendry *et al* 2006), which does not contain singularities and for which scattering analysis does not require arguments to show the scattering is identically zero (Chen *et al* 2007). This is because the 2D electromagnetic, 2D acoustic and 3D acoustic cases directly involve solutions of the Helmholtz equation and therefore the Bessel and spherical Bessel functions that do not tend to zero as their argument approaches zero. By using this result, a class of rectangular cylindrical devices for noise shielding by using acoustic metamaterials was investigated by Liu and Huang (2010). Chen and Chan (2007) have developed a 3D spherical acoustic cloaking shell by applying the isomorphism between the acoustic and electric equations. The correspondences among cloaks in electromagnetism and acoustics are studied in the context of exploiting the scalar nature of acoustic wave equations. These waves are all governed by a scalar partial differential equation invariant under geometric transforms (Chiroiu and Chiroiu 2003). The Helmholtz equation is key to analyzing the band spectra of sound invisibility cloaks and the anomalous resonances of sound refracting coatings.

An interesting open question is the selection of the sound-soft boundary in order to demonstrate the function of acoustic scatterers. This boundary condition can be the approximation for a liquid–gas interface since experimental fluid systems are the most practical way to realize acoustic metamaterials (Fang *et al* 2006, Li and Chan 2004). Milton *et al* 2006 described conceptually how anisotropic effective mass can be achieved with spring-loaded masses, whereas Torrent and Sanchez-Dehesa (2007) showed how effectively density and bulk modulus can be controlled in an acoustic metamaterial by embedding solid inclusions in a fluid matrix.

In this paper, we apply the 3D concave-down transformation to design a spherical cloak which surrounds a noisy machine; see figure 1. The original domain is a sphere of radius R_2, consisting of alternating concentric layers made from piezoelectric (PZ) ceramics and epoxy resin, following a triadic Cantor sequence. After the transformation, the cloak contains a region $r < R_1$ that is filled with air and contains the noisy source, whereas the shell $R_1 < r < R_2$ is filled by the nonlinearly transformed material.

2. Transformational acoustics

A finite-size object surrounded by a coating consisting of a specially designed metamaterial would become invisible for electromagnetic waves at any frequency (Pendry *et al* 2006). In acoustics, the idea of the invisibility cloak is that the sound sees the space differently (Dupont *et al* 2011). For the sound, the concept of distance is modified by the acoustic properties of the regions through which the sound travels. In geometrical acoustics, we are used to the idea of the acoustical path; when traveling an infinitesimal distance ds, the corresponding acoustical path length is $c^{-1}\mathrm{d}s$, where $c^{-1} = \sqrt{\rho/\kappa}$ with ρ being the fluid density and κ the compression modulus of the fluid (Synge 1981, Seymour and Varley 1982, Munteanu and Donescu 2004).

The 3D acoustic equation for the pressure waves propagating in a bounded fluid region $\Omega \subset R^3$ is

$$\nabla \cdot (\underline{\underline{\rho}}^{-1} \nabla p) + \frac{\omega^2}{\kappa} p = 0, \tag{2.1}$$

where p is the pressure, $\underline{\underline{\rho}}$ is the rank-2 tensor of the fluid density, κ is the compression modulus of the fluid and ω is the wave frequency.

Let us consider the geometric transformation from the coordinate system (x', y', z') of the compressed space to the original coordinate system (x, y, z), given by $x(x', y', z')$, $y(x', y', z')$ and $z(x', y', z')$. The change of coordinates is characterized by the transformation of the differentials through the Jacobian $J_{xx'}$ of this transformation, i.e.

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = J_{xx'} \begin{pmatrix} \mathrm{d}x' \\ \mathrm{d}y' \\ \mathrm{d}z' \end{pmatrix}, \quad J_{xx'} = \frac{\partial(x, y, z)}{\partial(x', y', z')}. \tag{2.2}$$

From the geometrical point of view, the change of coordinates implies that, in the transformed region, one can work with an associated metric tensor (Guenneau *et al* 2011, Zolla *et al* 2007)

$$T = \frac{J_{xx'}^{\mathrm{T}} J_{xx'}}{\det(J_{xx'})}. \tag{2.3}$$

In terms of the acoustic parameters, one can replace the material from the original domain (homogeneous and isotropic) by an equivalent compressed one that is inhomogeneous (its characteristics depend on the spherical (r', θ', ϕ') coordinates) and anisotropic (described by a tensor), and whose properties, in terms of $J_{x'x}$, are given by

$$\underline{\underline{\rho'}} = J_{x'x}^{-\mathrm{T}} \cdot \rho \cdot J_{x'x}^{-1} \cdot \det(J_{x'x}), \quad \kappa' = \kappa \det(J_{x'x}), \tag{2.4}$$

or, equivalently, in terms of $J_{xx'}$

$$\underline{\underline{\rho'}} = \frac{J_{xx'}^{\mathrm{T}} \cdot \rho \cdot J_{x'x}}{\det(J_{xx'})}, \quad \kappa' = \frac{\kappa}{\det(J_{xx'})}. \tag{2.5}$$

Here, $\underline{\underline{\rho'}}$ is a second-order tensor. When the Jacobian matrix is diagonal, (2.4) and (2.5) can be more easily written. Multiplying (2.1) by a test function φ and integrating by parts, one obtains (Dupont *et al* 2011)

$$-\int_{\Omega} (\nabla_{(x,y,z)}\varphi \cdot \rho^{-1} \nabla_{(x,y,z)} p)\, \mathrm{d}V + \int (\omega^2 \kappa^{-1} p\varphi)\, \mathrm{d}V = 0. \tag{2.6}$$

In (2.6) the surface integral, corresponding to a Neumann integral over the boundary $\partial\Omega$, is zero. By applying the coordinate transformation $(x, y, z) \rightarrow (x', y', z')$ to (2.6) and using (2.2), one obtains

$$-\int_{\Omega} (J^{\mathrm{T}}_{x'x} \nabla_{(x',y',z')} \varphi \cdot \underline{\underline{\rho}}^{-1} J^{\mathrm{T}}_{x'x} \nabla_{(x,y,z)} p) \det(J_{xx'}) \, \mathrm{d}V' + \int (\det(J_{xx'}) \omega^2 \kappa^{-1} p\varphi) \, \mathrm{d}V' = 0, \tag{2.7}$$

in terms of $J_{xx'}$, and

$$-\int_{\Omega} \left((\nabla_{(x',y',z')} \varphi)^{\mathrm{T}} \frac{J_{x'x} \underline{\underline{\rho}}^{-1} J^{\mathrm{T}}_{x'x}}{\det(J_{x'x})} \nabla_{(x',y',z')} p \right) \mathrm{d}V' + \int \left(\frac{\kappa^{-1}}{\det(J_{x'x})} \omega^2 p\varphi \right) \mathrm{d}V' = 0, \tag{2.8}$$

in terms of $J_{x'x}$.

The geometric transformation may be linear or nonlinear. Qiu *et al* (2009) classified the geometric transformation functions in terms of the negative (i.e. concave-down) or the positive (i.e. concave-up) sign of the second-order derivative of this function. All transformations, i.e. linear, concave-up and concave-down transformations, are perfect cloaks for the exact inhomogeneous design.

The concave-down nonlinear transformation compresses a sphere of radius R_2 in the original space Ω into a shell region $R_1 < r' < R_2$ in the compressed space Ω' as

$$r(\beta) = \frac{R_2^{\beta+1}}{R_2^{\beta} - R_1^{\beta}} \left(1 - \left(\frac{R_1}{r'} \right)^{\beta} \right), \tag{2.9}$$

where β denotes the degree of the nonlinearity in the transformation. By taking $\beta \rightarrow 0$ in (2.9), the linear case is obtained, namely

$$r(\beta) = \frac{R_2 \mathrm{Ln}(r'/R_1)}{\mathrm{Ln}(R_2/R_1)}. \tag{2.10}$$

All curves belonging to (2.9) have negative second-order derivative with respect to the physical space r'. This class of transformations is termed as the concave-down transformation. The transformation function (2.9) depends on the radial component r' in the spherical coordinate system (r', θ', ϕ') (Qiu *et al* 2009).

The concave-up nonlinear transformation compresses a sphere of radius R_2 in the original space Ω into a shell region $R_1 < r' < R_2$ in the compressed space Ω' as

$$r(\beta) = \frac{R_2 R_1^{\beta}}{R_2^{\beta} - R_1^{\beta}} \left(\left(\frac{r'}{R_1} \right)^{\beta} - 1 \right). \tag{2.11}$$

As $\beta \rightarrow 0$, one obtains again the linear case (2.10). This class of transformations is termed as the *concave-up* transformation because (2.11) has positive second-order derivatives.

All curves belonging to (2.9) have negative second-order derivative with respect to the physical space r'. This class of transformations is termed as the *concave-down* transformation. The nonlinear transformation function in (2.9) only depends on the radial component r' in the spherical coordinate system (r', θ', ϕ') (Qiu *et al* 2009). The cloak properties in the both transformed coordinates are given by (2.4) and (2.5), where $J_{r'r} = \partial r'/\partial r$.

Milton *et al* (2006) showed that geometric transformations cannot be applied to equations which are not invariant under coordinate transformations and, consequently, if cloaking exists for such equations (for example, the elasticity equations), it would be of a different nature from acoustic and electromagnetic. The existence of an acoustic cloaking indicates that cloaks might

possibly be built for other wave systems, including seismic waves that travel through the earth and the waves at the surface of the ocean (Cummer *et al* 2008).

Farhat *et al* (2009) discussed the special case of thin elastic plates, for which the elasticity tensor can be represented in a cylindrical basis by a diagonal matrix with two (spatially varying) nonvanishing entries. In a similar manner, Brun *et al* (2009) derived the elastic properties of a cylindrical cloak for in-plane coupled shear and pressure waves.

Indeed, the equations governing the propagation of elastodynamic waves with a time harmonic dependence are written, in a weak sense, as

$$\nabla \cdot C : \nabla u + \rho\omega^2 u = 0, \tag{2.12}$$

where ρ is the scalar density of an isotropic heterogeneous elastic medium, C is the fourth-order elasticity tensor, ω is the wave angular frequency and $u(x_1, x_2, x_3, t) = u(x_1, x_2, x_3)\exp(-\mathrm{i}\omega t)$ is the vector displacement. Milton *et al* (2006) showed that under a change of coordinates (x', y', z') to (x, y, z) such that $u'(x') = J^{-\mathrm{T}}_{x'x}u(x)$, $J_{x'x} = \frac{\partial(x',y',z')}{\partial(x,y,z)}$, equation (2.12) takes the form

$$\nabla'' \cdot (C' + S') : \nabla' u' + \underline{\underline{\rho'}}\omega^2 u' = D' : \nabla' u', \tag{2.13}$$

which preserves the symmetry of the new elasticity tensor $C' + S'$. Equation (2.13) contains two third-order symmetric tensors S' and D' with $D'_{pqr} = S'_{qrp}$ and a second-order tensor ρ'_{pq}.

3. Spherical acoustic cloak

Our intention is to replace a material made from concentric homogeneous and isotropic layers situated in the original spherical domain by an equivalent compressed inhomogeneous anisotropic material described by the transformation matrix (2.3). These kinds of materials are not naturally occurring. However, recent advances in metamaterials are encouraging for such an approach to constitutive parameters required for cloaking. Metamaterials are materials with subwavelength microstructures that are designed to have desired physical and acoustical properties. Despite the latest advances in metamaterials, we do not currently have the ability to manufacture a cloak with ideal constitutive parameters (McGuirk and Collins 2008).

Let us suppose that the original domain Ω is a sphere of radius R_2. The sphere consists of alternating concentric layers made from PZ ceramics and epoxy resin, following a triadic Cantor sequence up to the fourth generation (31 elements). A sketch of this material is represented in figure 2. The dashed regions are occupied by PZ ceramics of total volume V^{p} and boundary external surface S_1^{p}. The white regions are occupied by epoxy resins of total volume V^{e} and boundary external surface S_1^{e}. The lateral surfaces are S_2, while the interfaces between constituents are denoted by I^{pe}. Let the regions occupied by the plate be $\Omega = V^{\mathrm{p}} \cup V^{\mathrm{e}}$, where V^{p} and V^{e} are the regions occupied by the PZ and non-PZ (ER) layers, respectively. The boundary surface, S, of the domain Ω is partitioned as follows:

$$S = S_1^{\mathrm{p}} \cup S_1^{\mathrm{e}} \cup S_2, \quad S_1^{\mathrm{p}} \cap S_1^{\mathrm{e}} \cap S_2 = 0,$$

where $S_1^{\mathrm{p}} = \{x_3 = \pm h/2,\ 0 < x_1 < l\}$ is the boundary surface of V^{p} $S_1^{\mathrm{e}} = \{x_3 = \pm h/2,\ 0 < x_1 < l\}$ is the boundary surface of V^{e} and

$$S_2 = \{x_1 = 0,\ x_1 = l\ , -h/2 \leqslant x_3 \leqslant h/2\}.$$

Let the unit outward normal of S be n_i and the interfaces between constituents be I^{pe}.

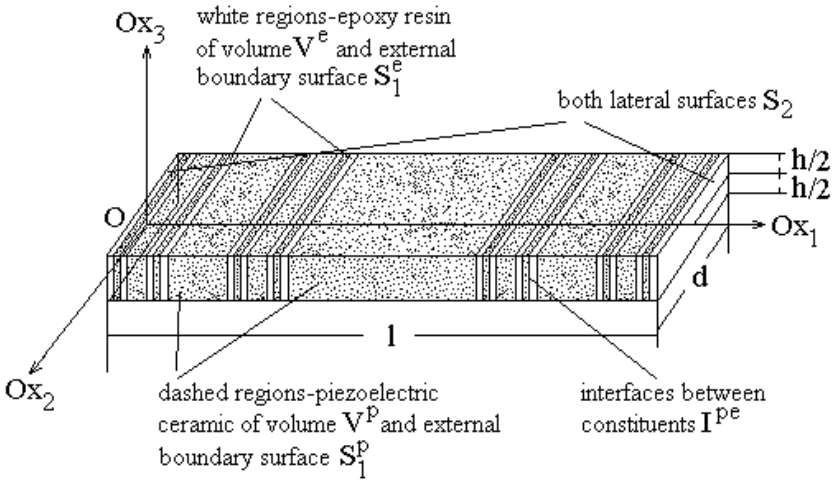


Figure 2. The Cantor-like structure (Chiroiu *et al* 2001).

The motivation of this choice goes back to Craciun *et al* (1992), Alippi (1982) and Alippi *et al* (1988, 1992), who showed experimental evidence of extremely low thresholds for the subharmonic generation of ultrasonic waves in 1D artificial PZ plates with Cantor-like structure, as compared with the corresponding homogeneous and periodical plates. An anharmonic coupling between the extended-vibration (phonon) and the localized-mode (fracton) regimes explained this phenomenon. The aforementioned authors proved that the large enhancement of nonlinear interaction results from the more favorable frequency and spatial matching of coupled modes (fractons and phonons) in the Cantor-like structure. The equations that govern the subharmonic ultrasonic wave phenomenon were solved by using the cnoidal method, which employed the cnoidal wave as the fundamental basis function (Chiroiu *et al* 2001, 2006). Cnoidal waves are much richer than sine waves, i.e. the modulus m of the cnoidal wave ($0 \leqslant m < 1$) can be varied to obtain a sine wave $m = 0$, Stokes wave ($m = 0.5$) or soliton ($m = 1$).

Next, the governing equations of this composite are written in the spirit of Truesdell (1974), Rogacheva (1994) and Landau and Lifshitz (1982, 1986). The quasistatic motion equations and constitutive laws read as

$$\rho \ddot{u}_i = t_{ij,j}, \quad \text{in } \Omega, \tag{3.1}$$

$$D_{i,i} = 0, \quad E_i + \varphi_{e,i} = 0, \quad \text{in } V^{\mathrm{p}}, \tag{3.2}$$

$$t_{ij} = \lambda^{\mathrm{p}} \varepsilon_{kk} \delta_{ij} + 2\mu^{\mathrm{p}} \varepsilon_{ij} - e_k^{\mathrm{p}} E_k \delta_{ij}, \quad \text{in } V^{\mathrm{p}} \tag{3.3}$$

$$t_{ij} = \lambda^{\mathrm{e}} \varepsilon_{kk} \delta_{ij} + 2\mu^{\mathrm{e}} \varepsilon_{ij}, \quad \text{in } V^{\mathrm{e}}, \tag{3.4}$$

$$D_i = \bar{\varepsilon}^{\mathrm{p}} E_i - e_i^{\mathrm{p}} \varepsilon_{kk}, \quad \text{in } V^{\mathrm{p}} \tag{3.5}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{in } \Omega. \tag{3.6}$$

Here, indices p and e denote the PZ and ER materials, respectively; ρ is the density; u_i, $i = 1, 2, 3$, are the components of the displacement vector; t_{ij}, $i = j = 1, 2, 3$, are the components of the stress tensor; D_i, $i = 1, 2, 3$, are the components of the electric induction vector; E_i, $i = 1, 2, 3$, are the components of the electric field and φ_{e} is the electric potential;

$\varepsilon_{ij}, i = j = 1, 2, 3$, are the components of the strain tensor; λ, μ are the Lamé constants; $\bar{\varepsilon}^{\mathrm{p}}$ is the dielectric constant and $e_i^{\mathrm{p}}(e_3^{\mathrm{p}} = e_2^{\mathrm{p}} = e_1^{\mathrm{p}})$ are the piezoelectricity coefficients. The coordinate x_1 is directed along the radial direction, x_3 is directed along the circumferential direction, whereas x_2 is located within the layer.

The scalar elastic potential φ, and the components ψ_1, ψ_2 and ψ_3 of the vectorial elastic potential, defined as

$$u_1 = \varphi_{,1} - \psi_{2,3}, \quad 4u_2 = \psi_{1,3} - \psi_{3,1}, \quad u_3 = \varphi_{,3} + \psi_{2,1}, \tag{3.7}$$

and the electric potential φ_{e} are expressed using the theta-function Θ (see e.g. Chiroiu *et al* 2001 and Kapelewski and Michalec 1991),

$$\varphi(x_1, x_2, x_3, t) = \varphi_0(t)\Delta(\log\Theta(x_1, x_2, x_3)), \quad \psi_i(x_1, x_2, x_3) = \psi_{i0}(t)\Delta(\log\Theta(x_1, x_2, x_3)),$$
$$i = 1, 2, 3, \quad \varphi_{\mathrm{e}}(x_1, x_2, x_3) = \varphi_{\mathrm{e}0}(t)\Delta(\log\Theta(x_1, x_2, x_3)). \tag{3.8}$$

On adopting the hypothesis of the theory of von Karman (Graff 1975), the theta function Θ is the solution of the von Karman equation

$$\nabla\cdot\zeta_{\mathrm{p,e}}^{-1}\nabla(\Delta\nabla\cdot\zeta_{\mathrm{p,e}}^{-1}\nabla\Theta) - \Lambda^{-1}\gamma_0^4\Theta = 0, \tag{3.9}$$

where $\zeta = E^{-1/2}$, E is the effective Young modulus of the composite, $\gamma_0^4 = \omega^2\rho h/D_0$, D_0 is the flexural rigidity of the plate, ρ its effective density, h its thickness, $\Lambda = \rho^{-1}$ and ω the frequency. Equation (3.9) can be factorized as a Helmholtz operator and an anti-Helmholtz operator (i.e. with an opposite sign for the spectral parameter)

$$(\nabla^2 + \gamma_0^2)(\nabla^2 - \gamma_0^2)\Theta = 0, \tag{3.10}$$

where for simplicity we have taken $\zeta = \Lambda = 1$. We write the Helmholtz equation in the coordinate system (x_1, x_2, x_3) as

$$\nabla\cdot(\zeta^{-1}\nabla\Theta) + \omega^2\Lambda^{-1}\Theta = 0. \tag{3.11}$$

Let us apply the concave-down transformation (2.9) to equation (3.9), which compresses the original domain Ω occupied by a sphere of radius R_2 into a shell region $R_1 < r' < R_2$ in the compressed space Ω', characterized by

$$\underline{\underline{\zeta}}_{\mathrm{p,e}}^{\prime-1}(r') = J_{rr'}^{\mathrm{T}}\zeta_{\mathrm{p},r}^{-1}(r)J_{rr'}/\det(J_{rr'}), \quad \underline{\underline{\Lambda}}^{\prime-1}(r') = J_{rr'}^{\mathrm{T}}\Lambda^{-1}(r)J_{rr'}/\det(J_{rr'}), \quad J_{rr'} = \partial r/\partial r'. \tag{3.12}$$

In the new coordinates, the transformed equation (3.9) now reads as

$$\nabla\cdot\underline{\underline{\zeta}}_{\mathrm{p,e}}^{-1}\nabla(\Delta_{33}\nabla\cdot\underline{\underline{\zeta}}_{\mathrm{p,e}}^{-1}\nabla\Theta') - \Lambda_{33}^{-1}\gamma_0^4\Theta' = 0, \tag{3.13}$$

where $\underline{\underline{\zeta}}_{\mathrm{p,e}}^{-1}$ is the upper diagonal part of the inverse of $\underline{\underline{\zeta}}$ and Λ_{33}^{-1} is the third diagonal entry of $\underline{\underline{\Lambda}}^{-1}$ (Guenneau *et al* 2011).

The cloak has inner radius $R_1 = 0.5\,\mathrm{m}$ and outer radius $R_2 = 1\,\mathrm{m}$. The concave-down transformation presents an overlap for all mapping curves for $\beta < 0.1$, which means the same results in applications. The effect of β on the amplitude of displacements, which vary from $-U$ to $U(U = \sqrt{u_1^2 + u_2^2 + u_3^2})$ inside the cloak $r \leqslant R_1$, is illustrated in figure 3.

It can be seen that as β increases, the amplitude increases significantly inside the region $r \leqslant R_1$ of the cloak. This is due to the fact that more energy is guided towards the inner boundary $r = R_1$, which in turn makes the cloaked object more *acoustically visible* to external incidences.

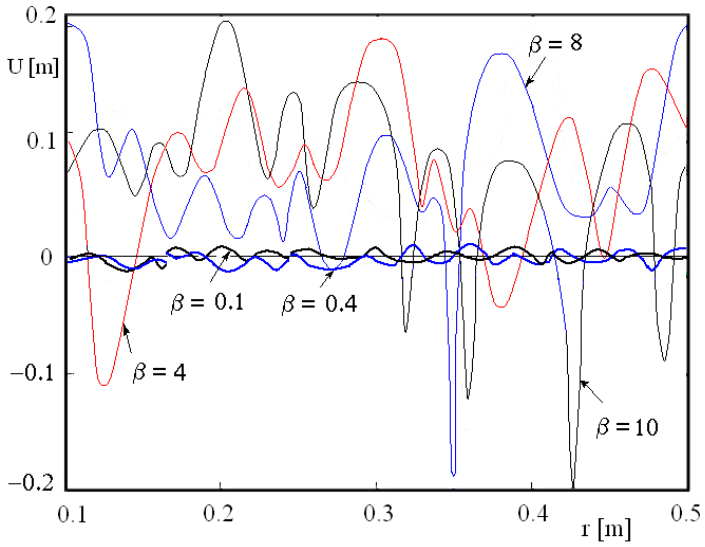


Figure 3. Variation of the displacement amplitude with respect to β in the region $r \leqslant R_1$.

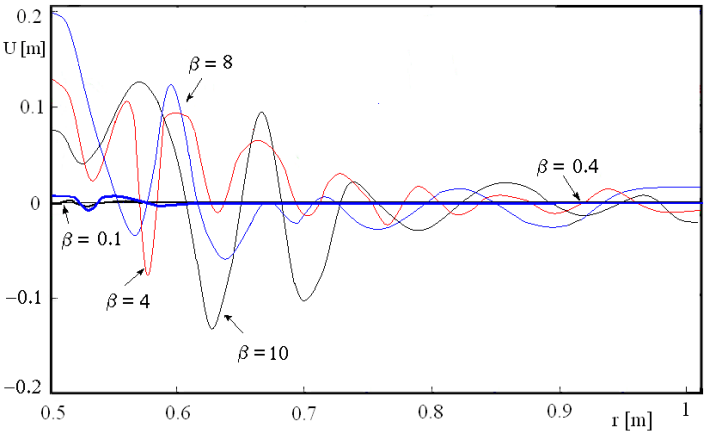


Figure 4. Variation of the displacement amplitude with respect to β in the region $R_1 < r < R_2$.

For $\beta = 0.1$ and 0.4, the acoustic invisibility is good. The effect of β on the amplitude of displacements in the shell region $R_1 < r < R_2$ is illustrated in figure 4. In a similar manner, as β increases, the amplitude increases significantly in the shell region of the cloak.

The absence of scattering of waves generated by an external source outside the cloak is observed in figure 5 for $\beta = 0.1$. The waves are smoothly bent around the central region inside the cloak. The results reported in figure 5 show that the wave field inside the cloak, i.e. the inner region of radius R_1 that surrounds the noisy machine, is completely isolated from the region situated outside the cloak. The waves generated by a noisy source are smoothly confined inside the inner region of the cloak, and the sound invisibility detected from the observer is proportional to β. The inner region is acoustically isolated and the sound is not detectable by an exterior observer because the amplitudes on the boundary vanish. The domain $r < R_1$

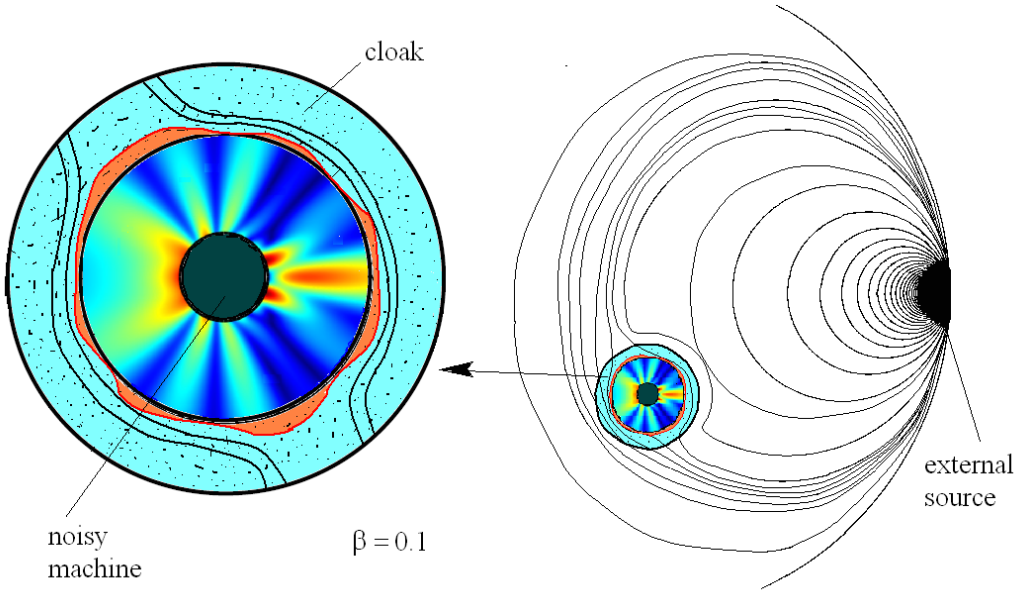


Figure 5. The wave fields inside and outside the cloak for $\beta = 0.1$.

is an acoustic invisible domain for exterior observers. The waves generated by the exterior source outside the cloak do not interact with the interior field of waves. A possible interaction or coupling between the internal and external wave fields is canceled out by the presence of the shell region $R_1 < r < R_2$ filled with metamaterial.

Hence, we can conclude that for the concave-down spherical cloaks, smaller values for β lead to a smaller disturbance in the acoustic fields in both the inner and the outer spaces $r < R_2$ and $r > R_2$, respectively.

4. Conclusions

In this paper, we have identified new aspects in the 3D spherical cloaking related to new reflectionless solutions which may exist for cloaking sonic systems that are not isomorphic to electromagnetism. The original domain consists of an alternation of concentric layers made from PZ ceramics and epoxy resin, following a triadic Cantor sequence. The spatial compression obtained by applying the concave-down transformation has led to an equivalent domain with an inhomogeneous and anisotropic distribution of the material parameters. This formulation was based on the pioneering work of Guenneau *et al* (2011), Qiu *et al* (2009) and Dupont *et al* (2011). However, the present study represents an application of the aforementioned analytical results, in the sense that a numerical implementation, which treated a new kind of metamaterial that might be useful in the design of elastic cloaking devices, was developed.

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References

Alippi A 1982 Nonlinear acoustic propagation in piezoelectric crystals *Ferroelectrics* **42** 109–16

Alippi A, Craciun F and Molinari E 1988 Stopband edges in the dispersion curves of Lamb waves propagating in piezoelectric periodical structures *Appl. Phys. Lett.* **53** 1806–8

Alippi A, Shkerdin G, Berttucci A, Craciun F, Molinari E and Petri A 1992 Low threshold subharmonic generation in composite structures with Cantor-like codes *Phys. Rev. Lett.* **69** 3318–21

Alu A and Engheta N 2003 Pairing an epsilon-negative slab with a mu-negative slab: resonance, tunneling and transparency *IEEE Trans. Antennas Propag.* **51** 2558–71

Alu A and Engheta N 2005 Achieving transparency with plasmonic and metamaterial coatings *Phys. Rev.* E **72** 016623

Brun M, Guenneau S and Movchan A B Achieving control of in-plane elastic waves *Appl. Phys. Lett.* **94** 061903

Chen H and Chan C T 2007 Acoustic cloaking in three dimensions using acoustic metamaterials *Appl. Phys. Lett.* **91** 183518

Chen H, Wu B I, Zhang B and Kong J A 2007 Electromagnetic wave interaction with a metamaterial cloak *Phys. Rev. Lett.* **99** 063903

Chiroiu V and Chiroiu C 2003 *Inverse Problems in Mechanics* (Bucharest: Publishing House of the Romanian Academy) (in Romanian)

Chiroiu V, Delsanto P P, Scalerandi M, Chiroiu C and Sireteanu T 2001 Subharmonic generation in piezoelectrics with Cantor-like structure *J. Phys. D: Appl. Phys.* **34** 1579–86

Chiroiu V, Donescu St and Munteanu L 2006 Subharmonic generation of Love waves in a ferrite-dielectric plate with Cantor-like structure *Proceedings of the Romanian Academy, Series A: Mathematics, Physics, Technical Sciences, Information Science (Bucharest)* **7** 47–54

Craciun F, Bettucci A, Molinari E, Petri A and Alippi A 1992 Direct experimental observation of fracton mode patterns in one-dimensional Cantor composites *Phys. Rev. Lett.* **68** 1555–8

Cummer S A and Schurig D 2007 One path to acoustic cloaking *New J. Phys.* **9** 45–52

Cummer S A, Popa B I, Schurig D, Smith D R, Pendry J, Rahm M and Starr A 2008 Scattering theory derivation of a 3D acoustic cloaking shell *Phys. Rev. Lett.* **100** 024301

Dupont G, Farhat M, Diatta A, Guenneau S and Enoch S 2011 Numerical analysis of three-dimensional acoustic cloaks and carpets *Wave Motion* **48** 483–96

Fang N, Xi D, Xu J, Ambati M, Srituravanich W, Sun C and Zhang X 2006 Ultrasonic metamaterials with negative modulus *Nat. Mater.* **5** 452–6

Farhat M, Guenneau S, Enoch S and Movchan A B 2009 Cloaking bending waves propagating in thin elastic plates *Phys. Rev.* B **79** 033102

Ffowcs Williams J E 1984 Review lecture: Anti-sound *Proc. R. Soc.* A **395** 63–88

Friot E and Bordier C 2004 Real-time active suppression of scattered acoustic radiation *J. Sound Vib.* **278** 563–580

Friot E, Guillermin R and Winninger M 2006 Active control of scattered acoustic radiation: a real-time implementation for a three-dimensional object *Acta Acoust.* **92** 278–88

Goethe J W 1810 *Zür Farbenlehre (Theory of Colours)* translated by C L Eastlake (Cambridge, MA: MIT Press)

Graff K F 1975 *Wave Motion in Elastic Solids* (New York: Dover)

Greenleaf A, Lassas M and Uhlmann G 2003 On nonuniqueness for Calderon ′s inverse problem *Math. Res. Lett.* **10** 685–93

Guenneau S, McPhedran R C, Enoch S, Movchan A B, Farhat M and Nicorovici N A 2011 The colours of cloaks *J. Opt.* **13** 024014

Hirsekorn M, Delsanto P P, Batra N K and Matic P 2004 Modelling and simulation of acoustic wave propagation in locally resonant sonic materials *Ultrasonics* **42** 231–5

Kapelewski J and Michalec J 1991 Soliton-like SAW in nonlinear isotropic piezoelectrics *Int. J. Eng. Sci.* **29** 285–91

Kohn R V, Shen H, Vogelius M S and Weinstein M I 2008 Cloaking via change of variables in electric impedance tomography *Inverse Probl.* **24** 015016

Landau L D and Lifshitz E M 1982 *Electrodynamics of Continuous Media* (Moscow: Nauka) (in russian) translated into English (Oxford: Pergamon Press)

Landau L D and Lifshitz E M 1986 *Theory of Elasticity* (Oxford: Pergamon)

Leonhardt U 2006 Optical conformal mapping *Science* **312** 1777–80

Li J and Chan C T 2004 Double-negative acoustic metamaterial *Phys. Rev.* E **70** 055602

Liu B and Huang J P 2010 Noise shielding using acoustic metamaterials *Commun. Theor. Phys.* **53** 560–4

McGuirk J S and Collins P J 2008 Controlling the transmitted field into a cylindrical cloak's hidden region *Opt. Express* **16** 17560

Miller D A B 2006 On perfect cloaking *Opt. Express* 12465

Milton G W 2007 New metamaterials with macroscopic behavior outside that of continuum elastodynamics *New J. Phys.* **9** 359–72

Milton G W and Nicorovici N A 2006 On the cloaking effects associated with anomalous localized resonance *Proc. R. Soc.* A **462** 3027–59

Milton G W, Briane M and Willis J R 2006 On cloaking for elasticity and physical equations with a transformation invariant form *New J. Phys.* **8** 248–68

Munteanu L and Chiroiu V 2010 On the dynamics of locally resonant sonic composites *Eur. J. Mech.* A **29** 871–78

Munteanu L and Donescu St 2004 *Introduction to Soliton Theory: Applications to Mechanics* (*Fundamental Theories of Physics* vol 143) (Dordrecht: Kluwer)

Nelson P A and Elliott S J 1992 *Active Control of Sound* (London: Academic Press) pp 290–93

Nicorovici N A, McPhedran R C and Milton G W 1994 Optical and dielectric properties of partially resonant composites *Phys. Rev.* B **490** 8479–82

Pendry J B, Shurig D and Smith D R 2006 Controlling electromagnetic fields *Science* **312** 1780–82

Qiu C W, Hu L, Zhang B, Wu B L, Johnson S G and Joannopoulos J D 2009 Spherical cloaking using nonlinear transformations for improved segmentation into concentric isotropic coatings *Opt. Express* **17** 13467

Rogacheva N N 1994 *The Theory of Piezoelectric Shells and Plates* (Boca Raton, FL: CRC Press)

Seymour B R and Varley E 1982 Exact solutions describing soliton-like interactions in a nondispersive medium *SIAM J. Appl. Math.* **42** 804–21

Synge J L 1981 On the vibrations of a heterogeneous string *Q. Appl. Math.* **39** 292–7

Torrent D and Sanchez-Dehesa J 2007 Acoustic metamaterials for new two-dimensional sonic devices *New J. Phys.* **9** 323–36

Truesdell C (ed) 1974 *Handbuch der Physics* vol VI a/4 (Berlin: Springer)

Wolf E and Habashy T 1993 Invisible bodies and uniqueness of the inverse scattering problem *J. Mod. Opt.* **40** 785–92

Zolla F, Guenneau S, Nicolet A and Pendry J B 2007 Electromagnetic analysis of cylindrical invisibility cloaks and the mirage effect *Opt. Lett.* **32** 1069–71