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## Squeezing the limit: quantum benchmarks for the teleportation and storage of squeezed states

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**Abstract.** We derive fidelity benchmarks for the quantum storage and teleportation of squeezed states of continuous variable systems, for input ensembles where the degree of squeezing  $s$  is fixed, no information about its orientation in phase space is given, and the distribution of phase-space displacements is a Gaussian. In the limit where the latter becomes flat, we prove analytically that the maximal classical achievable fidelity (which is  $1/2$  without squeezing, for  $s = 1$ ) is given by  $\sqrt{s}/(1+s)$ , vanishing when the degree of squeezing diverges. For mixed states, as well as for general distributions of displacements, we reduce the determination of the benchmarks to the solution of a finite-dimensional semidefinite program, which yields accurate, certifiable bounds thanks to a rigorous analysis of the truncation error. This approach may be easily adapted to more general ensembles of input states.

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## 1. Introduction

The storage and entanglement-assisted teleportation of quantum states are two of the central primitives of quantum information science. They have by now been accomplished with increasing precision in various experimental settings, with outstanding examples in the ‘continuous variable’ regime, adopting light modes or collective spins of atomic ensembles to, respectively, carry and store quantum information [1]–[5], or relying on motional atomic degrees of freedom [6, 7]. The need to certify success in such experiments, and to justify the use of the term *quantum* in setups such as ‘quantum memories’ and ‘quantum teleportations’, requires theoretical benchmarks which bound the performance of purely *classical* schemes [8]–[12]. Here, ‘classical’ refers to protocols where the quantum system is measured and later repared from information obtained in the measurement, which is in turn stored or transmitted by classical means. The fact that there are limitations to such ‘measure-and-prepare’ schemes immediately follows from the no-cloning principle. A precise notion of these limitations, however, depends on the figure of merit (usually the *fidelity*) as well as on the prior distribution, i.e. on the ensemble of quantum states to be stored or transmitted.

If, for a  $d$ -dimensional Hilbert space, the input ensemble is comprised of all the pure states distributed according to the Haar measure, the average fidelity achievable by a classical scheme is  $2/(d+1)$ , dropping to zero with increasing dimension  $d$  [13, 14]. Clearly, for *continuous variable systems* (where  $d = \infty$ ) not all pure states are experimentally accessible (as they constitute a set with infinitely many real parameters and, in principle, unbounded energy). The typical continuous variable implementations via electro-magnetic field modes or collective fluctuations in atomic ensembles favor Gaussian states of which coherent states are the simplest representatives. It was proven in [9] that the optimal fidelity for classical schemes is  $1/2$  if the input ensemble is made up of coherent states taken from a flat distribution in phase space. This

shows that such restrictions on the ‘alphabet’ of input states allows for classical schemes to achieve finite average fidelities, even though the dimension of the Hilbert space is infinite.

The present work deals with input ensembles including Gaussian *squeezed states*, for which one expects, in general, the constraint on classical schemes to become more and more severe as the degree of squeezing is increased. This study finds its motivation in recent and ongoing experimental attempts to teleport and store squeezed quantum states [15]–[18]: we will provide a means for certifying success in experiments.

Although the tools we shall develop are suitable for more general applications, encompassing non-Gaussian states or even finite-dimensional systems, the focus of the present paper will be Gaussian input ensembles with a fixed degree of squeezing  $s$ , ( $s$  being the factor by which the variance of one of the two canonical quadratures is reduced with respect to the vacuum level). This complements the results of [11], where  $s$  is assumed to be unknown, and generalizes the results of [9] on coherent states (special case  $s = 1$  in our notation). In brief, we have been able to obtain the following results:

- (i) For an ensemble of squeezed coherent states where, apart from the degree of squeezing, no *a priori* information is given about orientation and displacement, the maximal classical achievable fidelity is  $\sqrt{s}/(1+s)$ . The optimal measure-and-prepare scheme achieving this bound is realized by heterodyne detection followed by preparation of a coherent state.
- (ii) For an ensemble of squeezed states as in (i), but where each state is subject to additive Gaussian noise, analytical bounds are derived for the fidelity as well as for the maximal overlap achievable by classical schemes.
- (iii) For ensembles with a random orientation (phase covariance) but arbitrary distribution of displacements the maximal overlap is shown to be computable by means of a semidefinite program (SDP). Together with a rigorous bound on the truncation error in Fock space this yields reliable benchmarks.
- (iv) The SDP is applied to various cases of pure state ensembles (with and without Gaussian displacement distribution) and mixed state ensembles for parameter regimes relevant to present and future experiments.

Further technical details can be found in the appendices, where we present a characterization of (non-positive partial transpose (NPT) entanglement breaking) covariant channels and a discussion of the Choi-matrix formalism in infinite dimensions.

## 2. Notation and conventions

In this section, we briefly recapitulate some basic notation and useful concepts. For a more detailed exposition the reader is referred to [19]–[21].

Throughout the paper, we will deal with bosonic systems of  $n$  modes, each of which is assigned to an infinite-dimensional Hilbert space  $\mathcal{H} = L_2(\mathbb{R})$ .  $\mathfrak{B}(\mathcal{H})$  and  $\mathfrak{C}_1(\mathcal{H})$  will denote the set of bounded operators and the set of trace-class operators, respectively. Let us arrange the  $2n$  canonical operators in a vector  $R = (X_1, P_1, \dots, X_n, P_n)^T$  such that  $[R_j, R_k] = i\sigma_{jk}\mathbb{1}$ ,  $\sigma = i\bigoplus_{j=1}^n \sigma_y$  being the anti-symmetric symplectic form (whereas  $\sigma_y$  is the two-dimensional  $y$ -Pauli matrix). Notice that we set  $\hbar = 1$ .

As is customary, the Weyl displacement operator  $W_\xi$  will be defined as  $W_\xi = e^{i\xi \cdot \sigma R}$ , for  $\xi \in \mathbb{R}^{2n}$  so that  $W_\xi W_\eta = W_\eta W_\xi e^{-i\xi \cdot \sigma \eta}$ . The ‘characteristic function’  $\chi_A(\xi)$  of an operator  $A \in \mathfrak{C}_1(\mathcal{H}^{\otimes n})$  is defined as

$$\chi_A(\xi) = \text{Tr} [W_\xi A] . \quad (1)$$

In turn, the operator  $A$  is determined by its characteristic function according to the Fourier–Weyl relation

$$A = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{R}^{2n}} \chi_A(\xi) W_\xi^\dagger d^{2n}\xi , \quad (2)$$

which leads to the useful Parseval relation for  $A_1, A_2 \in \mathfrak{C}_1(\mathcal{H}^{\otimes n})$ :

$$\text{Tr}[A_1^\dagger A_2] = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{R}^{2n}} \overline{\chi_{A_1}(\xi)} \chi_{A_2}(\xi) d^{2n}\xi . \quad (3)$$

For a density operator  $\rho$ , we define a ‘covariance matrix’ (CM)  $\gamma_\rho$  with entries

$$(\gamma_\rho)_{jk} = \text{Tr} [\{R_j, R_k\} \rho] - 2\text{Tr} [R_j \rho] \text{Tr} [R_k \rho]$$

and a vector of first moments  $d_\rho$  with entries  $(d_\rho)_j = \text{Tr}[R_j \rho]$ . A state  $\rho$  is said to be Gaussian if its characteristic function  $\chi_\rho$  is a Gaussian, reading

$$\chi_\rho(\xi) = e^{i\xi \cdot \sigma d_\rho - (\sigma \xi) \cdot \gamma_\rho (\sigma \xi) / 4} .$$

The vacuum  $|0\rangle\langle 0|$  is a Gaussian state with  $\gamma_{|0\rangle\langle 0|} = \mathbb{1}$  and  $d_{|0\rangle\langle 0|} = 0$ .

### 3. Setup and figures of merit

Our goal is to quantify the limitations for measure-and-prepare schemes  $T$ , i.e. elements of the set  $\mathcal{E} \ni T$  of *entanglement-breaking channels* [22, 23], when acting on an ensemble  $\{\rho_\omega\}$  of input states characterized by a set of parameters  $\omega$ . Consider for instance an ensemble of pure squeezed Gaussian states with CM

$$\gamma = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} , \quad (4)$$

and displacement  $\xi$  and thus  $\omega = (s, \xi, \theta)$ . In order to fix a useful *figure of merit* we have to choose a functional  $F(T(\rho_\omega), \rho_\omega)$  which (i) measures, in some sense to be specified, the ability of  $T$  to ‘preserve’ the state  $\rho_\omega$  when applied to it and (ii) can be determined in experiments. Based on this choice, we can then either quantify the *worst case performance* or the *average case performance* of a channel  $T$ , where the latter depends on an *a priori* distribution  $q(\omega)$  over the parameter space. The corresponding benchmarks are then obtained by taking the supremum over all  $T \in \mathcal{E}$  leading to the definitions:

$$F_0(T) := \inf_{\rho_\omega} F(T(\rho_\omega), \rho_\omega), \quad F_0 := \sup_{T \in \mathcal{E}} F_0(T), \quad (5)$$

$$\bar{F}(T) := \int d\omega q(\omega) F(T(\rho_\omega), \rho_\omega), \quad \bar{F} := \sup_{T \in \mathcal{E}} \bar{F}(T). \quad (6)$$

We will restrict our attention to ensembles  $\{\rho_\omega\}$  with a fixed degree of squeezing  $s$ , so that the benchmarks will be functions of  $s$  and we will occasionally write  $\bar{F}(s)$ ,  $F_0(s)$  to emphasize this

dependence. Notice that  $s$  is a pure number (in practice,  $s$  is measured as the variance of one of the field quadratures in units of vacuum noise).

As usual, for the ideal scenario of pure states  $\rho_\omega$ , we use the *fidelity*  $F(T(\rho_\omega), \rho_\omega) = \text{Tr}[T(\rho_\omega)\rho_\omega]$ . Clearly, in practice it is more realistic to assume that the initial pure states  $\rho_\omega$  undergo a noisy channel  $\mathcal{N}$  and become mixed before entering a quantum memory or a teleportation scheme. A possible option for  $F$  would then be the ‘Uhlmann fidelity’ [24] between two mixed states, given by  $f(\rho_1, \rho_2) = \text{Tr}\left[\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\right]^2$ . Major drawbacks of this choice are that its nonlinearity leads to a very involved theoretical optimization and that such a quantity is exceedingly difficult to measure in experiments (without invoking a plethora of extra assumptions). We will present later on an analytical result adopting Uhlmann fidelity, but we will mainly follow a different route and use instead the *overlap*

$$F(T(\rho_\omega), \rho_\omega) = \text{Tr}[T(\mathcal{N}(\rho_\omega))\rho_\omega]. \quad (7)$$

This quantity is easier to determine in experiments ( $\rho_\omega$  may be interpreted as an observable) and by definition  $F_0$  and  $\bar{F}$  are proper benchmarks in the sense that beating their values means outperforming any classical scheme. Strictly speaking, of course, this overlap measures how close the output of  $T$  is to the initial noiseless state rather than to the input.

The derivation of the benchmarks simplifies considerably if the probability measure  $q$  is invariant with respect to a symmetry group  $G$ , i.e. if  $q(\omega) = q(g(\omega))$  for all  $g \in G$  and  $\rho_{g(\omega)} = U_g \rho_\omega U_g^\dagger$  for some unitary representation  $U_g$  of  $G$ . A standard argument [13, 25] then implies that, w.l.o.g. the channels  $T$  in the optimization of  $F_0$  and  $\bar{F}$  can be taken covariant with respect to  $G$ , in the sense that, for all density operators  $\rho$  and all  $g \in G$ , one has

$$U_g T(\rho) U_g^\dagger = T(U_g \rho U_g^\dagger). \quad (8)$$

For a compact symmetry group (with Haar measure  $dg$ ) the argument is straight forward since we can replace every single  $T$  by a covariant counterpart

$$\tilde{T}(\rho) := \int dg U_g^\dagger T(U_g \rho U_g^\dagger) U_g, \quad (9)$$

which performs at least as well as  $T$  if the chosen functional  $F$  is concave, as is the case for all the instances we discussed above. Moreover, if  $T$  is an element of a convex set closed under the group action, like the set of entanglement breaking channels, then so is  $\tilde{T}$ . If the orbit of  $G$  covers the entire parameter space then  $F(\tilde{T}(\rho_\omega), \rho_\omega)$  becomes state independent and we find  $F_0 = \bar{F}$ , i.e. the average case and the worst case performance become the same.

The argument becomes more subtle if  $G$  is not compact, as in the case where no *a priori* information is given about the displacement  $\xi$ . In this case  $F_0$  is still well defined but already  $\bar{F}$  has to be discussed more carefully, since it formally requires an average over a ‘flat’ distribution in phase space. Nevertheless, an analogous argument goes through and we can w.l.o.g. take  $T$  to be *phase-space covariant*, i.e. for all  $\xi$  and density operators  $\rho$

$$W_\xi T(\rho) W_\xi^\dagger = T(W_\xi \rho W_\xi^\dagger). \quad (10)$$

Clearly, due to non-compactness one cannot make the averaging procedure explicit, but has rather to invoke an invariant mean whose existence is guaranteed by the axiom of choice. For a more formal discussion of these matters see [26, 27].

#### 4. Analytical benchmarks for squeezed states under uniform rotations and displacements

##### 4.1. Classical benchmark based on the input–output overlap

Let us now focus on the specific case of an initial single-mode squeezed state  $\rho_s$  with CM

$$\gamma_{\rho_s} = \begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix},$$

which is uniformly displaced and rotated in phase space, and undergoes an additional noisy Gaussian channel  $\mathcal{N}$  adding classical noise with variance  $\eta$ . The input ensemble is given by  $\mathcal{N}(\rho_\omega)$  with  $\rho_\omega = W_\xi U_\theta \rho_s U_\theta^\dagger W_\xi^\dagger$ , where  $U_\theta = \exp[i\theta \hat{n}]$  denotes a phase-space rotation (whereas  $W_\xi$  is a displacement operator, defined in section 2, with  $\xi \in \mathbb{R}^2$ ), and the channel  $\mathcal{N}$  acts on CMs as  $\gamma \mapsto \gamma + \eta \mathbb{1}$  while leaving first moments invariant [21]. No *a priori* information is given about  $\theta \in [0, 2\pi]$ , whereas the degree of squeezing  $s$  and the noise  $\eta$  are fixed and known *a priori*. Note that  $\eta$  is the additive noise in units of vacuum energy  $\hbar\omega_0$  ( $\omega_0$  being the frequency of the mode at hand).

As pointed out earlier, we will use the overlap with  $\rho_\omega$  as figure of merit:

$$F_0(s, \eta) = \sup_{T \in \mathcal{E}} \inf_{\xi, \theta} \text{Tr}[T(\mathcal{N}(\rho_\omega))\rho_\omega],$$

where, due to the symmetry (which is preserved by  $\mathcal{N}$ ), one has  $F_0 = \bar{F}$ .

In order to determine the best possible overlap achievable by a measure-and-prepare scheme on such an ensemble, we will first prove an upper bound by enlarging the set of allowed channels to the set  $\mathcal{T} \supseteq \mathcal{E}$  of *NPT-entanglement breaking* channels, defined as the channels which remain valid quantum operations when concatenated with time-reversal (if applied locally, these channels turn the input state into a positive partial transport (PPT), but not necessarily separable, state). More precisely, if  $\vartheta$  denotes time-reversal, a channel  $T$  is ‘NPT-entanglement breaking’ if and only if its composition with time-reversal  $T \circ \vartheta$  is completely positive. This class of channels includes, for instance, all measure-and-prepare schemes which are assisted by PPT-bound entanglement shared between sender and receiver.

In a second step, we will show that the obtained bound is tight even within the set  $\mathcal{E}$ , as it turns out to be achievable by heterodyne measurement followed by preparation of a coherent state.

**Theorem 1 (Benchmark—mixed states).** *Let  $\mathcal{T}$  be either the set of all measure-and-prepare schemes or the set of NPT-entanglement breaking channels. Within these sets the maximal achievable overlap for the teleportation or storage of mixed squeezed coherent states with squeezing  $s$  and additive noise  $\eta$  is given by*

$$\bar{F}(s, \eta) = \sup_{T \in \mathcal{T}} \inf_{\xi, \theta} \text{Tr}[T(\mathcal{N}(\rho_\omega))\rho_\omega] = \left[ \left(1 + \frac{\eta}{2} + \frac{1}{s}\right) \left(1 + \frac{\eta}{2} + s\right) \right]^{-1/2}. \quad (11)$$

**Proof.** We first prove the upper bound. As already indicated above we can restrict ourselves to the set  $\tilde{\mathcal{T}}$  of *phase-space covariant* channels satisfying

$$\tilde{T}(W_\xi A W_\xi^\dagger) = W_\xi \tilde{T}(A) W_\xi^\dagger, \quad (12)$$

for all Weyl operators  $W_\xi$ ,  $\xi \in \mathbb{R}^2$  and all  $A \in \mathfrak{C}_1(L_2(\mathbb{R}))$ . This leads to  $F_0 = \bar{F}$  and lifts the need for the minimization over displacements [26, 27]. That is, we are left to determine

$$\sup_{\tilde{T} \in \tilde{\mathcal{T}}} \inf_{\theta \in [0, 2\pi]} \text{Tr}[U_\theta \rho_s U_\theta^\dagger \tilde{T}(\mathcal{N}(U_\theta \rho_s U_\theta^\dagger))], \quad (13)$$

(in other words, the infimum is only taken over all noisy squeezed coherent states centered at the origin, i.e. with vanishing first moments).

Furthermore, NPT-entanglement breaking phase-space covariant channels have a particular simple form in the Heisenberg picture, where we will denote the map under consideration by  $\tilde{T}^*$ . By lemma 4 (see appendix A) they act on Weyl operators as

$$\tilde{T}^*(W_\xi) = t(\xi) W_\xi, \quad (14)$$

where

$$t(\xi) = \text{Tr}[\tau W_{\sqrt{2}\xi}], \quad (15)$$

for some density matrix  $\tau$ . Conversely, every such  $\tau$  yields an admissible phase-space covariant channel  $\tilde{T}$ . This allows us to recast the optimization over channels into one over density operators, as follows.

Exploiting the Parseval relation (3) back and forth, we get for any density matrix  $\rho$  whose characteristic function is a Gaussian with CM  $\gamma$  and vanishing first moments:

$$\begin{aligned} \text{Tr}[\tilde{T}(\mathcal{N}(\rho_\omega))\rho_\omega] &= \frac{1}{2\pi} \int d^2\xi \text{Tr}[\tau W_{\sqrt{2}\xi}] e^{-(1/2)\xi \cdot (\gamma + \frac{\eta}{2}\mathbb{1})\xi} \\ &= \frac{1}{2\pi} \int d^2\xi \text{Tr}[\rho' W_{\sqrt{2}\xi}] \text{Tr}[\tau W_{\sqrt{2}\xi}] = \frac{1}{2} \text{Tr}[\rho' \tau], \end{aligned} \quad (16)$$

where  $\rho'$  is now a centered Gaussian state with CM  $\gamma + \frac{\eta}{2}\mathbb{1}$  and  $\tau$  is defined by equation (15) (and we took advantage of the fact that the characteristic function of centered Gaussian states is real and has a purely quadratic dependence on  $\xi$ ).

To compute equation (13) it is now convenient to reintroduce an average over rotations (instead of an infimum, which gives the same value due to the optimization over  $T$ ), and write equation (13) as:

$$\sup_{\tau} \text{Tr} \left[ \frac{1}{4\pi} \left( \int_0^{2\pi} U_\theta \rho'_s U_\theta^\dagger d\theta \right) \tau \right] = \left\| \frac{1}{4\pi} \int_0^{2\pi} U_\theta \rho'_s U_\theta^\dagger d\theta \right\|_\infty, \quad (17)$$

where  $\rho'_s$  is the centered Gaussian state with CM  $\gamma_{\rho_s} + \frac{\eta}{2}\mathbb{1}$ . Note that in the last step we used the fact that the noisy channel  $\mathcal{N}$  is invariant under phase-space rotations. The operator norm above (largest eigenvalue) can be promptly determined since averaging over rotations just sets all off-diagonal elements in the Fock basis to zero, so that one simply has to resort to the expression of the state  $\rho'_s$  in the Fock basis, to obtain:

$$\begin{aligned} \bar{F}(s, \eta) &\leq \left\| \frac{1}{4\pi} \int_0^{2\pi} d\phi U_\phi \rho'_s U_\phi^\dagger \right\|_\infty \\ &= \max_n \frac{1}{2} \langle n | \rho'_s | n \rangle = \frac{1}{2} \langle 0 | \rho'_s | 0 \rangle = \left[ \left( 1 + \frac{\eta}{2} + \frac{1}{s} \right) \left( 1 + \frac{\eta}{2} + s \right) \right]^{-1/2}, \end{aligned}$$

where the rhs is attained within the set of NPT-entanglement breaking channels.

To show that this upper bound is actually tight for measure-and-prepare schemes as well, we will now explicitly point out that a specific, simple measure-and-prepare scheme attains the bound<sup>6</sup>. Consider a heterodyne measurement, i.e. a positive operator valued measure (POVM)  $\{W_\xi|0\rangle\langle 0|W_\xi^\dagger/2\pi\}$ , where the outcome  $\xi$  is followed by the preparation of a coherent state  $W_\xi|0\rangle$ , so that the whole map  $H$  reads

$$H(\rho) = \frac{1}{2\pi} \int d^2\xi \langle 0|W_\xi^\dagger \rho W_\xi|0\rangle W_\xi|0\rangle\langle 0|W_\xi^\dagger. \quad (18)$$

This process acts on CMs as  $\gamma \mapsto \gamma + 2 \cdot \mathbb{1}$  (while leaving first moments unaffected), so that one can easily determine the input–output overlap for any state in our ensemble using again Parseval’s relation (3) and solving the resulting Gaussian integrals [28]

$$\begin{aligned} \text{Tr}[\rho_\omega H(\mathcal{N}(\rho_\omega))] &= 2 \left[ \text{Det} \left( \gamma_{\rho_s} + (\gamma_{\rho_s} + (2 + \eta) \cdot \mathbb{1}) \right) \right]^{-1/2} \\ &= \left[ \left( 1 + \frac{\eta}{2} + \frac{1}{s} \right) \left( 1 + \frac{\eta}{2} + s \right) \right]^{-1/2}, \end{aligned}$$

which is independent of  $\xi$  and  $\theta$  and achieves the above bound, thus completing the proof.  $\square$

Clearly, for  $\eta = 0$  one recovers the following fidelity benchmark for pure states:

**Corollary 1 (Benchmark—pure states).** *Let  $\mathcal{T}$  be either the set of all measure-and-prepare schemes or the larger set of NPT-entanglement breaking channels. Within these sets, the maximal achievable fidelity for the teleportation or storage of squeezed coherent states with squeezing  $s$  and subject to uniformly distributed rotations and displacements in phase space is given by*

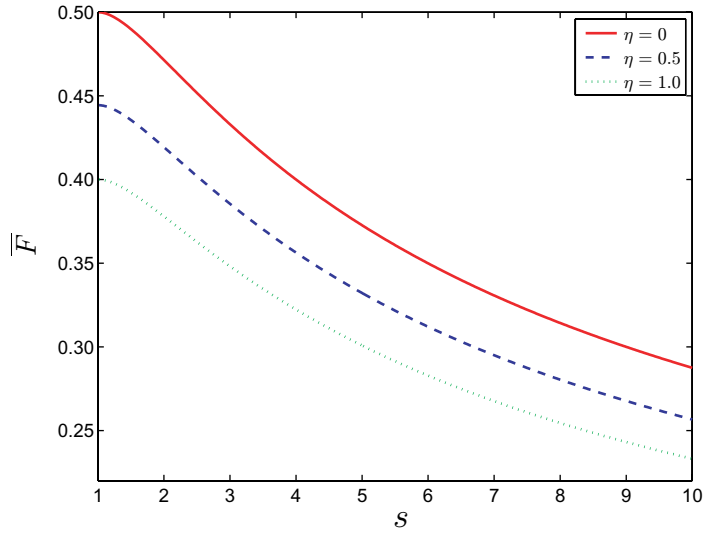
$$\bar{F}(s) = F_0(s) = \sup_{T \in \mathcal{T}} \inf_{\xi, \theta} \text{Tr}[\rho_\omega T(\rho_\omega)] = \frac{\sqrt{s}}{1+s}. \quad (19)$$

Notice that the overlap achieved by the ideal quantum channel (identity) in the general noisy case is  $[(\frac{\eta}{2} + \frac{1}{s})(\frac{\eta}{2} + s)]^{-1/2}$ . Clearly, the optimal classical performance degrades with increasing noise (this is obviously the case for quantum strategies as well). The obtained classical benchmark on the overlap as a function of the squeezing  $s$  is plotted in figure 1 for pure and mixed states (i.e. respectively, for  $\eta = 0$  and  $\eta > 0$ ).

Quite remarkably, the optimal strategy (heterodyning followed by generation of coherent states) does not depend on the degree of squeezing (nor on the additive noise  $\eta$ ), and is in fact the very same strategy which is optimal for a flat distribution of coherent states [8] (for coherent states, such optimality extends to rotationally invariant, Gaussian distributions of displacements as well [9]).

These analytical findings, though obtained for the ideal case of ‘flat’ displacement distributions, fully highlight the importance of randomizing the input phase in practical instances. In fact, for pure states and a flat distribution of displacements but fixed phase, the benchmark can be promptly inferred from the coherent states’ case, and is just  $1/2 \geq \sqrt{s}/(1+s)$ , regardless of the degree of squeezing  $s$  assumed. Randomizing the input phase appears to be very helpful to provably reach the quantum regime in experiments (and,

<sup>6</sup> An alternative way of noting that this bound remains the same when restricting to measure-and-prepare schemes is to realize that (i)  $\tau = |0\rangle\langle 0|$  (a Gaussian!) yields the maximum in equation (17) and (ii) for two-mode Gaussian states (bearing in mind the Choi-matrix of  $T$ ) the PPT-criterion is necessary and sufficient for separability.



**Figure 1.** Fidelity benchmarks  $\bar{F}(s, \eta)$  for measure-and-prepare schemes on ensembles with squeezing  $s$ , flatly distributed displacements and random phase-space orientations, as a function of the squeezing  $s$  (before additive noise is applied). The continuous (red) curve refers to pure states ( $\eta = 0$ ), whereas the dashed (blue) and dotted (green) curves refer to mixed states, for  $\eta = 0.5$  and  $\eta = 1$ , respectively. Notice that, as one should expect, the pure state case is an upper bound for mixed states for the figure of merit  $\bar{F}$ . As stated in theorem 2, the pure states case is also an upper bound for the Uhlmann fidelity benchmark of such mixed states.

not surprisingly, the advantage granted by random rotations becomes more and more relevant with increasing squeezing): for pure states and degrees of squeezing within experimental reach ( $s \lesssim 10$ ), the difference is around 0.2 and might easily turn out to be crucial for experimental success. As we will see later on (see section 5.3), and again not surprisingly, the same holds true for the randomization of displacements.

#### 4.2. Classical benchmark based on the Uhlmann fidelity

For the same ensemble of mixed state as in the preceding section, we shall present a further analytical benchmark in the form of an upper bound, this time adopting the Uhlmann fidelity  $f(T(\rho), \rho)$  as figure of merit:

**Theorem 2 (Benchmark—mixed state fidelity).** *Let  $\mathcal{T}$  be either the set of all measure-and-prepare schemes or the set of NPT-entanglement breaking channels. Within these sets the maximal achievable worst case (or average) Uhlmann fidelity for the teleportation or storage of mixed squeezed coherent states with squeezing  $s$  and arbitrary additive noise channel  $\mathcal{N}$  is bounded as follows:*

$$\sup_{T \in \mathcal{E}} \inf_{\xi, \theta} f(T(\mathcal{N}(\rho_\omega)), \mathcal{N}(\rho_\omega)) \leq \frac{\sqrt{s}}{1+s} \quad \forall \eta \in \mathbb{R}. \quad (20)$$

**Proof.** Concavity of the fidelity allows us again to restrict to phase-space covariant channels. All these channels commute with channels of the form  $\mathcal{N}^*(W_\xi) = W_\xi \exp[-\|\xi\|^2 \eta/4]$  ( $\mathcal{N}^*$  standing for channel  $\mathcal{N}$  in Heisenberg picture). The result follows then from the contractivity of cp-maps with respect to the fidelity, together with the pure state result of corollary 1 (equation (19)).  $\square$

## 5. Quantum benchmarks derived by semidefinite-programming

### 5.1. Problem settings

So far, we treated input ensembles of squeezed states which are displaced in phase space according to a ‘flat’ distribution. Needless to say, this is an idealization as one cannot implement an arbitrarily large displacement in practice.

To be more realistic we have to treat an input ensemble of squeezed states whose first moments are essentially contained in a finite region of phase space, e.g. due to a sufficiently rapidly decaying probability distribution as in [8, 9].

This section will deal with this scenario by resorting to numerical means, as a purely analytical treatment appears to be far too involved for finite, non-flat distributions of displacements (where the restrictions to phase-space covariant channels is no longer optimal). We will show that the problem of computing benchmarks of the desired kind can be cast into a SDP. As such, it comes with a guarantee of computing the correct value, as SDPs come in pairs of a *primal* and a *dual* problem which yield converging upper and lower bounds to the sought solution [29]. As the original SDP is in an infinite-dimensional space, truncation will be necessary and will induce errors. However, we will provide a rigorous bound to the truncation errors so that the finally derived benchmarks are reliable and rigorous, constituting upper bounds to the actual optimal classical figures of merit.

The figure of merit we use is again the average overlap

$$\bar{F} = \sup_{T \in \mathcal{E}} \int d\omega q(\omega) \text{Tr} \left[ T \left( \mathcal{N}(\rho_\omega) \right) \rho_\omega \right],$$

where  $\mathcal{N}$  is a noisy channel that describes the noise suffered by  $\rho_\omega$  before entering the storage or teleportation device, which for our purposes should be

- (i) a channel which allows for the computation of the matrix elements in the Fock basis  $\langle k | \mathcal{N}(\rho_\omega) | l \rangle$ , e.g. a Gaussian channel [21]
- (ii) rotationally covariant, i.e.  $\mathcal{N}(U_\theta \rho U_\theta^\dagger) = U_\theta \mathcal{N}(\rho) U_\theta^\dagger$  for all  $\rho, \theta$ .

Fortunately, channels representing attenuation, amplification and thermal noise are all of this type, the attenuation channel being probably the most relevant here, as it models losses of photons/excitations which are the dominant decoherence process in the practical realizations we have in mind, involving traveling waves of light. Thus we will adopt it in the following, denoting it by  $\mathcal{N}_\lambda$ . The dimensionless parameter  $\lambda \in [0, 1]$  represents the transmittivity (in intensity) such that the channel acts on, respectively, covariant matrix  $\gamma_\rho$  and vector of first moments  $d_\rho$  as:  $\gamma_\rho \mapsto \lambda \gamma_\rho + (1 - \lambda) \mathbb{1}$  and  $d_\rho \mapsto \sqrt{\lambda} d_\rho$ . Another crucial assumption for our method is that the input distribution  $q$  is uniform in the angle  $\theta$ . Moreover, we will consider ensembles with constant degree of squeezing  $s$  (although this is not necessary for the method). Hence  $q$  can be

considered as a probability distribution which depends only on the displacement  $\xi$  and therefore

$$\bar{F} = \sup_{T \in \mathcal{E}} \int_{\theta \in [0, 2\pi]} \int_{\xi \in \mathbb{R}^2} q(\xi) \text{Tr} [T(\mathcal{N}_\lambda(U_\theta \rho_{s,\xi} U_\theta^\dagger))(U_\theta \rho_{s,\xi} U_\theta^\dagger)] \frac{d\theta}{2\pi} d\xi, \quad (21)$$

where  $\rho_{s,\xi}$  is a pure squeezed states with degree of squeezing  $s$  and displacement  $\xi$ . Note that  $\bar{F}$  in this way becomes a functional of  $\lambda$ ,  $s$  and of the distribution  $q$ .

### 5.2. Reduction to a finite dimensional SDP

In this subsection, we will see how we can reduce the quantum benchmark  $\bar{F}$  in equation (21) to a finite dimensional SDP. We first show that equation (21) can be reduced to an infinite dimensional SDP problem. To this end, we use a simple correspondence between the set of all entanglement breaking channels and the set of bipartite separable positive operators [30] which is nothing but the Choi–Jamiołkowski isomorphism [31], albeit for an infinite-dimensional system (see appendix B for a proof):

**Theorem 3 (Choi–Jamiołkowski).** *Suppose  $\mathfrak{B}(\mathcal{H})$  is the space of all bounded operators and  $\mathfrak{C}_1(\mathcal{H})$  is the space of all trace class operators on a separable Hilbert space  $\mathcal{H}$ . Then, for all entanglement breaking channels  $\Psi$  on  $\mathcal{H}$ , there exists a unique separable positive bounded operator  $\Omega(\Psi)$  on  $\mathcal{H} \otimes \mathcal{H}$  such that  $\text{Tr}_B(\Omega(\Psi)) = \mathbb{1}_A$  and*

$$\text{Tr}(B\Psi(A)) = \text{Tr}(\Omega(\Psi)A \otimes B), \quad (22)$$

for all  $A \in \mathfrak{C}_1(\mathcal{H})$  and  $B \in \mathfrak{B}(\mathcal{H})$ . Conversely, for a separable positive bounded operator  $\Omega$  on  $\mathcal{H} \otimes \mathcal{H}$  satisfying  $\text{Tr}_B(\Omega) = \mathbb{1}_A$ , there exists a unique channel  $\Psi(\Omega)$  such that it is entanglement breaking and satisfies equation (22).

By means of the above theorem, we immediately derive an upper bound to the quantity  $\bar{F}$  of equation (21) in an infinite-dimensional SDP form by enlarging the set of positive separable operators (denoted by Sep) to the set of positive operators with PPT  $\Omega^\Gamma \geq 0$ :

$$\begin{aligned} \bar{F} &= \sup_{\Omega \in \mathfrak{B}(\mathcal{H} \otimes \mathcal{H})} \left\{ \text{Tr}(\Omega \eta) \mid \Omega \in \text{Sep}, \text{Tr}_B \Omega = \mathbb{1}_A \right\} \\ &\leq \sup_{\Omega \in \mathfrak{B}(\mathcal{H} \otimes \mathcal{H})} \left\{ \text{Tr}(\Omega \eta) \mid \Omega \geq 0, \Omega^\Gamma \geq 0, \text{Tr}_B \Omega = \mathbb{1}_A \right\}. \end{aligned} \quad (23)$$

In the above equations  $\eta$  is a state on  $\mathcal{H} \otimes \mathcal{H}$  defined by

$$\eta \stackrel{\text{def}}{=} \int_{\theta \in [0, 2\pi]} \int_{\xi \in \mathbb{R}^2} q(\xi) U_\theta \mathcal{N}_\lambda(\rho_{s,\xi}) U_\theta^\dagger \otimes (U_\theta \rho_{s,\xi} U_\theta^\dagger) \frac{d\theta}{2\pi} d\xi, \quad (24)$$

where we exploited the rotational covariance of  $\mathcal{N}_\lambda$ .

Here, we should remark that, if  $\lambda = 1$  and  $q(\xi) = \frac{\alpha}{\pi} \exp[-\alpha \|\xi\|^2]$  for  $\alpha \geq 0$ , the inequality in equation (23) is actually an equality. This is because, under such assumptions, we can choose an optimal  $\Omega$  to be Gaussian [9] and the PPT condition  $\Omega^\Gamma \geq 0$  is necessary and sufficient for a two-mode Gaussian state  $\Omega$  to be separable [32, 33]. The parameter  $(1/2\alpha)$  represents the variance of the distribution of states in phase space and can be expressed as an energy, in practical instances, by reintroducing the factor  $\hbar\omega_0$ , where  $\omega_0$  stands for the frequency of the considered mode.

In the remaining part of this subsection, we will transform the above infinite-dimensional SDP into a finite-dimensional SDP. We will need the following statement:

**Lemma 1 (operator norm bound).** *If a positive separable operator  $\Omega \in \mathfrak{B}(\mathcal{H} \otimes \mathcal{H})$  satisfies  $\text{Tr}_B \Omega = \mathbb{1}_A$ , then,  $\Omega$  also satisfies  $\|\Omega\|_\infty \leq 1$ , where  $\|\cdot\|_\infty$  is the operator norm.*

**Proof.** From the proof of theorem 3,  $\Omega$  can be written as  $\Omega(\Psi) = \int_{\mathcal{X}} M(dx) \otimes \sigma(x)$  by using a POVM  $\{M(dx)\}$  and a set of states  $\{\sigma(x)\}$ . Then, for any normalized state  $|\Phi\rangle$  on  $\mathcal{H} \otimes \mathcal{H}$ , we can bound  $\langle \Phi | \Omega | \Phi \rangle$  as follows:

$$\begin{aligned} \langle \Phi | \Omega | \Phi \rangle &= \int_{\mathcal{X}} \text{Tr}(M(dx) \otimes \sigma(x) |\Phi\rangle \langle \Phi|) \\ &\leq \int_{\mathcal{X}} \text{Tr}(M(dx) \rho_A) = 1, \end{aligned}$$

where  $\rho_A$  is defined as  $\text{Tr}_B(|\Phi\rangle \langle \Phi|)$  and we used that  $\sigma(x) \leq \mathbb{1}$  together with  $M(\mathcal{X}) = \mathbb{1}$ . Therefore, we have  $\|\Omega\|_\infty = \sup_{\Phi} \langle \Phi | \Omega | \Phi \rangle \leq 1$ .  $\square$

Notice that, even if  $\|\Omega\|_\infty \leq 1$  was shown for the corresponding operator of an entanglement breaking channel, this fact may not be true for other channels (in general we can only show that the geometric measure [34]  $G(\Omega(T)) \stackrel{\text{def}}{=} \sup_{\sigma \in \text{Sep}} \text{Tr}(\Omega(T)\sigma) \leq 1$ ).

Before we derive a finite SDP problem from equation (23), we observe the following crucial fact: by means of the action of the group integral over  $\{U_\theta \otimes U_\theta\}_{\theta \in [0, 2\pi]}$ ,  $\eta$  is block diagonalized as

$$\eta = \sum_{c=0}^{\infty} Q_c \left[ \int_{\xi \in \mathbb{R}^2} q(\xi) \mathcal{N}_\lambda(\rho_{s,\xi}) \otimes \rho_{s,\xi} d\xi \right] Q_c, \quad (25)$$

where  $Q_c \stackrel{\text{def}}{=} \sum_{k+l=c} |k\rangle \langle k| \otimes |l\rangle \langle l|$  is a finite-dimensional projector. Thus, defining  $P_c \stackrel{\text{def}}{=} \sum_{i=0}^c Q_i$ , we obtain  $\eta - P_c \eta P_c = \sum_{i=c+1}^{\infty} Q_i \eta Q_i \geq 0$  for all  $c$ . We arrive at the finite SDP upper bound now as follows: suppose a separable positive  $\Omega$  satisfies  $\text{Tr}_B \Omega = \mathbb{1}_A$ . Then,

$$\begin{aligned} \text{Tr}(\Omega \eta) &= \text{Tr}(\Omega P_c \eta P_c) + \text{Tr}(\Omega (\eta - P_c \eta P_c)) \\ &\leq \text{Tr}(\Omega P_c \eta P_c) + \|\Omega\|_\infty \text{Tr}(\eta - P_c \eta P_c) \\ &\leq \text{Tr}(\Omega P_c \eta P_c) + \text{Tr}(\eta - P_c \eta P_c) \\ &= \text{Tr}(\Omega P_c \eta P_c) - \text{Tr}(P_c \eta P_c) + 1, \end{aligned} \quad (26)$$

where we used the positivity of  $\eta - P_c \eta P_c$  and lemma 1.

Finally, by means of equation (26), we can derive an upper bound to the first line of equation (23). Denoting with  $R_c = (\sum_{i=0}^c |i\rangle \langle i|) \otimes (\sum_{i=0}^c |i\rangle \langle i|)$  the projection onto the subspace with a photon number smaller than  $c$  in each mode, with support  $\text{supp} R_c$ , we obtain

$$\begin{aligned} \bar{F} &\leq \sup_{\Omega \in \mathfrak{B}(\mathcal{H})} \left\{ \text{Tr}(\Omega P_c \eta P_c) \mid \Omega \in \text{Sep}, \text{Tr}_B \Omega = \mathbb{1}_A \right\} + 1 - \text{Tr}(P_c \eta P_c) \\ &= \sup_{\Omega \in \mathfrak{B}(\mathcal{H})} \left\{ \text{Tr}(R_c \Omega R_c P_c \eta P_c) \mid \Omega \in \text{Sep}, \text{Tr}_B \Omega = \mathbb{1}_A \right\} + 1 - \text{Tr}(P_c \eta P_c) \\ &\leq \sup_{\Omega \in \mathfrak{B}(\text{supp} R_c)} \left\{ \text{Tr}(\Omega P_c \eta P_c) \mid \Omega \in \text{Sep}, \text{Tr}_B \Omega \leq \mathbb{1}_A \right\} + 1 - \text{Tr}(P_c \eta P_c) \\ &\leq \sup_{\Omega \in \mathfrak{B}(\text{supp} R_c)} \left\{ \text{Tr}(\Omega P_c \eta P_c) \mid \Omega \geq 0, \Omega^\Gamma \geq 0, \text{Tr}_B \Omega \leq \mathbb{1}_A \right\} + 1 - \text{Tr}(P_c \eta P_c). \end{aligned} \quad (27)$$

The above upper bound now only involves a finite-dimensional SDP, as the original infinite-dimensional  $\Omega$  has been replaced by the finite-dimensional  $R_c \Omega R_c$ , with the same type of constraints (note that  $R_c$  was introduced to preserve the separability of the operator). Similarly, the term  $\text{Tr}(P_c \eta P_c)$  is just a trace of a finite-dimensional matrix, so that we can numerically compute every term of the last formula of equation (27) for as large a truncation parameter  $c$  as our computer allows. Thus, equation (27) enables us to efficiently calculate an upper bound for the quantum benchmark  $\bar{F}$  for any probability density  $q(\xi)$ , that is, ultimately, *for any rotationally-invariant input ensemble*. Moreover, since in the limit of large  $c$  we obtain the second formula of equation (23), we can expect that this upper bound reaches the exact value for this bound for sufficiently large  $c$ . Finally, we rephrase equation (27) in the form of a theorem:

**Theorem 4 (Benchmark—SDP).** *With the above definitions, for any probability density  $q(\xi)$  on  $\mathbb{R}^2$  and rotationally covariant noise channel  $\mathcal{N}$ , we have*

$$\bar{F} \leq \sup_{\Omega \in \mathfrak{B}(\text{supp } R_c)} \left\{ \text{Tr}(\Omega P_c \eta P_c) \middle| \Omega \geq 0, \Omega^\Gamma \geq 0, \text{Tr}_B \Omega \leq \mathbb{1}_A \right\} + 1 - \text{Tr}(P_c \eta P_c). \quad (28)$$

### 5.3. Results of numerical calculations

In this subsection, we present results of numerical calculations which were produced using inequality (28). In order to reduce the memory requirements, we further exploit the block diagonal structure of  $\eta$  in equation (25) and impose  $\sum_{j=0}^{2c} Q_j \Omega Q_j = \Omega$  for the implementation. The evaluation of the bound then splits into two terms:

$$\begin{aligned} \bar{F}_{\text{finite}} &\stackrel{\text{def}}{=} \sup_{\Omega \in \mathfrak{B}(\text{supp } R_c)} \left\{ \text{Tr}(\Omega P_c \eta P_c) \middle| \Omega \geq 0, \Omega^\Gamma \geq 0, \text{Tr}_B \Omega \leq \mathbb{1}_A \right\}, \\ \epsilon_{\text{error}} &\stackrel{\text{def}}{=} 1 - \text{Tr}(P_c \eta P_c). \end{aligned}$$

Denoting the rhs of (28) by  $\bar{F}_{\text{infinite}}$  we have

$$\bar{F}_{\text{infinite}} = \bar{F}_{\text{finite}} + \epsilon_{\text{error}}. \quad (29)$$

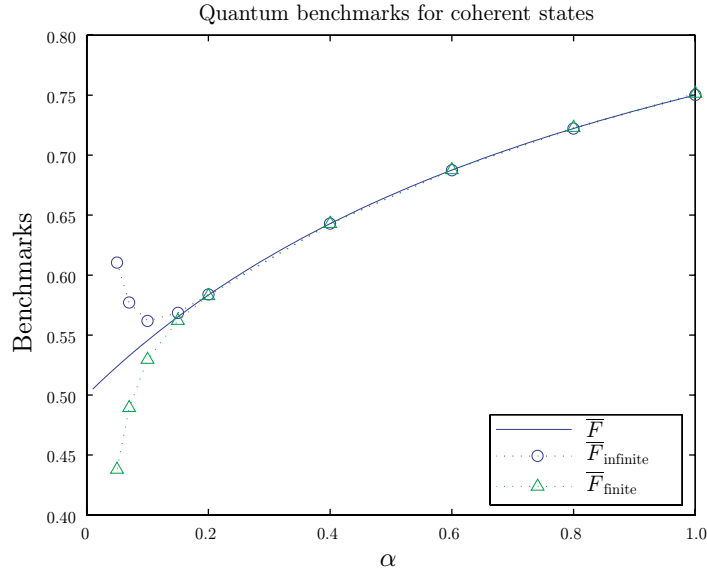
Our computation proceeds along the following steps:

- (i) Compute matrix elements of  $\langle k_1 | \mathcal{N}_\lambda(\rho_{s,\xi}) | l_1 \rangle$  and  $\langle k_2 | \rho_{s,\xi} | l_2 \rangle$  for all  $k_1, k_2, l_1, l_2$  satisfying  $k_1 + k_2 \leq c$  and  $l_1 + l_2 \leq c$ ;  $c$  is a fixed maximum photon number practically bounded by the computer memory. For the calculation, we used an analytical formula for matrix elements of single-mode Gaussian states as a *finite sum* over Hermite polynomials (see equation (4.10) of [35]).
- (ii) Evaluate the following integral over  $\xi$

$$\int_{\xi \in \mathbb{R}^2} q(\xi) \langle k_1 | \mathcal{N}_\lambda(\rho_{s,\xi}) | l_1 \rangle \langle k_2 | \rho_{s,\xi} | l_2 \rangle d\xi,$$

by quasi-Monte Carlo with the Halton sequence [36].

- (iii) From equation (25), we derive  $P_c \eta P_c$ .
- (iv) Finally, we solve the SDP leading to  $\bar{F}_{\text{infinite}}$ .



**Figure 2.** A plot comparing the known optimal classical bound  $\bar{F} = \frac{2\alpha+1}{2(\alpha+1)}$ , to the upper bound  $\bar{F}_{\text{infinite}}$  derived by our numerical calculation truncated at  $c = 35$  Fock states (with corrected truncation error), and the result of the finite SDP  $\bar{F}_{\text{finite}}$  (without corrected truncation error) for an ensemble of coherent states, no noise ( $\lambda = 1$ ), and a distribution of displacements  $q(\xi) = \frac{\alpha}{\pi} \exp[-\alpha \|\xi\|^2]$ .

Here, all the above numerical calculations were implemented in Matlab, and the SDP is solved using the Matlab toolbox SeDuMi version 1.1 [37]. In the above numerical calculation, we do not use any approximation except for the quasi-Monte Carlo integral.

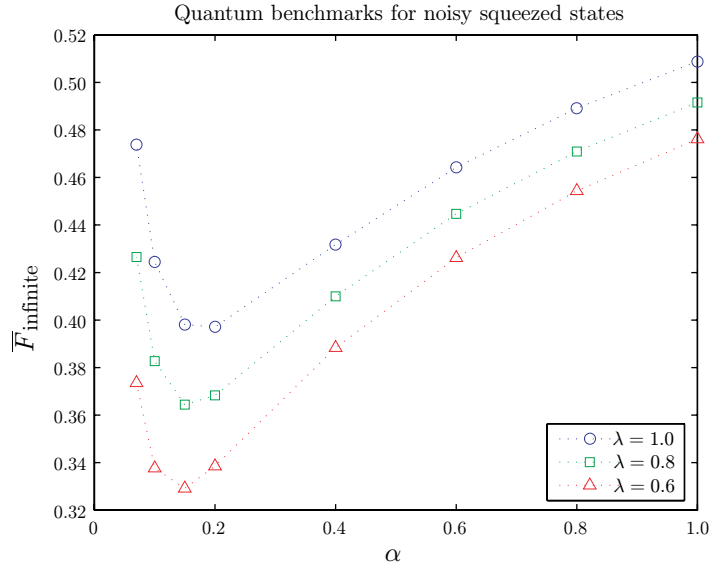
As a first application, in order to provide the reader with a convincing test for our numerical calculations, we show that our method reproduces the result of the optimal quantum benchmark for displaced coherent states derived in [9]. Suppose  $\lambda = s = 1$  and the input ensemble is distributed according to  $q(\xi) = \frac{\alpha}{\pi} \exp[-\alpha \|\xi\|^2]$ . That is we deal with an ensemble of coherent states whose centres are distributed according to a Gaussian distribution with variance  $1/(2\alpha)$ . Hence, using the invariance of the ensemble under rotations we obtain

$$\eta = \int_{\xi \in \mathbb{R}^2} \frac{\alpha}{\pi} \exp[-\alpha \|\xi\|^2] \rho_{1,\xi} \otimes \rho_{1,\xi} d\xi. \quad (30)$$

The quantum benchmark  $\bar{F}$  in this case was shown [9] to be (note that our definition of the parameter  $\alpha$  is different by the factor 2 from the definition in [9])

$$\bar{F} = \frac{2\alpha+1}{2(\alpha+1)}. \quad (31)$$

So we can compare our upper bound  $\bar{F}_{\text{infinite}}$  derived by SDP with the optimal bound, which is shown in figure 2 for the maximum photon number  $c = 35$ . As expected the numerics satisfies  $\bar{F}_{\text{infinite}} > \bar{F} > \bar{F}_{\text{finite}}$  but we observe in figure 2 that as long as  $\alpha \geq 0.2$ , the values of both  $\bar{F}_{\text{infinite}}$  and  $\bar{F}_{\text{finite}}$  are essentially indistinguishable from the optimal bound  $\bar{F}$ , whereas for  $\alpha \leq 0.2$  we find noticeable differences. In other words, in equation (29) the term  $\epsilon_{\text{error}}$ , which originates from the truncation of the dimension, becomes dominant in this region. As a result,



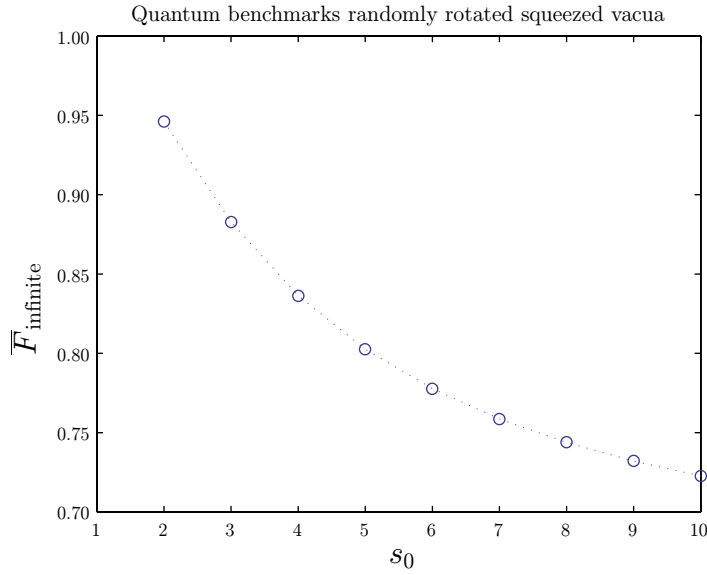
**Figure 3.** Results of numerical calculations of the quantum benchmark  $\bar{F}_{\text{infinite}}$  for an ensemble of noisy squeezed states displaced according to a Gaussian probability distribution  $q(\xi) = \frac{\alpha}{\pi} \exp[-\alpha \|\xi\|^2]$ , as a function of  $\alpha$ . The Fock space was truncated at  $c = 30$  states, the squeezing parameter is  $s = 8$  and loss factors are  $(1 - \lambda) = 0.4, 0.2, 0$  for, respectively, triangles (red), squares (green) and circles (blue).

the minimum value of  $\bar{F}_{\text{infinite}}$  is around  $\alpha = 0.15$ , i.e.  $\bar{F}_{\text{infinite}}$  is not monotonically decreasing with respect to  $\alpha$ .

From this result, we may expect that our upper bounds are also almost optimal in other situations (different  $q, s, \mathcal{N}$ ) as long as  $\epsilon_{\text{error}}$  is small with respect to  $\bar{F}_{\text{finite}}$ .

Next, we consider the case of an ensemble of noisy squeezed states with a fixed squeezing parameter  $s$  and again a Gaussian prior distribution for  $\xi$ . We show the results of our calculation in figure 3. Here, we chose  $c = 30$ ,  $s = 8$ , and plot for  $\lambda = 0.6, 0.8, 1$ . In this figure, we can observe that, as expected,  $\bar{F}_{\text{infinite}}$  decreases with decreasing  $\lambda$ . Due to the increasing contribution of  $\epsilon_{\text{error}}$  for decreasing  $\alpha$ , the best (lowest) values of our benchmarks are achieved, under realistic losses, for  $\alpha \simeq 0.15$ . This value, though maybe non-optimal, is suitable for comparisons with realistic experimental situations. A simple, preliminary analysis of possible experimental noise conditions indicates that the values of the benchmark for such parameters could be beaten by forthcoming experiments aimed at the teleportation or storage of squeezed states. Let us mention that single states have already been teleported or stored, both with light modes [15, 16] and with atomic memories [17, 18]. The experimental demonstration of the transmission of a full ensembles of squeezed states is yet to come, but techniques are ripe for it to be realized in both settings: our method is ready for the analysis of such developments by direct comparison.

Finally, we consider the simple case of an ensemble of randomly rotated squeezed vacuum states without displacement in phase space. That is, we choose  $q(\xi) = \delta(\|\xi\|^2)$ , shown in figure 4. Here, we chose  $c = 30$  and plot for  $2 \leq s \leq 10$ . Note that for  $s = 1$  there is only one state in the ensemble, the vacuum state, so that  $\bar{F} = 1$  in this case. Figure 4 clearly shows that, if the displacements in phase space are not randomized, the classical benchmarks we derived



**Figure 4.** The results for the quantum benchmark  $\bar{F}_{\infty}$  for an ensemble of uniformly rotated squeezed vacuum states with no displacements applied, i.e. for  $q(\xi) = \delta(\|\xi\|^2)$ . The Fock space was truncated at  $c = 30$  states and the states were assumed to be noiseless ( $\lambda = 1$ ).

increase significantly and beating them in current experiments would be a daunting challenge: distributing the displacements of the input ensemble is thus definitely, at present, a technical necessity in order to achieve a certifiable quantum performance<sup>7</sup>.

## 6. Summary and conclusions

In this work, we have derived upper bounds on the performance of classical ‘measure and prepare’ protocols for the quantum storage and teleportation of ensembles of squeezed states of continuous variable systems. These bounds may be employed as benchmarks to discern truly quantum mechanical performances in experiments aimed at the realization of such processes. Motivated by currently available experimental capabilities and concrete setups, we have considered ensembles comprised of states with fixed squeezing, but allowed for the possibility of random rotations and translations of those states in phase space.

Fully analytical benchmarks have been obtained for pure squeezed states as well as mixed states obtained from the application of additive thermal noise, for the case of completely unknown rotations and translations in phase space. In these ideal cases, the benchmarks decrease monotonically with increasing degree of squeezing  $s$ , vanishing for  $s \rightarrow \infty$ : in this sense, increasing the squeezing reveals the infinite-dimensional character of the Hilbert space. Furthermore, we have presented a numerical technique that first reduces the problem to a finite-dimensional setting with rigorous error bounds, and then allows for the numerical solution of

<sup>7</sup> We mention that an analytical upper bound to a fidelity benchmark for an ensemble of squeezed states with distributed displacements but *fixed phase of squeezing*, i.e. with no random rotations, has been derived in [38]. These bounds in a sense complement our numerical findings for non-displaced states.

the remaining finite-dimensional problem using semidefinite programming. The so obtained bounds are rigorous and this approach may be applied to more general settings, even beyond the Gaussian regime, for variable squeezing and arbitrarily distributed displacements, with the only proviso that the distribution over rotations should be still rotationally symmetric in phase space, to maintain covariance under rotations (which was essential in deriving rigorous bounds for the truncated problem). Let us also emphasize that our numerical strategy can be reliably applied to finite sets of states as well (rather than to continuous distributions), which are what, strictly speaking, is sampled in actual experiments.

Our numerical results strongly indicate that allowing for randomized displacements is crucial to outperform optimal classical strategies with current quantum technologies and that, on the other hand, if such a randomization is allowed, realistic setups might be able to enter the quantum regime with squeezed states. Likewise, randomized phases, other than providing much more ‘appealing’ evidence for the teleportation or storage of the states (since, in such a case, states with a varying structure of second moments would be transmitted), seem to be needed for the benchmarks to be beaten.

Overall, the results presented here will provide a useful resource for assessing the quantum mechanical character of experiments aimed at demonstrating quantum storage and transmission of squeezed states. In a future work, these methods will be applied to concrete experiments.

## Acknowledgments

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*Note added.* We acknowledge that, very recently (after the completion of this study and during the writing of the present paper), another work appeared ([12]), where benchmarks based on the Uhlmann fidelity for rotationally covariant ensembles were studied, and the reduction of the benchmark estimation to an SDP was independently derived.

## Appendix A. Phase-space covariant channels

In this appendix, we collect useful results about ‘phase-space covariant’ channels, i.e. channels covariant under the action of the Weyl–Heisenberg group of displacement operators  $\{W_\xi, \xi \in \mathbb{R}^{2n}\}$ . For a phase-space covariant channel one has

$$T(W_\xi^\dagger \hat{O} W_\xi) = W_\xi^\dagger T(\hat{O}) W_\xi, \quad \forall \hat{O} \in \mathfrak{B}(L_2(\mathbb{R})^{\otimes n}) \quad \text{and} \quad \forall \xi \in \mathbb{R}^{2n}.$$

[ $\mathfrak{B}(L_2(\mathbb{R})^{\otimes n})$  being the set of bounded linear operators on  $n$  copies of the bosonic Hilbert space.] These results were already used (though not explicitly detailed) in [25, 26]. Let us mention that Markovian master equations giving rise to phase-space covariant channels were characterized in [39], whereas the classical capacity and output entropies of phase-space covariant channels were addressed in [40]. The latter also contains an alternative proof of lemma 4.

Central to the present analysis is the characterization of the class of ‘linear bosonic channels’ in terms of their action on Weyl operators, given in [41], which we report here without

proof in the form of the following lemma. Recall that  $T^*$  stands for the operation  $T$  in the Heisenberg picture.

**Lemma 2 (Linear bosonic channels).** A map  $T^*(W_\xi) := f(\xi)W_{X\xi}$  is a quantum channel iff  $f$  is the quantum characteristic function with respect to a modified symplectic form  $\tilde{\sigma} := \sigma - X^T \sigma X$ . That is,  $f: \mathbb{R}^{2n} \rightarrow \mathbb{C}$  has to be continuous,  $f(0) = 1$  and every matrix with entries

$$M_{kl} = f(\xi^{(k)} - \xi^{(l)}) \exp \left[ \frac{i}{2} \xi^{(k)} \cdot \tilde{\sigma} \xi^{(l)} \right] \quad (\text{A.1})$$

has to be positive semidefinite for all  $\xi^{(k)}, \xi^{(l)} \in \mathbb{R}^{2n}$ .

Note that condition (A.1) for  $\tilde{\sigma} = \sigma$  ( $\tilde{\sigma} = 0$ ) is equivalent to  $f$  being a quantum (classical) characteristic function.

The following characterization of phase-space covariance ensues:

**Lemma 3 (phase-space covariant channels).** Every phase-space covariant channel is uniquely characterized by a classical characteristic function  $f$  and acts in the Heisenberg picture as

$$T^*(W_\xi) = f(\xi)W_\xi. \quad (\text{A.2})$$

**Proof.** Consider the action of  $T^*$  on a Weyl operator  $W_\xi$ . Exploiting the Weyl relations, phase-space covariance (with respect to any  $W_\eta$ ) leads to

$$W_\eta T^*(W_\xi) = e^{i\xi \cdot \sigma \eta} T^*(W_\xi) W_\eta. \quad (\text{A.3})$$

It is straightforward to check that this entails

$$[T^*(W_\xi)W_\xi^\dagger, W_\eta] = 0, \quad \forall \xi, \eta \in \mathbb{R}^{2n}. \quad (\text{A.4})$$

The irreducibility of the Weyl system then implies  $T^*(W_\xi)W_\xi^\dagger \propto \mathbb{1}$ . Denoting the proportionality constant by  $f(\xi)$ , we have a map of the form in lemma 2 with  $X = \mathbb{1}$ . Hence,  $f$  has to be a classical characteristic function.  $\square$

Let us denote by  $\vartheta$  the time reversal (or matrix transposition) operator. Every entanglement breaking channel, i.e. ‘measure-and-prepare scheme’ (see [23]), is such that  $T \circ \vartheta$  is completely positive (that is, the Choi matrix—or the Jamiolkowski state—has positive partial transpose). Moreover, time reversal acts very simply in phase space by flipping the sign of one of the two canonical quadratures (see, e.g. [32]):  $\vartheta(W_\xi) = W_{Z\xi}$ , where  $Z = \bigoplus_{j=1}^n \sigma_z$  ( $\sigma_z$  being the Pauli  $z$  matrix). We will now apply this additional constraint to phase-space covariant channels, in order to achieve a stronger characterization:

**Lemma 4 (NPT-entanglement breaking channels).** A phase-space covariant channel  $T$  is such that  $T \circ \vartheta$  is completely positive iff it has the form

$$T^*(W_\xi) = f\left(\xi/\sqrt{2}\right)W_\xi, \quad (\text{A.5})$$

where  $f$  is any quantum characteristic function, i.e. there is a density operator  $\tau \in \mathfrak{C}_1(L_2(\mathbb{R})^{\otimes n})$  such that  $f(\xi) = \text{Tr}[\tau W_\xi]$ .

**Proof.** Using lemma 3 we obtain

$$\vartheta \circ T^*(W_\xi) = f(\xi)\vartheta(W_\xi) = f(\xi)W_{Z\xi}, \quad (\text{A.6})$$

where  $Z$  reverses the momenta so that  $Z^T \sigma Z = -\sigma$ . Following lemma 2 this map is completely positive if and only if positivity of equation (A.1) holds for  $\tilde{\sigma} = 2\sigma$ . The result follows then by substituting  $\xi \rightarrow \xi/\sqrt{2}$ .  $\square$

## Appendix B. Proof of theorem 3

**Proof.** Suppose  $\Psi$  is an entanglement breaking channel and can be written [22, 23] as  $\Psi(\rho) = \int_{\mathcal{X}} \text{Tr}(M(dx)\rho)\sigma(x)$  for all  $\rho \in \mathfrak{C}_1(\mathcal{H})$ , where  $\sigma(x) \in \mathfrak{C}_1(\mathcal{H})$  are states and  $M$  is a POVM, i.e.  $M(X) \geq 0$  for all Borel subsets  $X$  of a complete separable metric space  $\mathcal{X}$  for which  $M(\mathcal{X}) = \mathbb{1}$ .

Then,  $\Omega(\Psi) = \int_{\mathcal{X}} M(dx) \otimes \sigma(x)$  satisfies  $\text{Tr}_B(\Omega(\Psi)) = \mathbb{1}_A$  and equation (22). Moreover, suppose there exists a  $\Omega'(\Psi)$  satisfying  $\text{Tr}_B(\Omega'(\Psi)) = \mathbb{1}_A$  and equation (22). Then, since  $\text{Tr}(\Omega'(\Psi)|i\rangle\langle j| \otimes |k\rangle\langle l|) = \text{Tr}(|k\rangle\langle l|\Psi(|i\rangle\langle j|)) = \text{Tr}(\Omega(\Psi)|i\rangle\langle j| \otimes |k\rangle\langle l|)$  for an orthonormal basis  $\{|i\rangle\}_i$  of  $\mathcal{H}$ , we immediately have  $\Omega' = \Omega$ .

Conversely, suppose there exists a separable positive bounded operator  $\Omega$  on  $\mathcal{H} \otimes \mathcal{H}$  satisfying  $\text{Tr}_B(\Omega) = \mathbb{1}_A$  and can be written as  $\Omega = \int |\Phi\rangle\langle\Phi| \otimes |\varphi\rangle\langle\varphi| \mu(d\Phi d\varphi)$  with measure  $\mu$ . We define a channel  $\Psi$  as  $\rho \rightarrow \int \text{Tr}(M(d\varphi)\rho)|\varphi\rangle\langle\varphi|$ , where  $M(d\varphi) \stackrel{\text{def}}{=} \int_{\Phi} |\Phi\rangle\langle\Phi| \mu(d\Phi d\varphi)$ . Then, since  $\text{Tr}_B(\Omega) = \mathbb{1}_A$ ,  $M$  is a POVM; that is,  $\Psi$  is an entanglement breaking channel. Evidently,  $\Psi$  satisfies equation (22). Moreover, suppose an entanglement breaking channel  $\Psi'$  also satisfies equation (22). Then, since  $\text{Tr}(B\Psi'(A)) = \text{Tr}(\Omega(\Psi)A \otimes B) = \text{Tr}(B\Psi(A))$  for all  $B \in \mathfrak{B}(\mathcal{H})$ ,  $\Psi'(A) = \Psi(A)$  for all  $A \in \mathfrak{C}_1(\mathcal{H})$ .  $\square$

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