REVIEW

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Review

Horndeski theory and beyond: a review

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Abstract
This article is intended to review the recent developments in the Horndeski theory and its generalization, which provide us with a systematic understanding of scalar–tensor theories of gravity as well as a powerful tool to explore astrophysics and cosmology beyond general relativity. This review covers the generalized Galileons, (the rediscovery of) the Horndeski theory, cosmological perturbations in the Horndeski theory, cosmology with a violation of the null energy condition, degenerate higher-order scalar–tensor theories and their status after GW170817, the Vainshtein screening mechanism in the Horndeski theory and beyond, and hairy black hole solutions.

Keywords: modified gravity, dark energy, inflation

1. Introduction

1.1. Modified gravity: why?

General relativity is doubtlessly a very successful theory, serving as the standard model of gravity. Nonetheless, modified theories of gravity have been explored actively for several reasons.

Probably the most major reason in recent years arises from the discovery of the accelerated expansion of the present universe [1, 2]. This may be caused by the (extremely fine-tuned) cosmological constant, but currently it would be better to have other possibilities at hand, and a long-distance modification of general relativity is one such possible alternative. Turning to the accelerated expansion of the early universe, it is quite likely that some scalar field, called the inflaton, provoked inflation [3–5] and there are a number of models in which the inflaton field is coupled non-minimally to gravity. Such inflation models are studied within the context of modified gravity.

In order to test gravity, we need to know predictions of theories other than general relativity. This motivation is becoming increasingly important after the first detection of gravitational waves [6]. In view of this, modified gravity is worth studying even if general relativity should turn out to be the correct (low-energy effective) description of gravity in the end.

Aside from phenomenology, pursuing consistent modifications of gravity helps us to learn more deeply about general relativity and gravity. For example, by trying to develop massive gravity one can gain a deeper understanding of general relativity and see how special a massless graviton is. Similarly, by studying gravity in higher (or lower) dimensions one can clarify how special gravity in four dimensions is. This motivation justifies us to study modified gravity even if we are driven by academic interest.

Finally, one should bear in mind that general relativity is incomplete anyway as a quantum theory and hence needs to be modified in the UV, though this subject is beyond the scope of this review.

1.2. Modified gravity: how?

Having presented some motivations, let us move to explaining how one can modify general relativity. According to Lovelock's theorem [7, 8], the Einstein equations (with a cosmological constant) are the only possible second-order Euler–Lagrange equations derived from a Lagrangian scalar density in four dimensions that is constructed solely from the metric $\mathcal{L} = \mathcal{L}[g_{\mu\nu}]$. To extend Einstein's theory of gravity, one needs to relax the assumptions of Lovelock's theorem. The simplest way would be just adding a new degree of freedom other than the metric, such as a scalar field. Higher-dimensional gravity may be described by an effective scalar–tensor theory in four dimensions via a dimensional reduction. Incorporating higher derivatives may lead to a pathological theory (as will be argued shortly) or something that can be recast in a scalar–tensor theory (e.g. R^2 gravity). Abandoning diffeomorphism invariance is also equivalent to introducing new degrees of freedom. Thus, modifying gravity amounts to changing the degrees of freedom in any case. In particular, many different theories of modified gravity can be described at least effectively by some additional scalar degree(s) of freedom on top of the usual two tensor degrees corresponding to gravitational waves. We therefore focus on *scalar–tensor theories* in this review.

1.3. Ostrogradsky instability

One of the guiding principles we follow when we seek for a 'healthy' extension of general relativity is to avoid what is called the Ostrogradsky instability [9, 10]. The theorem states that a system described by a non-degenerate higher-derivative Lagrangian suffers from ghost-like instabilities. We will demonstrate this below by using a simple example in the context of mechanics.

Let us consider the following Lagrangian involving a second derivative:

$$L = \frac{a}{2}\ddot{\phi}^2 - V(\phi), \tag{1}$$

where $a\,(\neq 0)$ is a constant and $V(\phi)$ is an arbitrary potential. The Euler–Lagrange equation derived from (1) is of fourth order: $a\ddddot{\phi} - \mathrm{d}V/\mathrm{d}\phi = 0$. To solve this we need four initial conditions, which means that we have in fact two dynamical degrees of freedom. According to the Ostrogradsky theorem, one of them must be a ghost. This can be seen as follows. By introducing an auxiliary variable, the Lagrangian (1) can be written equivalently as

$$\begin{aligned} L &= a\psi\ddot{\phi} - \frac{a}{2}\psi^2 - V(\phi) \\ &= -a\dot{\psi}\dot{\phi} - \frac{a}{2}\psi^2 - V(\phi) + a\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi\dot{\phi}\right). \end{aligned} \tag{2}$$

It is easy to see that the first line reproduces the original Lagrangian (1) after substituting the Euler–Lagrange equation for ψ, namely, $\psi = \ddot{\phi}$. The last term in the second line does not contribute to the Euler–Lagrange equation. In terms of the new variables defined as $q = (\phi + \psi)/\sqrt{2}$ and $Q = (\phi - \psi)/\sqrt{2}$, the Lagrangian (2) can be rewritten (up to a total derivative) in the form

$$L = -\frac{a}{2}\dot{q}^2 + \frac{a}{2}\dot{Q}^2 - U(q, Q). \tag{3}$$

This Lagrangian clearly shows that the system contains two dynamical degrees of freedom, one of which has a wrong sign kinetic term, signaling ghost instabilities. This is true irrespective of the sign of a.

Although we have seen the appearance of the Ostrogradsky instability in higher-derivative systems only through the above simple example, this is generically true in higher-derivative field theory. The theorem can be extended to the systems with third-order equations of motion [11]. In this review, we will therefore consider scalar–tensor modifications of general relativity that have second-order field equations. The most general form of the Lagrangian for the scalar–tensor theory having second-order field equations is known as the Horndeski theory [12], and it has been widely used in cosmology and astrophysics beyond general relativity in recent years.

An important postulate of the Ostrogradsky theorem is that the Lagrangian is non-degenerate. If this is not the case, one can reduce a set of higher-derivative field equations to a healthy second-order system. This point will also be discussed in the context of scalar–tensor theories.

1.4. Structure of the review

The outline of this article is as follows.

In the next section, we review aspects of the Horndeski theory, the most general scalar–tensor theory with second-order equations of motion. It is shown that the original form of the Horndeski action is indeed equivalent to its modern form frequently used in the literature (i.e. the generalized Galileons). A short status report is also given on the attempt to extend the Horndeski theory to allow for multiple scalar fields.

We then present some applications of the Horndeski theory to cosmology in section 3.

Scalar–tensor theories that are more general than Horndeski necessarily have higher-order equations of motion. Nevertheless, one can circumvent the Ostrogradsky instability if the system is degenerate, as argued above. This idea gives rise to new healthy scalar–tensor theories beyond Horndeski. We review the recent developments in this direction in section 4. The nearly simultaneous detection of gravitational waves GW170817 and the γ-ray burst GRB 170817A places a very tight constraint on the speed of gravitational waves. We mention the status of the Horndeski theory and its extensions after this event.

The Vainshtein screening mechanism is essential for modified gravity evading solar system constraints. In section 5, we describe this mechanism based on the Horndeski theory and theories beyond Horndeski, emphasizing in particular the interesting phenomenology of partial breaking of Vainshtein screening inside matter in degenerate higher-order scalar–tensor theories.

Black hole solutions in the Horndeski theory and beyond are summarized very briefly in section 6.

Finally, we draw our conclusion in section 7.

This review only covers scalar–tensor theories. For a more comprehensive review including other types of modified gravity, see [13, 14]. We will not describe much about cosmological tests of gravity, which are covered by excellent reviews such as [15–17].

2. Horndeski theory

2.1. From Galileons to Horndeski theory

To introduce the Horndeski theory in a pedagogical manner, we start from the Galileon theory (see also [18] for a review on the same subject). The Galileon [19] is a scalar field with the symmetry under the transformation $\phi \to \phi + b_\mu x^\mu + c$. This is called, by an analogy to a Galilei transformation in classical mechanics, the Galilean shift symmetry. In order to avoid ghost instabilities, we demand that ϕ's equation of motion is of second order. The most general Lagrangian (in four dimensions) having these properties is given by [19]

$$\begin{aligned}\mathcal{L} = c_1\phi + c_2X - c_3X\Box\phi + \frac{c_4}{2}\{X[(\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi] \\ +\Box\phi\partial^\mu\phi\partial^\nu\phi\partial_\mu\partial_\nu\phi - \partial_\mu X\partial^\mu X\} + \frac{c_5}{15}\{-2X[(\Box\phi)^3 \\ -3\Box\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi] \\ +3\partial^\mu\phi\partial_\mu X[(\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi] \\ +6\Box\phi\partial_\mu X\partial^\mu X - 6\partial^\mu\partial^\nu\phi\partial_\mu X\partial_\nu X\},\end{aligned} \tag{4}$$

where $X := -\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$ and $c_1, \cdots, c_5$ are constants. This can be written in a more compact form by making use of integration by parts as

$$\begin{aligned}\mathcal{L} = c_1\phi + c_2X - c_3X\Box\phi + c_4X[(\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi] \\ -\frac{c_5}{3}X[(\Box\phi)^3 - 3\Box\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi].\end{aligned} \tag{5}$$

Note that the field equation is of second order even though the Lagrangian depends on the second derivatives of the field.

The Lagrangian (5) describes a scalar field theory on a fixed Minkowski background. One can introduce gravity and consider a covariant version of (5) by promoting $\eta_{\mu\nu}$ to $g_{\mu\nu}$ and ∂_μ to ∇_μ. However, since covariant derivatives do not commute, the naive covariantization leads to higher derivatives in the field equations, which would be dangerous. For example, one would have derivatives of the Ricci tensor $R_{\mu\nu}$ from the term having the coefficient c_4,

$$c_4X\nabla^\mu[\nabla_\mu\nabla_\nu\nabla^\nu\phi - \nabla_\nu\nabla_\mu\nabla^\nu\phi] = -c_4X\nabla^\mu(R_{\mu\nu}\nabla^\nu\phi), \tag{6}$$

in the scalar field equation of motion. Such higher derivative terms can be canceled by adding curvature-dependent terms appropriately to equation (5). The covariant version of (5) that leads to second-order field equations both for the scalar field and the metric is given by [20]

$$\begin{aligned}\mathcal{L} = c_1\phi + c_2X - c_3X\Box\phi + \frac{c_4}{2}X^2R + c_4X[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}] \\ + c_5X^2G^{\mu\nu}\phi_{\mu\nu} - \frac{c_5}{3}X[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^\mu_\lambda],\end{aligned} \tag{7}$$

where R is the Ricci tensor, $G_{\mu\nu}$ is the Einstein tensor, $\phi_\mu := \nabla_\mu\phi$, $\phi_{\mu\nu} := \nabla_\mu\nabla_\nu\phi$ and now $X := -g^{\mu\nu}\phi_\mu\phi_\nu/2$. Here, the fourth term in the first line and the first term in the second line are the 'counter terms' introduced to remove higher derivatives in the field equations. The counter terms are unique. This theory is called the covariant Galileon. Since the field equations derived from the Lagrangian (7) involve first derivatives of ϕ, the Galilean shift symmetry is broken in the covariant Galileon theory. Only the second property of the Galileon, i.e. the second-order nature of the field equations, is maintained in the course of covariantization. The covariant Galileon theory (7) is formulated in four spacetime dimensions, but it can be extended to arbitrary dimensions [21].

The generalized Galileon [22] is a further generalization of the covariant Galileon [20, 21] retaining second-order field equations. More precisely, first one determines the most general scalar field theory on a fixed Minkowski background which yields a second-order field equation, assuming that the Lagrangian contains at most second derivatives of ϕ and is polynomial in $\partial_\mu\partial_\nu\phi$. One then promotes the theory to a covariant one in the same way as above by adding appropriate (unique) counter terms so that the field equations are of second order both for ϕ and the metric. The generalized Galileon can thus be obtained. It should be noted that this procedure can be done in arbitrary spacetime dimensions. In four dimensions, the Lagrangian for the generalized Galileon is given by [22]

$$\begin{aligned}\mathcal{L} = G_2(\phi,X) - G_3(\phi,X)\Box\phi + G_4(\phi,X)R + G_{4X}[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}] \\ + G_5(\phi,X)G^{\mu\nu}\phi_{\mu\nu} - \frac{G_{5X}}{6}[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^\mu_\lambda],\end{aligned} \tag{8}$$

where G_2, G_3, G_4, and G_5 are arbitrary functions of ϕ and X. Here and hereafter we use the notation $f_X := \partial f/\partial X$ and $f_\phi := \partial f/\partial\phi$ for a function f of ϕ and X.

The generalized Galileon (8) is now known as the Horndeski theory [12], i.e. *the most general scalar–tensor theory having second-order field equations in four dimensions*. However, Horndeski determined the theory starting from different assumptions than those made for deriving the generalized Galileon, and the original form of the Lagrangian [12] looks very different from (8):

$$\begin{aligned}\mathcal{L} = \delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\Big[\kappa_1\phi^\mu_\alpha R_{\beta\gamma}{}^{\nu\sigma} + \frac{2}{3}\kappa_{1X}\phi^\mu_\alpha\phi^\nu_\beta\phi^\sigma_\gamma + \kappa_3\phi_\alpha\phi^\mu R_{\beta\gamma}{}^{\nu\sigma} \\ +2\kappa_{3X}\phi_\alpha\phi^\mu\phi^\nu_\beta\phi^\sigma_\gamma\Big] \\ +\delta^{\alpha\beta}_{\mu\nu}\Big[(F+2W)R_{\alpha\beta}{}^{\mu\nu} + 2F_X\phi^\mu_\alpha\phi^\nu_\beta + 2\kappa_8\phi_\alpha\phi^\mu\phi^\nu_\beta\Big] \\ -6(F_\phi + 2W_\phi - X\kappa_8)\Box\phi + \kappa_9.\end{aligned} \tag{9}$$

Here, $\delta^{\alpha_1\alpha_2\ldots\alpha_n}_{\mu_1\mu_2\ldots\mu_n} := n!\delta^{[\alpha_1}_{\mu_1}\delta^{\alpha_2}_{\mu_2}\ldots\delta^{\alpha_n]}_{\mu_n}$ is the generalized Kronecker delta, and κ_1, κ_3, κ_8, and κ_9 are arbitrary functions of ϕ and X. We have another function $F = F(\phi, X)$, but this must satisfy $F_X = 2(\kappa_3 + 2X\kappa_{3X} - \kappa_{1\phi})$ and hence is not independent. We also have a function of ϕ, $W = W(\phi)$, which can be absorbed into the redefinition of F: $F_{\rm old} + 2W \to F_{\rm new}$. Thus, we have the same number of free functions of ϕ and X as in the generalized Galileon theory. Nevertheless, the equivalence between the two theories is apparently far from trivial.

In [23] it was shown that the generalized Galileon can be mapped to the Horndeski theory by identifying $G_i(\phi, X)$ as

$$G_2 = \kappa_9 + 4X\int^X {\rm d}X'(\kappa_{8\phi} - 2\kappa_{3\phi\phi}), \tag{10}$$

$$G_3 = 6F_\phi - 2X\kappa_8 - 8X\kappa_{3\phi} + 2\int^X {\rm d}X'(\kappa_8 - 2\kappa_{3\phi}), \tag{11}$$

$$G_4 = 2F - 4X\kappa_3, \tag{12}$$

$$G_5 = -4\kappa_1, \tag{13}$$

and performing integration by parts. Having thus proven that the generalized Galileon is indeed equivalent to the Horndeski theory, we can now use (8) as the Lagrangian for the most general scalar–tensor theory with second-order field equations. However, while the generalized Galileon is formulated in arbitrary dimensions, the higher-dimensional extension of the Horndeski theory has not been known so far and it is unclear whether or not the generalized Galileon gives the most general second-order scalar–tensor theory in higher dimensions. Note in passing that the lower-dimensional version of the Horndeski theory can be obtained straightforwardly [12].

The Horndeski theory was obtained already in 1974, but the paper [12] had long been forgotten until 2011 when it was rediscovered by [24]. Let us sketch (very briefly) the original derivation of the Horndeski theory. The starting point is a generic action of the form

$$S = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}(g_{\mu\nu}, g_{\mu\nu,\lambda_1}, \cdots, g_{\mu\nu,\lambda_1,\cdots,\lambda_p}, \phi, \phi_{,\lambda_1}, \cdots, \phi_{,\lambda_1,\cdots,\lambda_q}), \tag{14}$$

with $p, q \geqslant 2$ *in four dimensions*. The assumptions here should be contrasted with those in the generalized Galileon: Horndeski's derivation starts from the more general Lagrangian, but it is restricted to four dimensions. Varying the action with respect to the metric and the scalar field, we obtain the field equations: $\mathcal{E}_{\mu\nu}\left(:= 2(\sqrt{-g})^{-1}\delta S/\delta g^{\mu\nu}\right) = 0$ and $\mathcal{E}_\phi\left(:= (\sqrt{-g})^{-1}\delta S/\delta\phi\right) = 0$, where $\mathcal{E}_{\mu\nu}$ and $\mathcal{E}_\phi$ are assumed to involve at most second derivatives of $g_{\mu\nu}$ and ϕ. As a consequence of the diffeomorphism invariance of the action, the following 'Bianchi identity' holds:

$$\nabla^\nu\mathcal{E}_{\mu\nu} = -\nabla_\mu\phi\,\mathcal{E}_\phi. \tag{15}$$

In general, $\nabla^\nu\mathcal{E}_{\mu\nu}$ would be of third order in derivatives of $g_{\mu\nu}$ and ϕ. However, since the right-hand side contains at most second derivatives, $\nabla^\nu\mathcal{E}_{\mu\nu}$ must be of second order even though $\mathcal{E}_{\mu\nu}$ itself is of second order. This puts a tight restriction on the structure of $\mathcal{E}_{\mu\nu}$. Our next step is to construct the tensor $\mathcal{A}_{\mu\nu}$ satisfying this property. After a lengthy procedure one can determine the general form of $\mathcal{A}_{\mu\nu}$ in the end. (At this step the assumption on the number of spacetime dimensions is used.) Then, one further restricts the form of $\mathcal{A}_{\mu\nu}$ by requiring that $\nabla^\nu\mathcal{A}_{\mu\nu}$ is proportional to $\nabla_\mu\phi$ as implied by equation (15). The tensor $\mathcal{A}_{\mu\nu}$ thus obtained will be $\mathcal{E}_{\mu\nu}$. The final step is to seek the Lagrangian $\mathcal{L}$ that yields the Euler–Lagrange equations $\mathcal{E}_{\mu\nu} = 0$ and $\mathcal{E}_\phi = 0$. Fortunately enough, it turns out that the Euler–Lagrange equations derived from $\mathcal{L} = g^{\mu\nu}\mathcal{E}_{\mu\nu}$ reproduce the structure of $\mathcal{E}_{\mu\nu}$ and $\mathcal{E}_\phi$. This is how we can arrive at the Lagrangian (9).

By taking the functions in equation (8) appropriately, one can reproduce any second-order scalar–tensor theory as a specific case. For example, non-minimal coupling of the form $f(\phi)R$ can be obtained by taking $G_4 = f(\phi)$, and its limiting case $G_4 = \text{const} = M_{\text{Pl}}^2/2$ gives the Einstein–Hilbert term. Clearly, G_2 is the familiar term used in k-inflation [25]/k-essence [26, 27], and the G_3 term was investigated more recently in the context of kinetic gravity braiding [28]/G-inflation [29]. It is well known that $f(R)$ gravity (a theory whose Lagrangian is given by some function of the Ricci scalar) can be expressed equivalently as a second-order scalar–tensor theory and hence is a subclass of the Horndeski theory (see, e.g. [30, 31]). Non-minimal coupling of the form $G^{\mu\nu}\phi_\mu\phi_\nu$ has been studied often in the literature (see, e.g. [32]), and this term can be expressed in two ways, $G_4 = X$ or $G_5 = -\phi$, with integration by parts.

A nontrivial and interesting example is non-minimal coupling to the Gauss–Bonnet term,

$$\xi(\phi)\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{16}$$

No similar terms can be found in (8) or (9), but since the Horndeski theory is the most general scalar–tensor theory with second-order field equations and it is known that the term (16) yields the second-order field equations, this *must* be obtained somehow as a specific case of the Horndeski theory. In fact, one can reproduce (16) by taking [23]

$$G_2 = 8\xi^{(4)}X^2(3-\ln X), \quad G_3 = 4\xi^{(3)}X(7-3\ln X),$$
$$G_4 = 4\xi^{(2)}X(2-\ln X), \quad G_5 = -4\xi^{(1)}\ln X, \tag{17}$$

where $\xi^{(n)} := \partial^n\xi/\partial\phi^n$. To confirm that (17) is indeed equivalent to (16) at the level of the action is probably extremely difficult, but it is straightforward to see the equivalence if one works at the level of the equations of motion. Note in passing that a function of the Gauss–Bonnet term in a Lagrangian, $f(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$, can be recast into the form of (16) by introducing an auxiliary field, and hence it is also included in the Horndeski theory.

Another nontrivial example is the derivative coupling to the double dual Riemann tensor,

$$\phi_\mu\phi_\nu\phi_{\alpha\beta}L^{\mu\alpha\nu\beta}, \tag{18}$$

where

$$L^{\mu\alpha\nu\beta} := R^{\mu\alpha\nu\beta} + \left(R^{\mu\beta}g^{\nu\alpha} + R^{\nu\alpha}g^{\mu\beta} - R^{\mu\nu}g^{\alpha\beta} - R^{\alpha\beta}g^{\mu\nu}\right)$$
$$+ \frac{1}{2}R\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\nu\alpha}\right). \tag{19}$$

This can be reproduced simply by taking $G_5 = X$ [33].

There are other well-motivated models or scenarios which have some links to the Galileon/Horndeski theory. The Dvali–Gabadadze–Porrati (DGP) model [34] based on a five-dimensional braneworld scenario gives rise to the cubic Galileon interaction $\sim (\partial\phi)^2\Box\phi$ in its four-dimensional effective theory [35]. Actually, the Galileon was originally proposed as a generalization of the DGP effective theory. The Dirac–Born–Infeld (DBI) action, which is described by a particular form of $G_2(\phi, X)$ and often studied in the context of inflation [36, 37], can be obtained from a probe brane moving in a five-dimensional bulk spacetime. By extending the probe brane action, one can similarly derive the generalization of the Galileon whose action is of the particular Horndeski form involving G_3, G_4, and G_5 [38–41]. It is shown in [42] and revisited in [43] that some particular cases of the Horndeski action can be obtained through a Kaluza–Klein compactification of higher-dimensional Lovelock gravity. The non-minimal couplings in the Horndeski theory capture the essential structure of (the decoupling limit of) massive gravity [44, 45].

2.2. ADM decomposition

For the later purpose, it is convenient to perform a $3+1$ Arnowitt–Deser–Misner (ADM) decomposition [46] in the Horndeski theory. We take the unitary gauge in which ϕ is homogeneous on constant-time hypersurfaces, so that $\phi = \phi(t)$. (We assume that it is possible to take such a coordinate system.) The metric can be expressed as

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right), \quad (20)$$

where N is the lapse function, N_i $\left(=\gamma_{ij}N^j\right)$ is the shift vector, and γ_{ij} is the three-dimensional spatial metric. Then, since $X = \dot\phi^2(t)/(2N^2)$, the functions of ϕ and X can be regarded as those of t and N. We will need the extrinsic curvature of the spatial hypersurfaces,

$$K_{ij} := \frac{1}{2N}\left(\dot\gamma_{ij} - D_iN_j - D_jN_i\right), \quad (21)$$

where a dot denotes differentiation with respect to t, and D_i is the covariant derivative associated with γ_{ij}. The second derivatives of ϕ can be expressed using K_{ij}. For example, we have $\phi_{ij} = -(\dot\phi/N)K_{ij}$. The four-dimensional Ricci tensor can also be expressed using the extrinsic curvature and the three-dimensional Ricci tensor, $R^{(3)}_{ij}$. With some manipulation, we find that the Horndeski action in the ADM form is given by [47]

$$S = \int \mathrm{d}t\mathrm{d}^3x\sqrt{\gamma}N\Big[A_2(t,N) + A_3(t,N)K + B_4(t,N)R^{(3)} - (B_4 + NB_{4N})\left(K^2 - K_{ij}K^{ij}\right) + B_5(t,N)G^{(3)}_{ij}K^{ij} + \frac{NB_{5N}}{6}\left(K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{jk}K^i_k\right)\Big]. \quad (22)$$

Now we have four free functions of t and N, which are related to G_2, G_3, G_4, and G_5 as

$$A_2 = G_2 + \sqrt{X}\int^X \frac{G_{3\phi}}{\sqrt{X'}}\mathrm{d}X', \quad (23)$$

$$A_3 = \int^X G_{3X'}\sqrt{2X'}\mathrm{d}X' - 2\sqrt{2X}G_{4\phi}, \quad (24)$$

$$B_4 = G_4 - \frac{\sqrt{X}}{2}\int^X \frac{G_{5\phi}}{\sqrt{X'}}\mathrm{d}X', \quad (25)$$

$$B_5 = -\int^X G_{5X'}\sqrt{2X'}\mathrm{d}X'. \quad (26)$$

The above ADM form of the action is particularly useful for studying cosmology in the Horndeski theory.

Since the scalar field is apparently gone in the ADM description, one might wonder how one can understand from the action (22) that the theory has $(2+1)$ dynamical degrees of freedom. The point is that $\delta S/\delta N = 0$ gives the equation that determines N in terms of γ_{ij} and $\dot\gamma_{ij}$ rather than a constraint among γ_{ij} and $\dot\gamma_{ij}$, which signals an extra degree of freedom. Note that this remains true even if one generalizes the ADM action to

$$S = \int \mathrm{d}t\mathrm{d}^3x\sqrt{\gamma}N\big[\cdots + B_4(t,N)R^{(3)} + C_4(t,N)\left(K^2 - K_{ij}K^{ij}\right) + B_5(t,N)G^{(3)}_{ij}K^{ij} + C_5(t,N)\left(K^3 + \cdots\right)\big], \quad (27)$$

where B_4, B_5, C_4, and C_5 are *independent* functions. This idea hints at the possibility of generalizing the Horndeski theory while retaining the number of dynamical degrees of freedom. Indeed, the Gleyzes–Langlois–Piazza–Vernizzi (GLPV) generalization of the Horndeski theory was noticed in this way [47, 48]. We will come back to the GLPV theory in section 4.1. See also [49–53] for a further generalization of the ADM description of scalar–tensor theories.

2.3. Multi-scalar generalization

Having determined the most general *single*-scalar–tensor theory with second-order field equations, it is natural to explore its multi-scalar generalization. However, so far no complete multi-scalar version of the Horndeski theory has been obtained. Let us summarize the current status of attempts to generalize the Galileon/Horndeski theory to multiple scalar fields.

The Galileon is generalized to mixed combinations of p-form fields in [54], a special case of which is the bi- and multi-Galileon theory [55, 56]. The multi-Galileon theory can also be derived from a probe brane embedded in a higher-dimensional bulk by extending the method of [38] to the case with higher co-dimensions [41, 57]. The multi-Galileon theory can be promoted to involve arbitrary functions of the N scalar fields ϕ^I $(I = 1,\cdots,N)$ and their kinetic terms $-\partial_\mu\phi^I\partial^\mu\phi^J/2$ [58, 59]. Similarly to the single-field Galileon, the multi-Galileon can be covariantized, while maintaining the second-order nature, to give [58]

$$\mathcal{L} = G_2 - G_{3I}\Box\phi^I + G_4R + G_{4,\langle IJ\rangle}\left(\Box\phi^I\Box\phi^J - \nabla_\mu\nabla_\nu\phi^I\nabla^\mu\nabla^\nu\phi^J\right) + G_{5I}G^{\mu\nu}\nabla_\mu\nabla_\nu\phi^I - \frac{1}{6}G_{I,\langle JK\rangle}\big[\Box\phi^I\Box\phi^J\Box\phi^K - 3\Box\phi^{(I}\nabla_\mu\nabla_\nu\phi^J\nabla^\mu\nabla^\nu\phi^{K)} + 2\nabla_\mu\nabla_\nu\phi^I\nabla^\nu\nabla^\lambda\phi^J\nabla_\lambda\nabla^\mu\phi^K\big], \quad (28)$$

where G_2, G_{3I}, G_4, and G_{5I} are arbitrary functions of ϕ^I and $X^{IJ} := -g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J/2$, and we defined the symmetrized derivative for any function f of X^{IJ} as $f_{,\langle IJ\rangle} := (\partial f/\partial X^{IJ} + \partial f/\partial X^{JI})/2$. For these functions we must require that

$$G_{3I,\langle JK\rangle}, \quad G_{4,\langle IJ\rangle,\langle KL\rangle}, \quad G_{5I,\langle JK\rangle}, \quad G_{5I,\langle JK\rangle,\langle LM\rangle} \quad (29)$$

are symmetric in all of their indices $I, J, \cdots$, so that the field equations are of second order. This may be regarded as the multi-scalar version of the Lagrangian (8), obtained by generalizing the multi-Galileon theory on a fixed Minkowski background.

Unlike the case of the single-field Galileon, the Lagrangian (28) does *not* give the most general multi-scalar–tensor theory with second-order field equations. Indeed, the probe-brane derivation of the multi-field DBI-type Galileon [60, 61] yields the terms that cannot be described by (28) but nevertheless have second-order field equations [62]. Thus, the 'Galileon route' is unsuccessful.

In contrast, Ohashi *et al* closely followed the original derivation of the Horndeski theory and derived the most general second-order *equations of motion* for *bi*-scalar–tensor theory [63]. However, the corresponding action has not been obtained so far.

More recently, new terms for the multi-Galileon that had been overlooked was proposed in [64], and their covariant completion was obtained in [65]. These new terms can reproduce the multi-field DBI Galileon [65]. However, whether or not the most general second-order multi-scalar–tensor theory is described by the Lagrangian (28) plus these new terms remains an open question.

3. Horndeski theory and cosmology

A great variety of dark energy/modified gravity models have been proposed so far to account for the present accelerated expansion of the universe. Also in the context of the accelerated expansion of the early universe, namely, inflation, gravity modification is now a popular way of building models. (Actually, one of the earliest proposals of inflation already invoked higher curvature terms [3, 66].) The Horndeski theory provides us with a useful tool to study such cosmologies in a unifying way. In this section, we review the applications of the Horndeski theory to cosmology.

3.1. Structure of the background equations

Let us review the derivation of the field equations for a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric and a homogeneous scalar field,

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad \phi = \phi(t). \tag{30}$$

Here, $N(t)$ can be set to 1 by redefining the time coordinate, but we need to retain it for the moment in order to derive the background equation corresponding to the Friedmann equation (the *tt* component of the gravitational field equations).

First, let us consider the universe filled only with the scalar field. Substituting this metric and the scalar field to equation (8), we get the action of the form

$$S = \int \mathrm{d}t\mathrm{d}^3x\, L(N, \dot{N}; a, \dot{a}, \ddot{a}; \phi, \dot{\phi}, \ddot{\phi}), \tag{31}$$

where a dot denotes differentiation with respect to t. Varying this action with respect to N, a, and ϕ, and then setting $N = 1$, we obtain the following set of background equations:

$$\mathcal{E}(H; \phi, \dot{\phi}) := -\frac{1}{a^3}\frac{\delta S}{\delta N} = 0, \tag{32}$$

$$\mathcal{P}(H, \dot{H}; \phi, \dot{\phi}, \ddot{\phi}) := \frac{1}{3a^2}\frac{\delta S}{\delta a} = 0, \tag{33}$$

$$\mathcal{E}_\phi(H, \dot{H}; \phi, \dot{\phi}, \ddot{\phi}) := \frac{1}{Na^3}\frac{\delta S}{\delta \phi} = 0, \tag{34}$$

with $H := \dot{a}/a$ being the Hubble parameter. Equations (32) and (33) correspond respectively to the *tt* and *ij* components of the gravitational field equations, and equation (34) is the equation of motion for ϕ. Explicit expressions for $\mathcal{E}$, $\mathcal{P}$, and $\mathcal{E}_\phi$ are found in [23].

Although the action apparently depends on $\dot{N}$, $\ddot{a}$, and $\ddot{\phi}$, all the higher derivatives are canceled in the field equations, leading to the second-order system as expected. In particular, $\mathcal{E}(H; \phi, \dot{\phi}) = 0$ is the constraint equation. It is interesting to see that $\mathcal{P}$ and $\mathcal{E}_\phi$ depend on both $\dot{H}$ and $\ddot{\phi}$ in general. This implies the kinetic mixing of gravity and the scalar field, which does not occur in Einstein gravity (plus G_2). (In Einstein gravity, $\mathcal{P}$ (respectively $\mathcal{E}_\phi$) is independent of $\ddot{\phi}$ (respectively $\dot{H}$).) In the traditional scalar–tensor theory whose Lagrangian is of the form $\mathcal{L} = G_2(\phi, X) + G_4(\phi)R$, this mixing can be undone by moving to the Einstein frame through the conformal transformation of the metric, $\tilde{g}_{\mu\nu} = C(\phi)g_{\mu\nu}$. However, once $G_3(\phi, X)$ is introduced, the mixing becomes essential and cannot be removed by such a field redefinition. This nature is called *kinetic gravity braiding* [28, 67].

Equations (32)–(34) are useful for studying the background dynamics of inflation (and its alternatives), because the early universe can be described generically by a gravity-scalar system. However, if one considers the late-time universe based on the Horndeski theory, it is necessary to introduce the other kinds of matter (dark matter, baryons, and radiation). In that case, assuming that the matter is minimally coupled to gravity, the background equations are given by

$$\mathcal{E} = -\rho, \quad \mathcal{P} = -p, \quad \mathcal{E}_\phi = 0, \tag{35}$$

where ρ and p are respectively the energy density and pressure of the matter.

3.2. Cosmological perturbations

3.2.1. Linear perturbations and stability. Linear perturbations around an FLRW background are important in two ways. First, one can judge the stability of a given cosmological model by studying linear perturbations. Second, linear perturbations can be used to test modified theories of gravity against cosmological observations. For these purposes let us derive the quadratic action for linear cosmological perturbations.

Linear perturbations around an FLRW background can be decomposed into scalar, vector, and tensor components according to their transformation properties under three-dimensional spatial rotations (see, e.g. [68, 69]). The vector perturbations are less interesting because they are non-dynamical in scalar–tensor theories as well as in Einstein gravity. We therefore focus on scalar and tensor perturbations.

Thanks to the general covariance, one may make use of the gauge transformation, $t \to t - T(t, \vec{x})$, $\vec{x} \to \vec{x} - \vec{\xi}(t, \vec{x})$, to remove some of the perturbation variables. For example, fluctuations in the scalar field, $\delta\phi(t, \vec{x})$, transform as

$$\delta\phi \to \delta\phi + \dot{\phi}T, \tag{36}$$

and thus we are allowed to take $\delta\phi = 0$ by choosing the time coordinate appropriately. This is called the *unitary gauge*, which is particularly useful and hence we will use it for the moment.

Now all the fluctuations are in the metric, and in the ADM form we parametrize them as

$$N = 1 + \delta N, \quad N_i = \partial_i \psi, \quad \gamma_{ij} = a^2 e^{2\zeta} (e^h)_{ij}, \tag{37}$$

where

$$(e^h)_{ij} := \delta_{ij} + h_{ij} + \frac{1}{2} h_{ik} h_{kj} + \cdots . \tag{38}$$

Here, δN, ψ, and ζ are scalar perturbations and h_{ij} are tensor perturbations (gravitational waves) satisfying the transverse and traceless conditions, $\partial^i h_{ij} = 0 = h^i_i$. Writing the spatial metric as $\gamma_{ij} = a^2 e^{2\zeta} (e^h)_{ij}$ rather than $\gamma_{ij} = a^2 [(1+2\zeta)\delta_{ij} + h_{ij}]$ simplifies the computation of the action for the cosmological perturbations. Note that the spatial gauge transformation was used to put γ_{ij} into the form given above. Substituting the metric (37) to the action and expanding it to second order in perturbations, we obtain

$$S^{(2)} = S^{(2)}_{\rm tensor} + S^{(2)}_{\rm scalar}, \tag{39}$$

with

$$S^{(2)}_{\rm tensor} = \frac{1}{8} \int dt d^3x a^3 \left[\mathcal{G}_T \dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2} (\partial_k h_{ij})^2 \right] \tag{40}$$

and

$$S^{(2)}_{\rm scalar} = \int dt d^3x a^3 \left[-3\mathcal{G}_T \dot{\zeta}^2 + \frac{\mathcal{F}_T}{a^2} (\partial\zeta)^2 + \Sigma \delta N^2 - 2\Theta \delta N \frac{\partial^2 \psi}{a^2} + 2\mathcal{G}_T \dot{\zeta} \frac{\partial^2 \psi}{a^2} + 6\Theta \delta N \dot{\zeta} - 2\mathcal{G}_T \delta N \frac{\partial^2 \zeta}{a^2} \right]. \tag{41}$$

The coefficients are given explicitly by

$$\mathcal{G}_T := 2 \left[G_4 - 2XG_{4X} - X \left(H\dot{\phi} G_{5X} - G_{5\phi} \right) \right], \tag{42}$$

$$\mathcal{F}_T := 2 \left[G_4 - X \left(\ddot{\phi} G_{5X} + G_{5\phi} \right) \right], \tag{43}$$

$$\begin{aligned} \Sigma := {} & XG_{2X} + 2X^2 G_{2XX} + 12H\dot{\phi} XG_{3X} + 6H\dot{\phi} X^2 G_{3XX} \\ & - 2XG_{3\phi} - 2X^2 G_{3\phi X} - 6H^2 G_4 + 6 \Big[H^2 \left(7XG_{4X} + 16X^2 G_{4XX} \right. \\ & \left. + 4X^3 G_{4XXX} \right) - H\dot{\phi} \left(G_{4\phi} + 5XG_{4\phi X} + 2X^2 G_{4\phi XX} \right) \Big] \\ & + 30H^3 \dot{\phi} XG_{5X} + 26H^3 \dot{\phi} X^2 G_{5XX} + 4H^3 \dot{\phi} X^3 G_{5XXX} \\ & - 6H^2 X \left(6G_{5\phi} + 9XG_{5\phi X} + 2X^2 G_{5\phi XX} \right), \end{aligned} \tag{44}$$

$$\begin{aligned} \Theta := {} & -\dot{\phi} XG_{3X} + 2HG_4 - 8HXG_{4X} - 8HX^2 G_{4XX} + \dot{\phi} G_{4\phi} + 2X\dot{\phi} G_{4\phi X} \\ & - H^2 \dot{\phi} \left(5XG_{5X} + 2X^2 G_{5XX} \right) + 2HX \left(3G_{5\phi} + 2XG_{5\phi X} \right), \end{aligned} \tag{45}$$

which depend on time in general.

We see that time derivatives of δN and ψ do not appear in the quadratic action for the scalar perturbations (41). Therefore, variation with respect to δN and ψ yields the constraint equations,

$$\Sigma \delta N - \Theta \frac{\partial^2 \psi}{a^2} + 3\Theta \dot{\zeta} - \mathcal{G}_T \frac{\partial^2 \zeta}{a^2} = 0, \tag{46}$$

$$\Theta \delta N - \mathcal{G}_T \dot{\zeta} = 0. \tag{47}$$

These equations allow us to express δN and ψ in terms of ζ. Then, one can remove δN and ψ from equation (41) and obtain the action written solely in terms of ζ (the curvature perturbation in the unitary gauge):

$$S^{(2)}_\zeta = \int dt d^3x a^3 \left[\mathcal{G}_S \dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2} (\partial\zeta)^2 \right], \tag{48}$$

where

$$\mathcal{G}_S := \frac{\Sigma}{\Theta^2} \mathcal{G}_T^2 + 3\mathcal{G}_T, \tag{49}$$

$$\mathcal{F}_S := \frac{1}{a} \frac{d}{dt} \left(\frac{a}{\Theta} \mathcal{G}_T^2 \right) - \mathcal{F}_T. \tag{50}$$

The general quadratic actions (40) and (48) were derived in [23].

It is instructive to check here that the standard textbook result is reproduced in the case of general relativity + a canonical scalar field, $G_2 = X - V(\phi)$, $G_4 = M_{\rm Pl}^2/2$, $G_3 = G_5 = 0$. Obviously, we have $\mathcal{G}_T = \mathcal{F}_T = M_{\rm Pl}^2$. Since $\Sigma = X - 3M_{\rm Pl}^2 H^2$ and $\Theta = M_{\rm Pl}^2 H$, we have $\mathcal{G}_S = X/H^2$ and $\mathcal{F}_S = M_{\rm Pl}^2 (-\dot{H}/H^2) = M_{\rm Pl}^2 \epsilon$, where $\epsilon := -\dot{H}/H^2$ is the slow-roll parameter. Using the background equation, $-M_{\rm Pl}^2 \dot{H} = X$, it turns out that $\mathcal{G}_S = \mathcal{F}_S = M_{\rm Pl}^2 \epsilon$, and thus the standard result is obtained.

The propagation speeds of the tensor and scalar modes are given respectively by

$$c_{\rm GW}^2 := \frac{\mathcal{F}_T}{\mathcal{G}_T}, \tag{51}$$

$$c_s^2 := \frac{\mathcal{F}_S}{\mathcal{G}_S}. \tag{52}$$

These quantities must be positive, $c_{\rm GW}^2 > 0$, $c_s^2 > 0$, because otherwise each perturbation mode exhibits an exponential growth. This is called the gradient instability. This instability is dangerous in particular for short-wavelength modes, because the time scale of the instability is proportional to the wavelength.

In addition to the above stability conditions, we require that $\mathcal{G}_T > 0$ and $\mathcal{G}_S > 0$ in order to guarantee the positivity of the kinetic terms for h_{ij} and ζ, i.e. the absence of ghost instabilities. To sum up, for a given cosmological model to be stable, one must demand that

$$\mathcal{G}_T > 0, \quad \mathcal{F}_T > 0, \quad \mathcal{G}_S > 0, \quad \mathcal{F}_S > 0. \tag{53}$$

The equation of motion derived from (48) is

$$\frac{1}{a^3 \mathcal{G}_S} \frac{d}{dt} \left(a^3 \mathcal{G}_S \frac{d\zeta}{dt} \right) - \frac{c_s^2}{a^2} \partial^2 \zeta = 0. \tag{54}$$

On large (superhorizon[1]) scales, one may ignore the second term and obtain the solution

$$\zeta(t,\vec{x}) \simeq C(\vec{x}) + D(\vec{x}) \int^t \frac{\mathrm{d}t'}{a^3(t')\mathcal{G}_S(t')}, \tag{55}$$

where C and D are integration functions. It is natural to assume that all the time-dependent functions (except, of course, for the scale factor, $a \sim \mathrm{e}^{Ht}$) vary slowly during inflation, and hence $\mathcal{G}_S \simeq$ const. If this is the case, the second term in equation (55) decays rapidly and thus can be neglected, leading to the conservation of ζ on superhorizon scales, $\dot{\zeta} \simeq 0$, in the Horndeski theory [70, 71]. This is the generalization of the standard result. Note, however, that even in the case of general relativity + a canonical scalar field the ultra slow-roll/non-attractor phase of inflation can appear, in which we have $\mathcal{G}_S \propto a^{-6}$ and the second term grows [72, 73]. The non-attractor inflationary dynamics may be more complicated in the presence of the Galileon terms [74].

Similarly, for tensor perturbations we have the superhorizon solution,

$$h_{ij}(t,\vec{x}) \simeq C_{ij}(\vec{x}) + D_{ij}(\vec{x}) \int^t \frac{\mathrm{d}t'}{a^3(t')\mathcal{G}_T(t')}, \tag{56}$$

where C_{ij} and D_{ij} are integration functions, and we see that the second term corresponds to the decaying mode. However, $\mathcal{G}_T$ can vary rapidly in time in some ultra slow-roll models of inflation with non-minimal couplings between gravity and the scalar field (i.e. non-constant G_4 and G_5). In such a model, the would-be decaying tensor mode can grow in a similar manner to the aforementioned growth of ζ [75].

For the purpose of computing the power spectra of primordial perturbations from inflation, it is convenient to recast the quadratic actions (40) and (48) into the canonically normalized form. For ζ we introduce the new time coordinate defined by $\mathrm{d}y := (c_s/a)\mathrm{d}t$ and variable

$$u := z\zeta, \quad z := \sqrt{2}a(\mathcal{F}_S\mathcal{G}_S)^{1/4}. \tag{57}$$

Then, we have

$$S_\zeta^{(2)} = \frac{1}{2}\int \mathrm{d}y\mathrm{d}^3x \left[(u')^2 - (\partial u)^2 + \frac{z''}{z}u^2\right], \tag{58}$$

where a prime here denotes differentiation with respect to y. This is of the familiar 'Sasaki–Mukhanov' form. Tensor perturbations can be analyzed in a similar way [23].

The power spectrum can be evaluated by following the standard procedure to quantize u [69]. Let us assume for simplicity that the time-dependent coefficients in the quadratic action vary very slowly during inflation. In such a 'slow-varying' limit, the power spectra for the curvature and tensor perturbations are given respectively by

$$\mathcal{P}_\zeta = \frac{\mathcal{G}_S^{1/2}}{2\mathcal{F}_S^{3/2}}\frac{H^2}{4\pi^2}, \tag{59}$$

[1] Since the sound speed c_s is different from 1 in general, the horizon scale here should be understood as the sound horizon scale. The same remark applies to the tensor modes.

$$\mathcal{P}_h = \frac{8\mathcal{G}_T^{1/2}}{\mathcal{F}_T^{3/2}}\frac{H^2}{4\pi^2}, \tag{60}$$

evaluated at the (sound) horizon crossing time. The tensor-to-scalar ratio $r := \mathcal{P}_h/\mathcal{P}_\zeta$ is given by

$$r = 16\left(\frac{\mathcal{F}_S}{\mathcal{F}_T}\right)^{3/2}\left(\frac{\mathcal{G}_S}{\mathcal{G}_T}\right)^{-1/2}. \tag{61}$$

The standard expression $r = 16\epsilon$ can be reproduced by substituting $\mathcal{G}_T = \mathcal{F}_T = M_{\mathrm{Pl}}^2$ and $\mathcal{G}_S = \mathcal{F}_S = M_{\mathrm{Pl}}^2\epsilon$, but in general the tensor-to-scalar ratio and the consistency relation can be non-standard in the Horndeski theory.

3.2.2. Beyond linear order.

With increasingly precise measurements of cosmic microwave background (CMB) anisotropy, it is important to study non-Gaussian signatures of primordial perturbations from inflation. For this purpose we need to compute the action to cubic (and higher) order in perturbations. Following the seminal work by Maldacena [76], this program can be carried out in the context of the Horndeski theory.

The cubic action for the scalar perturbations in the Horndeski theory is given in [77, 78]. It was pointed out in [79] that no new operators appear compared to simpler k-inflation [80, 81], though we have four free functions in the theory so that there is a larger degree of freedom in adjusting the coefficients of each term in the cubic action. Shapes of non-Gaussianities have been investigated in more detail in [82, 83].

The cubic action for the tensor perturbations is presented in [84], where it was found that only two independent operators appear, including the one that is already present in general relativity. The non-Gaussian contribution from the new term in the Horndeski theory might in principle be detectable through CMB B-mode polarization if the corresponding coefficient is extremely large [85].

Cross-bispectra among tensor and scalar perturbations were computed in [86].

3.3. NEC-violating cosmologies and their stability

In this subsection, we discuss cosmological consequences of violation of the null energy condition (NEC) based on the Horndeski theory. See also [87] for a mini-review on the same subject.

The NEC demands the following bound on the energy momentum tensor:

$$T_{\mu\nu}k^\mu k^\nu \geqslant 0 \tag{62}$$

for any null vector k^μ. In the context of cosmology, this condition is equivalent to

$$\rho + p \geqslant 0, \tag{63}$$

and then in general relativity the NEC implies that

$$\dot{H} \leqslant 0 \tag{64}$$

through the Einstein equations. As there is no clear distinction between the energy-momentum tensor of the scalar field and the 'left-hand side' (i.e. the geometrical part) of the gravitational field equations in scalar–tensor theories, in the following we mean equation (64) by the NEC.

In usual inflationary cosmology with a canonical scalar field, we have $-2M_{\rm Pl}^2\dot{H} = \rho + p = \dot{\phi}^2 \geqslant 0$, and thus the NEC is automatically satisfied. This in particular implies that the spectrum of primordial tensor modes, $\mathcal{P}_h = 2H^2/\pi^2 M_{\rm Pl}^2$, evaluated at the horizon crossing, must be red. If the spectrum of the tensor modes had a blue tilt, the NEC might be violated during inflation due to some non-standard mechanism.

Although inflation is a very attractive scenario, inflationary spacetime is past-incomplete [88–90] (see also [91]), which motivates non-singular alternatives to inflation such as bouncing models (see, e.g. [92–96] for a review). In non-singular cosmologies, there must be some interval during which the NEC is violated. A non-canonical scalar field or some other kind of matter is required to realize such non-singular alternatives.

It is therefore interesting to explore the possibilities of NEC-violating cosmology in scalar–tensor theories. Since we would expect that the energy conditions are somehow related to the stability of spacetime, the key question now is: can we construct *stable* NEC-violating cosmology?

To answer this question, let us first consider Einstein gravity plus $G_2(\phi, X)$. The background equations in this case read

$$3M_{\rm Pl}^2 H^2 = 2XG_{2X} - G_2, \tag{65}$$

$$-M_{\rm Pl}^2\left(3H^2 + 2\dot{H}\right) = G_2, \tag{66}$$

and hence $-M_{\rm Pl}^2\dot{H} = XG_{2X}$. The NEC can therefore be violated if the function $G_2(\phi, X)$ is chosen so that $G_{2X} < 0$ can occur. However, in this theory we have $\mathcal{F}_S = M_{\rm Pl}^2(-\dot{H}/H^2)$, which implies that NEC-violating solutions are unstable. Thus, in this simplest case the NEC is closely related to stability.

The situation drastically changes if one adds G_3 and the other more general terms that are included in the Horndeski theory, because the general expressions for the stability conditions, (42), (43), (49), and (50), are not correlated with the sign of $\dot{H}$. Therefore, one can construct NEC-violating stages that are nevertheless stable within the Galileon and Horndeski theories [97] (see, however, [98]). This opens up Pandora's box of non-singular bouncing cosmology [99–101] as well as blue gravitational waves from inflation [29] (see, however, [102]).

A novel NEC-violating cosmological scenario called the *Galilean genesis* was proposed based on the cubic Galileon theory [103]. The Lagrangian for this scenario is given by

$$\mathcal{L} = \frac{M_{\rm Pl}^2}{2}R - {\rm e}^{2\phi/f}X - \frac{X}{\Lambda^3}\Box\phi + \frac{X^2}{\Lambda^3 f}, \tag{67}$$

where f and Λ are positive constants having the dimension of mass. This theory admits the following approximate solution valid for $M_{\rm Pl}(-t) \gg (f/\Lambda)^{3/2}$:

$$a \simeq 1 + \frac{f^3}{8M_{\rm Pl}^2\Lambda^3}\frac{1}{(-t)^2}, \quad H \simeq \frac{f^3}{4M_{\rm Pl}^2\Lambda^3}\frac{1}{(-t)^3},$$
$${\rm e}^{\phi/f} \simeq \sqrt{\frac{3f}{2\Lambda^3}\frac{1}{(-t)}} \quad (t<0). \tag{68}$$

As seen from (68), the universe starts expanding from a low-energy, quasi-Minkowski state in the asymptotic past. Clearly, the NEC is violated. Nevertheless, we have

$$\mathcal{G}_S \simeq \mathcal{F}_S \simeq 12M_{\rm Pl}^4(-t)^2\left(\frac{\Lambda}{f}\right)^3 > 0, \tag{69}$$

showing that this solution is stable. The Galilean genesis thus has the potential to be an interesting alternative to inflation. This scenario has further been generalized and investigated in more detail in [104–118].

As an alternative to inflation, the genesis phase described by (68) is supposed to be matched onto a radiation-dominated universe across the reheating stage at $t \sim -(f/\Lambda)^{3/2}/M_{\rm Pl}$. Or, one may consider the initial genesis phase followed by inflation as an 'early-time completion' of the inflationary scenario [119]. In any case, a problem arises in considering the whole history of such a singularity-free universe: gradient instabilities show up at some moment in history. Not only the Galilean genesis but also bouncing models have the same problem. Several examples show that instabilities may occur at the transition from the NEC-violating phase to some subsequent phase or even in a far future after the transition [100, 119–125]. This implies that the instabilities may not be related directly to the violation of the energy condition.

In fact, it can be proven that the appearance of gradient instabilities is generic to all non-singular cosmological solutions in the Horndeski theory [126, 127] (see also [128, 129]). As we have seen, one can construct an NEC-violating solution that is stable during a finite interval. What we will observe below is that such a solution is however unstable once the whole history is concerned. The key inequality follows from equation (50) and the stability conditions:

$$\frac{{\rm d}\xi}{{\rm d}t} > a\mathcal{F}_T > 0 \quad (-\infty < t < \infty), \tag{70}$$

where $\xi := a\mathcal{G}_T^2/\Theta$. For a stable, non-singular cosmological solution we have $a \geqslant$ const, $\mathcal{G}_T > 0$, and $|\Theta| < \infty$. Therefore, ξ must be a monotonically increasing function of time that never crosses zero, which means that $\xi \to$ const as $t \to \infty$ or $-\infty$. (Note that Θ and hence ξ can take either sign.) Integrating equation (70) from $-\infty$ to some t and from some t to ∞, one obtains

$$\xi(t) - \xi(-\infty) > \int_{-\infty}^{t} a\mathcal{F}_T{\rm d}t, \quad \xi(\infty) - \xi(t) > \int_{t}^{\infty} a\mathcal{F}_T{\rm d}t. \tag{71}$$

At least one of the integrals must be convergent, otherwise the stability conditions would be violated at some moment in the entire history of the universe.

By designing the functions in the Horndeski action so that either of the integrals in (71) is convergent, it is indeed possible to construct a stable, non-singular cosmological solution [127, 130]. However, the convergent integral indicates that the spacetime is geodesically incomplete for the propagation of gravitons [131]. This can be understood by moving to the Einstein frame for gravitons via disformal transformation [132]. Moreover, the normalization of vacuum quantum fluctuations tells us that they would grow and diverge if $\mathcal{F}_T$ approaches zero sufficiently fast either in the asymptotic past or the future, which implies that the tensor sector is pathological. If one requires geodesic completeness for gravitons and thereby avoids this subtle behavior, then stable, non-singular cosmologies are prohibited within the Horndeski theory. Note that this no-go theorem cannot tell when the gradient instability shows up. That moment may be in the remote future from the early NEC-violating phase.

Several comments are now in order. First, the no-go theorem for non-singular cosmologies can be extended to include multiple components other than the Horndeski scalar [65, 131, 133] and to the spatially open universe [134]. Second, there is some debate about the zero-crossing of Θ and the validity of the use of the curvature perturbation in the unitary gauge ζ [122, 135–138]. Third, from the effective field theory viewpoint, the strong coupling scale may cut off the instabilities [139] (see also [140]). Finally, the no-go theorem can be circumvented in scalar–tensor theories beyond Horndeski [131, 138, 141–145][2].

3.4. Inclusion of matter

So far we have considered cosmological perturbations in the universe dominated by ϕ, bearing the application to the early universe in mind. Let us extend the previous results to include the other kind of matter, since in the late universe matter perturbations would also be important.

3.4.1. Stability conditions. As additional matter, we are interested in an irrotational, barotropic perfect fluid minimally coupled to the metric. Such a fluid can be mimicked by a k-essence field χ whose action S_m is of the form

$$S_m = \int \mathrm{d}^4x\sqrt{-g}P(Y), \quad Y := -\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi. \quad (72)$$

Introducing the k-essence field as a perfect fluid is a concise and useful technique to treat the fluid at the action level. The energy-momentum tensor of χ is given by $T_{\mu\nu} = 2P_Y\partial_\mu\chi\partial_\nu\chi + Pg_{\mu\nu}$, from which we see that the energy density, pressure, and four-velocity of this fluid are expressed as $\rho = 2YP_Y - P$, $p = P$, and $u_\mu = -\partial_\mu\chi/\sqrt{2Y}$. The background equation for χ, which is equivalent to $\nabla_\nu T^\nu_\mu = 0$, reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3P_Y\dot{\chi}\right) = 0 \quad \Rightarrow \quad \ddot{\chi} + 3c_m^2H\dot{\chi} = 0, \quad (73)$$

where

$$c_m^2 := \frac{\dot{p}}{\dot{\rho}} = \frac{P_Y}{P_Y + 2YP_{YY}} \quad (74)$$

is the sound speed squared of the matter. For $P \propto Y^n$, we have $w := p/\rho = \mathrm{const} = 1/(2n-1)(= c_m^2)$ [146, 147]. This implies that one must be careful when taking the limit of pressureless dust, $c_m^2, w \to 0$, which is singular (see [148]). We therefore assume that $c_m^2 \neq 0$ for the moment.

Expanding the action to second order in perturbations, we obtain $S^{(2)} = S^{(2)}_{\text{tensor}} + S^{(2)}_{\text{scalar}} + S^{(2)}_m$, where $S^{(2)}_{\text{tensor}}$ and $S^{(2)}_{\text{scalar}}$ are given by equations (40) and (41), respectively. The contribution from the matter action, $S^{(2)}_m$, is given by

$$S^{(2)}_m = \int \mathrm{d}t\mathrm{d}^3x\frac{a^3P_Y}{c_m^2}\left[Y\delta N^2 - \dot{\chi}\left(\delta N - 3c_m^2\zeta\right)\dot{\delta\chi} + c_m^2\dot{\chi}\frac{\partial^2\psi}{a^2}\delta\chi + \frac{1}{2}\dot{\delta\chi}^2 - \frac{c_m^2}{2a^2}(\partial\delta\chi)^2\right], \quad (75)$$

where $\delta\chi = \delta\chi(t,\vec{x})$ is a fluctuation of χ. Since the tensor sector remains unaltered by the inclusion of the matter, we focus on the scalar sector.

It follows from $\delta S^{(2)}/\delta(\delta N) = 0$ and $\delta S^{(2)}/\delta\psi = 0$ that

$$\Sigma\delta N - \Theta\frac{\partial^2\psi}{a^2} + 3\Theta\dot{\zeta} - \mathcal{G}_T\frac{\partial^2\zeta}{a^2} + \frac{YP_Y}{c_m^2}\delta N - \frac{P_Y}{2c_m^2}\dot{\chi}\dot{\delta\chi} = 0, \quad (76)$$

$$\Theta\delta N - \mathcal{G}_T\dot{\zeta} - \frac{P_Y}{2}\dot{\chi}\delta\chi = 0. \quad (77)$$

Similarly to the previous analysis without χ, one can eliminate δN and ψ from the quadratic action by using equation (77). The reduced action written solely in terms of ζ and $\delta\chi$ takes the form [149]

$$S^{(2)} = \int \mathrm{d}t\mathrm{d}^3xa^3\left[G_{AB}\dot{q}^A\dot{q}^B - \frac{1}{a^2}F_{AB}\partial q^A\cdot\partial q^B + \cdots\right], \quad (78)$$

where

$$q^A := \left(\zeta, \frac{\mathcal{G}_T}{\Theta}\frac{\delta\chi}{\dot{\chi}}\right), \quad (79)$$

and

$$G_{AB} = \begin{pmatrix} \mathcal{G}_S + Z & -Z \\ -Z & Z \end{pmatrix}, \quad F_{AB} = \begin{pmatrix} \mathcal{F}_S & -c_m^2Z \\ -c_m^2Z & c_m^2Z \end{pmatrix}, \quad (80)$$

with

$$Z := \left(\frac{\mathcal{G}_T}{\Theta}\right)^2\frac{\rho + p}{2c_m^2}. \quad (81)$$

Here we only write the terms that are relevant to ghost and gradient instabilities. To avoid ghost instabilities we require

[2] As will be argued in section 4.1, only theories that can be generated from the Horndeski theory via disformal transformation (108) are phenomenologically viable. Since the disformal transformation is just a field redefinition, one may wonder why the the no-go theorem can be evaded in theories beyond (and disformally related to) Horndeski. The trick is that the disformal transformation that generates the theories admitting stable non-singular cosmology is singular at some moment [131].

that G_{AB} is a positive definite matrix. This is equivalent to $\mathcal{G}_S > 0$ and $Z > 0$. The propagation speeds of the two scalar modes are determined by solving $\det(v^2 G_{AB} - F_{AB}) = 0$, yielding $v^2 = (\mathcal{F}_S - c_m^2 Z)/\mathcal{G}_S$ and $v^2 = c_m^2$. Thus, the stability conditions in the presence of an additional perfect fluid are summarized as

$$\mathcal{G}_S > 0, \quad \rho + p > 0, \quad c_m^2 > 0, \quad \mathcal{F}_S > \frac{1}{2}\left(\frac{\mathcal{G}_T}{\Theta}\right)^2 (\rho + p). \tag{82}$$

It can be seen that the conditions imposed on the fluid component are quite reasonable.

3.4.2. Matter density perturbations. In late-time cosmology, we are often interested in the evolution of the density perturbations of pressureless matter on subhorizon scales. The analysis is usually done in the Newtonian gauge, in which the metric takes the form

$$\mathrm{d}s^2 = -[1 + 2\Phi(t, \vec{x})]\mathrm{d}t^2 + a^2[1 - 2\Psi(t, \vec{x})]\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{83}$$

with the non-vanishing scalar field fluctuation,

$$\phi = \phi(t) + \delta\phi(t, \vec{x}). \tag{84}$$

One can move from the unitary gauge to the Newtonian gauge by performing the coordinate transformation $t_N = t - T$ such that

$$\Phi = \delta N + \dot{T}, \quad \Psi = -\zeta - HT, \quad 0 = \psi - T, \quad \delta\phi = 0 + \dot{\phi}T. \tag{85}$$

The fluctuation of χ in the Newtonian gauge is given by

$$\delta\chi_N = \delta\chi + \dot{\chi}T. \tag{86}$$

Substituting equations (85) and (86) to equations (41) and (75), we obtain the Newtonian gauge expression for the quadratic action. As we are interested in the quasi-static evolution of the perturbations inside the (sound) horizon, we assume that $\dot{\varepsilon} \sim H\varepsilon \ll \partial\varepsilon/a$ ($\varepsilon = \Phi, \Psi, H\delta\phi/\dot{\phi}$). We will take the pressureless limit $c_m^2 \to 0$, which is apparently singular. Therefore, we carefully retain the would-be singular terms in this limit. The resultant action in the quasi-static approximation is given by

$$\begin{aligned} S_{\mathrm{QS}}^{(2)} = \int \mathrm{d}t\mathrm{d}^3x \Big\{ & a\left[\mathcal{F}_T(\partial\Psi)^2 - 2\mathcal{G}_T\partial\Phi\partial\Psi + b_0 H^2(\partial T)^2 \right. \\ & \left. -2b_1 H\partial T\partial\Psi - 2b_2 H\partial T\partial\Phi\right] + \frac{a^3 P_Y}{2} \\ & \left[-\frac{1}{a^2}(\partial\delta\chi_N)^2 + \frac{1}{c_m^2}\left(\dot{\delta\chi}_N - \dot{\chi}\Phi\right)^2\right]\Big\}, \end{aligned} \tag{87}$$

where $T = \delta\phi/\dot{\phi}$ and the coefficients are defined as

$$b_0 := \frac{1}{H^2}\left[\dot{\Theta} + H\Theta + H^2(\mathcal{F}_T - 2\mathcal{G}_T) - 2H\dot{\mathcal{G}}_T + YP_Y\right], \tag{88}$$

$$b_1 := \frac{1}{H}\left[\dot{\mathcal{G}}_T + H(\mathcal{G}_T - \mathcal{F}_T)\right], \tag{89}$$

$$b_2 := \frac{1}{H}\left(H\mathcal{G}_T - \Theta\right). \tag{90}$$

In equation (88) one may replace YP_Y with $\rho/2$ in the pressureless limit. Note that there could be terms of the form $m^2\varepsilon^2$ (without spatial derivatives) which are larger than $\mathcal{O}(H^2\varepsilon^2)$ and could be as large as $\mathcal{O}((\partial\varepsilon)^2)$, but for simplicity we ignored such terms.

The equations of motion derived from (87) are

$$\delta\Psi: \quad \partial^2\left(\mathcal{F}_T\Psi - \mathcal{G}_T\Phi - b_1 HT\right) = 0, \tag{91}$$

$$\delta\Phi: \quad \partial^2\left(\mathcal{G}_T\Psi + b_2 HT\right) = \frac{a^2 P_Y}{2c_m^2}\left(\dot{\chi}\dot{\delta\chi}_N - \dot{\chi}^2\Phi\right)\left(= \frac{\delta\rho}{2}\right), \tag{92}$$

$$\delta T: \quad \partial^2\left(b_0 HT - b_1\Psi - b_2\Phi\right) = 0, \tag{93}$$

and

$$\delta\chi_N: \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{a^3 P_Y}{c_m^2}\left(\dot{\delta\chi}_N - \dot{\chi}\Phi\right)\right] = aP_Y\partial^2\delta\chi_N, \tag{94}$$

where the right-hand side of equation (92) may be replaced with the density perturbation by noting that

$$\delta\rho = \frac{P_Y}{c_m^2}\left(\dot{\chi}\dot{\delta\chi}_N - \dot{\chi}^2\Phi\right). \tag{95}$$

Thus, the apparently singular behavior of this equation in the $c_m^2 \to 0$ limit can be eliminated. Taking $b_0 = b_1 = b_2 = 0$ in equations (91)–(93), the standard result in Einstein gravity can be recovered.

Solving the algebraic equations (91)–(93), one arrives at the modified Poisson equation [150],

$$\frac{1}{a^2}\partial^2\Phi = 4\pi G_{\mathrm{eff}}(t)\delta\rho, \quad 8\pi G_{\mathrm{eff}} := \frac{b_0\mathcal{F}_T - b_1^2}{b_0\mathcal{G}_T^2 + 2b_1 b_2\mathcal{G}_T + b_2^2\mathcal{F}_T}. \tag{96}$$

The effective gravitational coupling G_{eff} can be different from the Newton constant, and, as seen below, it affects the evolution of the density perturbations. The ratio

$$\eta(t) := \frac{\Psi}{\Phi} = \frac{b_0\mathcal{G}_T + b_1 b_2}{b_0\mathcal{F}_T - b_1^2} \tag{97}$$

is also an important quantity because $\eta \neq 1$ if gravity is nonstandard. Since the bending of light depends on $\Phi + \Psi$, weak-lensing observations are useful to test $\eta \neq 1$. Note that the above expressions cannot be used for $f(R)$ and chameleon models of dark energy, because we dropped the mass term for simplicity. See [151–153] for the limits of the quasi-static assumption and [150, 154, 155] for the complete expression of the equations (without using the quasi-static approximation).

The equation of motion (94) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\delta\rho\right) - 3c_m^2 H\left(a^3\delta\rho\right) = aP_Y\dot{\chi}\partial^2\delta\chi_N, \tag{98}$$

where we used (73). Using (73) again, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{a^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\delta\rho\right)-3c_m^2H\left(a^3\delta\rho\right)\right]\right\}=a^3\partial^2\left(c_m^2\delta\rho+2YP_Y\Phi\right). \tag{99}$$

Now we can take the limit $c_m^2\to 0$ and $2YP_Y\to\rho$ safely to get

$$\ddot{\delta}+2H\dot{\delta}=\frac{1}{a^2}\partial^2\Phi, \tag{100}$$

where $\delta:=\delta\rho/\rho$. Thus, as expected, the familiar evolution equation for the density contrast δ is recovered from the equation of motion for $\delta\chi_N$ in the pressureless limit. Combining this with the modified Poisson equation (96), one can derive the closed-form evolution equation for δ.

Instead of introducing the k-essence field χ, one may replace the second line in equation (87) with

$$-a^3\Phi\delta\rho, \tag{101}$$

and use the usual fluid equations for a pressureless dust. This is a simpler procedure to arrive at the same result.

4. Beyond Horndeski

So far we have considered the most general scalar–tensor theory having second-order equations of motion and its application to cosmology. Thanks to this second-order nature, the theory is obviously free of the Ostrogradsky instability. However, it should be emphasized that the second-order equations of motion are *not* the necessary conditions for the absence of the Ostrogradsky instability in theories with multiple fields.

To see this, let us consider a simple toy model in mechanics whose Lagrangian is given by [156]

$$L=\frac{a}{2}\ddot{\phi}^2+b\ddot{\phi}\dot{q}+\frac{c}{2}\dot{q}^2+\frac{1}{2}\dot{\phi}^2-\frac{1}{2}\phi^2-\frac{1}{2}q^2. \tag{102}$$

Here, the coefficients a, b, and c are assumed to be constants. The Euler–Lagrange equations are of higher order in general:

$$a\ddddot{\phi}+b\dddot{q}-\ddot{\phi}-\phi=0, \tag{103}$$

$$b\dddot{\phi}+c\ddot{q}+q=0, \tag{104}$$

implying that the system contains an extra degree of freedom and hence suffers from the Ostrogradsky ghost. However, if the kinetic matrix constructed from the highest derivative terms,

$$M=\begin{pmatrix} a & b\\ b & c\end{pmatrix}, \tag{105}$$

is degenerate, i.e. $ac-b^2=0$, then the system contains only two degrees of freedom. Indeed, if $ac-b^2=0$ is satisfied, we can combine the equations of motion (103) and (104) to reduce the number of derivatives. First, $c\times$(103)$-b\times\mathrm{d}$(104)$/\mathrm{d}t$ gives

$$\ddot{\phi}+\frac{b}{c}\dot{q}+\phi=0. \tag{106}$$

Then, d(106)/dt is used to remove $\dddot{\phi}$ from equation (104), yielding

$$\left(1-\frac{b^2}{c^2}\right)\ddot{q}-\frac{b}{c}\dot{\phi}+\frac{1}{c}q=0. \tag{107}$$

We thus arrive at the two second-order equations of motion (106) and (107) for ϕ and q. This shows that the degenerate system is free of the Ostrogradsky ghost and hence is healthy despite the higher-order Euler–Lagrange equations.

In this section, we will briefly explore such healthy degenerate higher-order theories containing the metric and a scalar field and extend the Horndeski theory. The reader is referred to [157, 158] for a more complete review on this topic.

4.1. Degenerate higher-order scalar–tensor theories

The first example of degenerate higher-order scalar–tensor (DHOST) theories beyond Horndeski [159] was obtained by performing a disformal transformation [160]

$$g_{\mu\nu}\to\tilde{g}_{\mu\nu}=C(\phi,X)g_{\mu\nu}+D(\phi,X)\phi_\mu\phi_\nu. \tag{108}$$

This is a generalization of the familiar conformal transformation, $\tilde{g}_{\mu\nu}=C(\phi)g_{\mu\nu}$. The disformal transformation (108) is invertible if

$$C(C-XC_X+2X^2D_X)\neq 0. \tag{109}$$

Since the disformal transformation contains derivatives of ϕ, the theory transformed from Horndeski has higher-order field equations[3]. Nevertheless, it is a degenerate theory with $(2+1)$ degrees of freedom because an invertible field redefinition does not change the number of physical degrees of freedom [162–164]. This example implies the existence of a wider class of healthy scalar–tensor theories than the Horndeski class.

Degenerate higher-order scalar–tensor theories have been constructed and investigated systematically in [156, 165–169]. Let us follow [156] and consider the extension of Horndeski's G_4 Lagrangian given by

$$\mathcal{L}=f(\phi,X)R+\sum_{I=1}^{5}A_I(\phi,X)L_I, \tag{110}$$

where

$$\begin{aligned} &L_1=\phi_{\mu\nu}\phi^{\mu\nu},\quad L_2=(\Box\phi)^2,\quad L_3=\Box\phi\phi^\mu\phi^\nu\phi_{\mu\nu},\\ &L_4=\phi^\mu\phi_{\mu\alpha}\phi^{\alpha\nu}\phi_\nu,\quad L_5=(\phi^\mu\phi^\nu\phi_{\mu\nu})^2. \end{aligned} \tag{111}$$

These five constituents exhaust all the possible quadratic terms in second derivatives of ϕ, and the Horndeski theory is the special case with $A_2=-A_1=f_X$ and $A_3=A_4=A_5=0$. The scalar field (respectively, the metric) corresponds to ϕ (respectively, q) in the previous mechanical toy model. By inspecting the structure of the highest derivative terms in (110)[4], one finds that the degeneracy conditions are given by three equations relating the six functions in the Lagrangian, leaving

[3] If both C and D depend only on ϕ, the transformed field equations remain of second order and so the Horndeski theory is mapped to Horndeski [161].

[4] We require the degeneracy in any coordinate systems. It is argued in [170] that one can relax this requirement and consider theories that are degenerate when restricted to the unitary gauge.

three arbitrary functions (except for some special cases). The degenerate theories whose Lagrangian is of the form (110) are called quadratic DHOST theories. Note that one is free to add to (110) the Horndeski terms $G_2(\phi,X) - G_3(\phi,X)\Box\phi$, because these two terms are nothing to do with the degeneracy conditions.

Quadratic DHOST theories are classified into several subclasses [156, 166, 167]. Of particular importance among them is the so-called class Ia, which is characterized by

$$A_2 = -A_1, \tag{112}$$

$$A_4 = \frac{1}{2(f+2XA_1)^2}[8XA_1^3 + (3f+16Xf_X)A_1^2 - X^2fA_3^2 + 2X(4Xf_X-3f)A_1A_3 + 2f_X(3f+4Xf_X)A_1 + 2f(Xf_X-f)A_3 + 3ff_X^2], \tag{113}$$

$$A_5 = -\frac{(f_X + A_1 + XA_3)\left(2fA_3 - f_XA_1 - A_1^2 + 3XA_1A_3\right)}{2(f+2XA_1)^2}, \tag{114}$$

with $f + 2XA_1 \neq 0$. (Recall that we are using the notation $X := -g^{\mu\nu}\phi_\mu\phi_\nu/2$.) The arbitrary functions are thus taken to be f, A_1, and A_3. Cosmology in this class of DHOST theories has been studied in [171–173], where it is demonstrated that the apparently higher-order equations of motion can be reduced to a second-order system for the scale factor and the scalar field.

Clearly, the Horndeski theory is included in class Ia. Another important particular case is (a subclass of) the GLPV theory [47, 48], satisfying

$$A_2 = -A_1 = f_X + XA_3 \quad \Rightarrow \quad A_4 = -A_3, \quad A_5 = 0. \tag{115}$$

In this case one has two arbitrary functions f and A_3, and the Horndeski theory is reproduced by further taking $A_3 = 0$. The Lagrangian for the GLPV theory is written explicitly as

$$\mathcal{L}_{\rm GLPV} = fR + f_X\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] + A_3\left\{X\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] + \Box\phi\phi^\mu\phi^\nu\phi_{\mu\nu} - \nabla_\mu X\nabla^\mu X\right\}. \tag{116}$$

Interestingly, the second line (with $A_3 = \text{const}$) can be obtained from a naive, minimal covariantization of the Galileon theory (see the second line in equation (4)). In section 2.1 we introduced the counter term to cancel the higher derivatives which appear upon covariantization. However, this example shows that without the counter term we still have a healthy degenerate theory [174].

A notable property of DHOST theories in class Ia is that all the Lagrangians can be mapped into a Horndeski Lagrangian through a disformal transformation (108) [167]. In other words, one can remove two of the three functions of ϕ and X in the quadratic DHOST sector by using the two functions, $C(\phi,X)$ and $D(\phi,X)$, in the disformal transformation, to move into a 'Horndeski frame' with a single function $G_4(\phi,X)$ at quadratic order. At this point it is worth emphasizing that class Ia DHOST theories in the presence of minimally coupled matter are equivalent to Horndeski with disformally coupled matter, but *not* to Horndeski with minimally coupled matter. This fact is crucial in particular to the screening mechanism discussed in the next section.

Subclasses other than class Ia are phenomenologically unacceptable. In these subclasses, the gradient terms in the quadratic actions for scalar and tensor cosmological perturbations have opposite signs (i.e. either of the two modes is unstable), or tensor perturbations are non-dynamical [169, 175]. Therefore, only the DHOST theories disformally related to Horndeski can be viable.

In this subsection we have focused for simplicity on DHOST theories whose Lagrangian is a quadratic polynomial in $\phi_{\mu\nu}$. One can do similar manipulation to construct cubic DHOST theories as an extension of Horndeski's G_5 Lagrangian (i.e. DHOST theories whose Lagrangian is a cubic polynomial in $\phi_{\mu\nu}$) [168], though their classification is much more involved. Cubic DHOST theories disformally disconnected to Horndeski also exhibit gradient instabilities in tensor or scalar modes.

One can go beyond the polynomial assumption and generate a novel family of DHOST theories from non-degenerate theories via a non-invertible disformal transformation with

$$D(\phi,X) = \frac{C(\phi,X)}{2X} + F(\phi), \tag{117}$$

where $C(\phi,X)$ and $F(\phi)$ are arbitrary (see equation (109)) [176, 177]. This is essentially the field redefinition used in the context of mimetic gravity [178, 179] (see [180] for a review). The idea behind this is that non-invertible field redefinition can change the number of dynamical degrees of freedom. New DHOST theories thus generated are not disformally connected to Horndeski in general. Such 'mimetic DHOST' theories suffer from gradient instabilities of tensor or scalar modes [176, 177] (see also [181, 182] for more about this instability issue).

The idea of degenerate theories can be extended to include more than one higher derivative field [183, 184], though it seems challenging to construct concrete nontrivial examples of a multi-scalar version of DHOST theories. Degenerate theories involving only the metric were explored in [185] under the name 'Beyond Lovelock gravity'.

4.2. After GW170817

Measuring the speed of gravitational waves $c_{\rm GW}$ can be a test for modified gravity theories [186–189]. Indeed, the nearly simultaneous detection of gravitational waves GW170817 and the γ-ray burst GRB 170817A [190–192] provides a tight constraint on $c_{\rm GW}$ and hence on scalar–tensor theories and other types of modified gravity. The limit on the difference between $c_{\rm GW}$ and the speed of light imposed by this recent event is[5]

$$-3\times10^{-15} < c_{\rm GW} - 1 < 7\times10^{-16}. \tag{118}$$

This constraint motivates us to identify the viable subclass of the Horndeski and DHOST theories as an alternative to dark

[5] The lower bound on $c_{\rm GW}$ can also be obtained from the argument on gravitational Cherenkov radiation, which can even be tighter than this [193, 194]. However, the frequencies concerned are much higher than those of LIGO observations.

energy satisfying $c_{\rm GW}=1$ [195–202] (see also [203, 204] for scalar–tensor theories that achieve $c_{\rm GW}=1$ dynamically).

We start with the Horndeski theory, in which the general form of the propagation speed of gravitational waves is given by equation (51):

$$c_{\rm GW}^2=\frac{G_4-X(\ddot{\phi}G_{5X}+G_{5\phi})}{G_4-2XG_{4X}-X(H\dot{\phi}G_{5X}-G_{5\phi})}. \quad (119)$$

In order for this to be equal to the speed of light irrespective of the background cosmological evolution, we require that

$$G_{4X}=0, \quad G_5=0. \quad (120)$$

Thus, the viable subclass within Horndeski is described by the Lagrangian

$$\mathcal{L}=G_2(\phi,X)-G_3(\phi,X)\Box\phi+G_4(\phi)R. \quad (121)$$

This excludes for instance the scalar field coupled to the Gauss–Bonnet term.

Let us then consider DHOST theories. It turns out that any term in the cubic DHOST Lagrangians leads to $c_{\rm GW}\neq 1$ (just as the G_5 term in the Horndeski theory does), and hence all cubic DHOST theories are ruled out. In quadratic DHOST theories, the action for the tensor perturbations is given by [169, 175]

$$S_{\rm tensor}^{(2)}=\frac{1}{4}\int {\rm d}t{\rm d}^3xa^3\left[(f+2XA_1)\dot{h}_{ij}^2-\frac{f}{a^2}(\partial_k h_{ij})^2\right]. \quad (122)$$

From this we see that the propagation speed of gravitational waves is

$$c_{\rm GW}^2=\frac{f}{f+2XA_1}, \quad (123)$$

and so we impose $A_1=0$. The viable subclass in DHOST theories thus reduces to

$$A_2=-A_1=0, \quad (124)$$

$$A_4=\frac{1}{2f}[-X^2A_3^2+2(Xf_X-f)A_3+3f_X^2], \quad (125)$$

$$A_5=-\frac{A_3\,(f_X+XA_3)}{f}. \quad (126)$$

We have two free functions $f(\phi,X)$ and $A_3(\phi,X)$ in addition to the lower-order Horndeski terms $G_2(\phi,X)$ and $G_3(\phi,X)$.

More recently, it was pointed out that gravitons can decay into ϕ in DHOST theories [205]. To avoid this graviton decay, it is further required that $A_3=0$. (Otherwise, gravitational waves would not be observed.) We thus finally have

$$A_4=\frac{3f_X^2}{2f}, \quad A_1=A_2=A_3=A_5=0. \quad (127)$$

Note that this subclass does not belong to Horndeski nor GLPV families (if $f_X\neq 0$).

As argued in the previous subsection, mimetic gravity can be viewed as a kind of DHOST theory. The implications of GW170817 for the mimetic class of DHOST theories have been discussed in [206–208].

It should be emphasized that in constraining scalar–tensor theories with gravitational waves we have assumed that the DHOST theory under consideration as an alternative description of dark energy is valid on much higher energy scales where LIGO observations are made ($\sim$100 Hz $\sim 10^{-13}$ eV). The validity of this assumption needs to be looked into carefully [209]. Similarly, gravity at much higher energies than this remains unconstrained. Therefore, modified gravity in the early universe is free from these gravitational wave constraints.

A final remark is that, even if $c_{\rm GW}=1$, f (or G_4) may depend on time, which gives rise to the extra contribution $\dot{f}/f$ to the friction term in the equation of motion for h_{ij}. This results in a modification of the amplitude of gravitational waves, which can be measurable [210–212].

5. Vainshtein screening

While a scalar–tensor theory as an alternative to dark energy is supposed to give rise to $\mathcal{O}(1)$ modification of gravity on cosmological scales, the extra force mediated by the scalar degree of freedom ϕ must be screened on small scales where general relativity has been tested to high precision. This occurs if ϕ is effectively massive in the vicinity of a source, or if ϕ is effectively weakly coupled to the source. The former case corresponds to the chameleon mechanism [213, 214], in which the effective potential for ϕ depends on the local energy density through the coupling of ϕ to matter. The latter case is based on the idea of Vainshtein [215] (see also [216]) and is called the Vainshtein mechanism. (There are other screening mechanisms called symmetron [217, 218] and k-Mouflage models [219], both of which effectively suppress the coupling to matter.) The Vainshtein mechanism is relevant to the Galileon theories, and below we will review this screening mechanism in the context of the generalized Galileon/Horndeski theory. See also [220] for a nice review on the Vainshtein mechanism.

5.1. A Vainshtein primer

We start with emphasizing the need for a screening mechanism, and then introduce the Vainshtein mechanism.

To see how gravity is modified around matter in a simple model, and as a result fails to satisfy the experimental constraints, let us consider the theory

$$S=\int {\rm d}^4x\sqrt{-g}\,[f(\phi)R+X]+S_m[g_{\mu\nu},\psi_m], \quad (128)$$

where the matter fields (denoted as ψ_m) are minimally coupled to the metric $g_{\mu\nu}$.

We investigate perturbations around a Minkowski background with a constant scalar ϕ,

$$g_{\mu\nu}=\eta_{\mu\nu}+M_{\rm Pl}^{-1}h_{\mu\nu}(t,\vec{x}), \quad \phi=\phi_0+\varphi(t,\vec{x}), \quad (129)$$

caused by the energy-momentum tensor for matter, $T_{\mu\nu}$ (the above theory admits the background solution $g_{\mu\nu}=\eta_{\mu\nu}$ and $\phi=\phi_0=$ const). Here we write $f(\phi_0)=M_{\rm Pl}^2/2$ and define the metric perturbations so that $h_{\mu\nu}$ has the dimension of mass. Expanding (128) to second order in perturbations, we obtain the effective Lagrangian for the description of weak gravitational fields as

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \xi h^{\mu\nu}X^{(1)}_{\mu\nu} + \frac{1}{2M_{\rm Pl}}h^{\mu\nu}T_{\mu\nu}, \tag{130}$$

where $\xi := M_{\rm Pl}^{-1}{\rm d}f/{\rm d}\phi|_{\phi=\phi_0}$,

$$X^{(1)}_{\mu\nu} := \eta_{\mu\nu}\Box\varphi - \varphi_{\mu\nu}, \tag{131}$$

and

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} := -\frac{1}{2}\Box h_{\mu\nu} + \partial^\lambda\partial_{(\mu}h_{\nu)\lambda} + \frac{1}{2}\eta_{\mu\nu}\Box h - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\partial_\rho h^{\lambda\rho} - \frac{1}{2}\partial_\mu\partial_\nu h \tag{132}$$

is the linearized Einstein tensor (divided by $M_{\rm Pl}$). Here, indices are raised and lowered by $\eta_{\mu\nu}$.

The third term in the Lagrangian (130) signals the mixing of the scalar degree of freedom with the graviton. This can be disentangled by making use of the field redefinition

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - 2\xi\varphi\eta_{\mu\nu}, \tag{133}$$

leading to

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}\tilde{h}^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} - \frac{1+6\xi^2}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2M_{\rm Pl}}\tilde{h}^{\mu\nu}T_{\mu\nu} - \frac{\xi}{M_{\rm Pl}}\varphi T. \tag{134}$$

The transformation (133) is equivalent to the linear part of the conformal transformation to the Einstein frame, $\tilde{g}_{\mu\nu} = C(\phi)g_{\mu\nu}$ with $C = f(\phi)/f(\phi_0)$. In the new frame we have the non-minimal coupling of the form φT, and the field equations are given by

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} = M_{\rm Pl}^{-1}T_{\mu\nu}, \tag{135}$$

$$(1+6\xi^2)\Box\varphi = M_{\rm Pl}^{-1}\xi T. \tag{136}$$

Thus, $\mathcal{O}(1)$ modification of gravity is expected for $\xi = \mathcal{O}(1)$.

To be more concrete, let us consider a spherical distribution of non-relativistic matter, $T_{\mu\nu} = \rho(r)\delta^0_\mu\delta^0_\nu$, with $\tilde{h}_{00} = -2\tilde{\Phi}(r)$ and $\tilde{h}_{ij} = -2\tilde{\Psi}(r)\delta_{ij}$. Then, the field equations read

$$\frac{1}{r^2}\left(r^2\tilde{\Psi}'\right)' = \frac{\rho}{2M_{\rm Pl}}, \tag{137}$$

$$\tilde{\Psi} - \tilde{\Phi} = 0, \tag{138}$$

$$\frac{1}{r^2}\left(r^2\varphi'\right)' = -\frac{\xi}{1+6\xi^2}\frac{\rho}{M_{\rm Pl}}, \tag{139}$$

where a prime stands for differentiation with respect to r. These equations can be integrated straightforwardly to give

$$M_{\rm Pl}^{-1}\tilde{\Phi}' = M_{\rm Pl}^{-1}\tilde{\Psi}' = (8\pi M_{\rm Pl}^2)^{-1}\frac{\mathcal{M}(r)}{r^2},$$
$$M_{\rm Pl}^{-1}\varphi' = -\frac{2\xi}{1+6\xi^2}\cdot(8\pi M_{\rm Pl}^2)^{-1}\frac{\mathcal{M}(r)}{r^2}, \tag{140}$$

where $\mathcal{M}(r)$ is the enclosed mass, $\mathcal{M}(r) := 4\pi\int^r\rho(s)s^2{\rm d}s$. It follows from (133) that the metric perturbations in the original frame are given by $\Phi = \tilde{\Phi} - \xi\varphi$ and $\Psi = \tilde{\Psi} + \xi\varphi$. Thus, the metric potentials outside the matter distribution are given by

$$\Phi = -\frac{G_N\mathcal{M}}{r}, \quad \Psi = \gamma\Phi, \tag{141}$$

with

$$8\pi G_N := \frac{1+8\xi^2}{1+6\xi^2}\frac{1}{M_{\rm Pl}^2}, \quad \gamma - 1 = -\frac{4\xi^2}{1+8\xi^2}. \tag{142}$$

For $\xi = \mathcal{O}(1)$ we have $\gamma - 1 = \mathcal{O}(1)$, which clearly contradicts the solar system experiments [221].

Now we add a Galileon-like cubic interaction to (134):

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}\tilde{h}^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} - \frac{1+6\xi^2}{2}(\partial\varphi)^2 - \frac{1}{2\Lambda^3}(\partial\varphi)^2\Box\varphi + \frac{1}{2M_{\rm Pl}}\tilde{h}^{\mu\nu}T_{\mu\nu} - \frac{\xi}{M_{\rm Pl}}\varphi T. \tag{143}$$

Then, the scalar field equation of motion becomes

$$(1+6\xi^2)\Box\varphi + \frac{1}{\Lambda^3}\left[(\Box\varphi)^2 - \varphi_{\mu\nu}\varphi^{\mu\nu}\right] = \frac{\xi}{M_{\rm Pl}}T, \tag{144}$$

which, for a spherical matter distribution such as a star, gives

$$(1+6\xi^2)r^2\varphi' + \frac{2}{\Lambda^3}r(\varphi')^2 = -\frac{\xi\mathcal{M}(r)}{4\pi M_{\rm Pl}}. \tag{145}$$

This equation can be solved algebraically, yielding

$$\varphi' = c\Lambda^3 r\left[-1 + \sqrt{1 - \frac{\xi}{c^2}\left(\frac{r_V}{r}\right)^3}\right], \tag{146}$$

where $c := (1+6\xi^2)/4$ is an $\mathcal{O}(1)$ constant and we defined

$$r_V := \left(\frac{\mathcal{M}}{8\pi M_{\rm Pl}\Lambda^3}\right)^{1/3}. \tag{147}$$

(We consider a stellar exterior so that now $\mathcal{M}(=\text{const})$ is the mass of the star.) For $r \gg r_V$, equation (146) reproduces (140). However, for $r \ll r_V$, we find

$$\varphi' \simeq (-\xi)^{1/2}\left(\frac{r}{r_V}\right)^{3/2}\tilde{\Phi}' \ll \tilde{\Phi}'$$
$$\Rightarrow \quad \frac{\Phi}{M_{\rm Pl}} \simeq \frac{\Psi}{M_{\rm Pl}} \simeq -\frac{G_N\mathcal{M}}{r}, \quad 8\pi G_N := \frac{1}{M_{\rm Pl}^2}. \tag{148}$$

It turns out that the nonlinear interaction introduced in (143) helps the recovery of standard gravity, and the solar system constraints can thus be evaded if r_V is sufficiently large. This is the *Vainshtein mechanism*, and r_V is called the *Vainshtein radius*, within which general relativity is reproduced. Although we are considering small perturbations, we see that

$$\frac{\Box\varphi}{\Lambda^3} \gtrsim \mathcal{O}(1) \quad \text{for} \quad r \lesssim r_V. \tag{149}$$

This tells us why nonlinearity is important even in a weak gravity environment.

If the scalar degree of freedom accounts for the present accelerating expansion of the universe, Λ is expected to be as small as

$$\Lambda \sim (M_{\rm Pl}H_0^2)^{1/3}, \tag{150}$$

where H_0 is the present Hubble scale. This is deduced from the estimate

$$M_{\rm Pl}^2 H_0^2 \sim \dot{\phi}^2 \sim \frac{\dot{\phi}^2\ddot{\phi}}{\Lambda^3}, \quad \ddot{\phi} \sim H_0\dot{\phi}. \tag{151}$$

For $M \sim M_\odot$, equation (147) with (150) gives

$$r_V \sim 100\,{\rm pc}, \tag{152}$$

which is much larger than the size of the solar system.

5.2. Vainshtein screening in Horndeski theory

We can repeat the same analysis in the Horndeski theory [33, 222–224]. We only consider the case with $G_5 = 0$ for a reason to be explained later. In order for the background $g_{\mu\nu} = \eta_{\mu\nu}$ where $\phi = \phi_0 = $ const is a solution, we require that $G_2(\phi_0, 0) = G_{2\phi}(\phi_0, 0) = 0$.

In substituting (129) to the Horndeski action (now $M_{\rm Pl}$ is defined by $G_4(\phi_0, 0) = M_{\rm Pl}^2/2$) and expanding it in terms of perturbations, one must carefully retain the nonlinear terms with second derivatives because they can be large on small scales as suggested by (149). More specifically, we have the terms of the following forms in the Lagrangian:

$$(\partial h_{\mu\nu})^2, \quad (\partial\varphi)^2, \quad (\partial\varphi)^2(\partial^2\varphi)^n, \quad h_{\mu\nu}(\partial^2\varphi)^n. \tag{153}$$

However, as we are interested in the Vainshtein mechanism, we ignore the mass term $K_{\phi\phi}\varphi^2$. We thus find (in the original frame) [222]

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{\eta}{2}(\partial\varphi)^2 + \frac{\mu}{\Lambda^3}\mathcal{L}_3^{\rm Gal} + \frac{\nu}{\Lambda^6}\mathcal{L}_4^{\rm Gal} - \xi h^{\mu\nu}X^{(1)}_{\mu\nu} - \frac{\alpha}{\Lambda^3}h^{\mu\nu}X^{(2)}_{\mu\nu} + \frac{1}{2M_{\rm Pl}}h^{\mu\nu}T_{\mu\nu}, \tag{154}$$

where

$$\mathcal{L}_3^{\rm Gal} := -\frac{1}{2}(\partial\varphi)^2\Box\varphi, \tag{155}$$

$$\mathcal{L}_4^{\rm Gal} := -\frac{1}{2}(\partial\varphi)^2\left[(\Box\varphi)^2 - \varphi_{\mu\nu}\varphi^{\mu\nu}\right], \tag{156}$$

$X^{(1)}_{\mu\nu}$ was already defined in equation (131), and

$$X^{(2)}_{\mu\nu} := \varphi^\alpha_\mu\varphi_{\alpha\nu} - \Box\varphi\varphi_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\left[(\Box\varphi)^2 - \varphi_{\alpha\beta}\varphi^{\alpha\beta}\right]. \tag{157}$$

We have defined the dimensionless parameters η, ξ, μ, ν, and α by

$$G_{4\phi} = M_{\rm Pl}\xi, \quad G_{2X} - 2G_{3\phi} = \eta, \quad G_{3X} - 3G_{4\phi X} = -\frac{\mu}{\Lambda^3}, \quad G_{4X} = \frac{M_{\rm Pl}\alpha}{\Lambda^3}, \quad G_{4XX} = \frac{\nu}{\Lambda^6}, \tag{158}$$

with Λ being some energy scale. These dimensionless parameters are assumed to be $\mathcal{O}(1)$ unless they vanish. The Lagrangian (154) describes the effective theory for the Vainshtein mechanism. Note that this effective theory has the Galilean shift symmetry, $\varphi \to \varphi + b_\mu x^\mu + c$.

One notices the presence of the new term representing the mixing of the scalar degree of freedom and the graviton: $h^{\mu\nu}X^{(2)}_{\mu\nu}$. This, as well as $h^{\mu\nu}X^{(1)}_{\mu\nu}$, can be demixed through the field redefinition [225]

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - 2\xi\varphi\eta_{\mu\nu} + \frac{2\alpha}{\Lambda^3}\partial_\mu\varphi\partial_\nu\varphi. \tag{159}$$

The new piece $(2\alpha/\Lambda^3)\partial_\mu\varphi\partial_\nu\varphi$ is equivalent to a disformal transformation. After this transformation the effective Lagrangian (154) reduces to

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}\tilde{h}^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} - \frac{\eta+6\xi^2}{2}(\partial\varphi)^2 + \frac{\mu+6\alpha\xi}{\Lambda^3}\mathcal{L}_3^{\rm Gal} + \frac{\nu+2\alpha^2}{\Lambda^6}\mathcal{L}_4^{\rm Gal} + \frac{1}{2M_{\rm Pl}}\tilde{h}^{\mu\nu}T_{\mu\nu} - \frac{\xi}{M_{\rm Pl}}\varphi T + \frac{\alpha}{M_{\rm Pl}\Lambda^3}\partial_\mu\varphi\partial_\nu\varphi T^{\mu\nu}. \tag{160}$$

Things are more transparent in this Einstein frame than in the original Jordan frame.

To see how the Vainshtein mechanism operates generically, let us again consider a spherically symmetric matter distribution. The field equation for φ,

$$(\eta+6\xi^2)\Box\varphi + \frac{\mu+6\alpha\xi}{\Lambda^3}\left[(\Box\varphi)^2 - \varphi_{\mu\nu}\varphi^{\mu\nu}\right] + \frac{\nu+2\alpha^2}{\Lambda^6}\left[(\Box\varphi)^3 - 3\varphi_{\mu\nu}\varphi^{\mu\nu}\Box\varphi + 2\varphi_{\mu\nu}\varphi^{\nu\lambda}\varphi^\mu_\lambda\right] = \frac{\xi}{M_{\rm Pl}}T + \frac{2\alpha}{M_{\rm Pl}}\varphi_{\mu\nu}T^{\mu\nu}, \tag{161}$$

can be written in the following form after being integrated once:

$$\frac{\eta+6\xi^2}{2}x + (\mu+6\alpha\xi)x^2 + (\nu+2\alpha^2)x^3 = -\xi A, \tag{162}$$

where we introduced the convenient dimensionless quantities

$$x(r) := \frac{1}{\Lambda^3}\frac{\varphi'}{r}, \quad A(r) := \frac{1}{M_{\rm Pl}\Lambda^3}\frac{\mathcal{M}(r)}{8\pi r^3}. \tag{163}$$

The field equations for $\tilde{h}_{\mu\nu}$ imply

$$\frac{1}{\Lambda^3}\frac{\tilde{\Phi}'}{r} = \frac{1}{\Lambda^3}\frac{\tilde{\Psi}'}{r} = A. \tag{164}$$

Since the special case with $\nu + 2\alpha^2 = 0$ was already essentially analyzed in the previous subsection, we focus on the generic case with $\nu + 2\alpha^2 \neq 0$. We have, for $A \gg 1$,

$$x \simeq \left(\frac{-\xi A}{\nu+2\alpha^2}\right)^{1/3}. \tag{165}$$

The metric perturbations in the Jordan frame are obtained from $\Phi = \tilde{\Phi} - \xi\varphi$ and $\Psi = \tilde{\Psi} + \xi\varphi - (\alpha/\Lambda^3)(\varphi')^2$, but we see from (165) that the extra scalar field contributions are small, yielding $M_{\rm Pl}^{-1}\Phi' \simeq M_{\rm Pl}^{-1}\Psi' \simeq G_N\mathcal{M}/r^2$ where

$8\pi G_N = M_{\rm Pl}^{-2} = [2G_4(\phi_0, 0)]^{-1}$. Since $A \propto r^{-3}$ outside the source, it is appropriate to define the Vainshtein radius $r_V := (\mathcal{M}/8\pi M_{\rm Pl}\Lambda^3)^{1/3}$ so that $A = (r_V/r)^3$. The nonlinearity in the scalar field equation of motion thus helps to suppress the force mediated by φ, so that standard gravity is recovered inside the Vainshtein radius r_V.

Though the expression is slightly more complicated, the complete effective Lagrangian from the Horndeski theory including the G_5 term can be obtained in the same way as above [222]. One then finds the quintic Galileon interaction for φ and another mixing term between φ and $h_{\mu\nu}$ in the effective Lagrangian. This mixing cannot be eliminated by a field redefinition [225]. It can be shown that the screened region outside the spherically symmetric matter distribution is unstable against linear perturbations in the presence of this mixing [222].

So far we have considered the simplest background solution, $g_{\mu\nu} = \eta_{\mu\nu}$ with $\phi = \phi_0 =$ const, and static, spherically symmetric perturbations on top of the background. The above analysis has been extended to a cosmological background with time-dependent ϕ_0 in [226]. On a cosmological background we start with the Newtonian gauge metric (83) rather than (129) because Lorentz invariance is spontaneously broken due to non-vanishing $\dot\phi_0(t)$. Keeping the appropriate nonlinear terms, the Lagrangian under the quasi-static approximation is computed as

$$\begin{aligned}\mathcal{L}_{\rm eff} = a\Big[&M^2\left(-c_{\rm GW}^2\Psi\partial^2\Psi + 2\Psi\partial^2\Phi\right)\\ &-\frac{\eta}{2}(\partial\varphi)^2 - 2M\left(\xi_1\Phi - 2\xi_2\Psi\right)X^{(1)}\Big]\\ &+\frac{\mu}{a\Lambda^3}\mathcal{L}_3^{\rm Gal} + \frac{\nu}{a^3\Lambda^6}\mathcal{L}_4^{\rm Gal}\\ &-\frac{2M}{a\Lambda^3}\left(\alpha_1\Phi - \alpha_2\Psi\right)X^{(2)} - a^3\Phi\delta\rho, \end{aligned} \tag{166}$$

where $\delta\rho$ is a density perturbation,

$$M^2 := \mathcal{G}_T, \quad X^{(1)} := \partial^2\varphi, \quad X^{(2)} := \frac{1}{2}\left[(\partial^2\varphi)^2 - \partial_i\partial_j\varphi\partial^i\partial^j\varphi\right], \tag{167}$$

and

$$\mathcal{L}_3^{\rm Gal} := -\frac{1}{2}(\partial\varphi)^2\partial^2\varphi, \quad \mathcal{L}_4^{\rm Gal} := -\frac{1}{2}(\partial\varphi)^2\left[(\partial^2\varphi)^2 - \partial_i\partial_j\varphi\partial^i\partial^j\varphi\right]. \tag{168}$$

The coefficients are (slowly varying) functions of time in general. Explicitly, we have

$$\frac{M\alpha_1}{\Lambda^3} := G_{4X} + 2XG_{4XX}, \quad \frac{M\alpha_2}{\Lambda^3} := G_{4X}, \quad \frac{\nu}{\Lambda^6} := G_{4XX}, \tag{169}$$

while

$$\begin{aligned}&M\xi_1 \simeq -XG_{3X} + G_{4\phi} + 2XG_{4\phi X}, \quad M\xi_2 \simeq G_{4\phi} - 2XG_{4\phi X},\\ &\frac{\mu}{\Lambda^3} \simeq -\left(G_{3X} - 3G_{4\phi X} + 2XG_{4\phi XX}\right),\end{aligned} \tag{170}$$

where to simplify the expressions we ignored $\ddot\phi_0$ and H in ξ_1, ξ_2, and μ. The explicit expression for η is not important here. Note that if $c_{\rm GW}^2 = 1$ then $\alpha_1 = \alpha_2 = \nu = 0$.

In the present case, we stay in the Jordan frame rather than try to disentangle the couplings between φ and the metric potentials such as $\Phi X^{(1)}$. For a spherical overdensity, the field equations derived from (166) are written as

$$y - c_{\rm GW}^2 z = -2\xi_2 x - \alpha_2 x^2, \tag{171}$$

$$z = A + \xi_1 x + \alpha_1 x^2, \tag{172}$$

and

$$\frac{\eta}{2}x - \xi_1 y + 2\xi_2 z - 2(\alpha_1 y - \alpha_2 z)x + \mu x^2 + \nu x^3 = 0, \tag{173}$$

where we introduced

$$\begin{aligned}&x := \frac{1}{\Lambda^3}\frac{\varphi'}{r}, \quad y := \frac{M}{\Lambda^3}\frac{\Phi'}{r}, \quad z := \frac{M}{\Lambda^3}\frac{\Psi'}{r},\\ &A := \frac{1}{M\Lambda^3}\frac{\mathcal{M}}{8\pi r^3}, \quad \mathcal{M} := 4\pi\int_0^r \delta\rho(t,s)s^2{\rm d}s,\end{aligned} \tag{174}$$

and took $a \to 1$ for simplicity. Using equations (171) and (172) one can remove y and z from equation (173) to get

$$\begin{aligned}&\left[c_1 + 2(\alpha_2 - c_{\rm GW}^2\alpha_1)A\right]x + c_2x^2 + \left[\nu + 4\alpha_1\alpha_2 - 2c_{\rm GW}^2\alpha_1^2\right]x^3\\ &= -(2\xi_2 - c_{\rm GW}^2\xi_1)A,\end{aligned} \tag{175}$$

where the coefficients c_1 and c_2 are written in terms of η, ξ_1, etc. This extends equation (162) to a time-dependent background with $\dot\phi_0 \neq 0$.

In theories with $c_{\rm GW}^2 = 1$, equation (175) becomes

$$c_1x + c_2x^2 = \left(\frac{\eta}{2} - \xi_1^2 + 4\xi_1\xi_2\right)x + \mu x^2 = -(2\xi_2 - \xi_1)A. \tag{176}$$

The Vainshtein radius is defined by $A(r_V) = 1$, and for $A \gg 1$ we have $x \sim A^{1/2} \ll A \simeq y \simeq z$. Thus, inside the Vainshtein radius the metric potentials obey

$$\Phi' = \Psi' = \frac{G_N\mathcal{M}}{r^2}, \quad 8\pi G_N := \frac{1}{2G_4(\phi_0(t))}. \tag{177}$$

The situation is similar to that for a static background, $\dot\phi_0 = 0$. However, it is interesting to see that, even in the minimally coupled theories with $G_4 =$ const, we still have $M\xi_1 \simeq -XG_{3X} \neq 0$ if $\dot\phi_0$ is non-vanishing, so that φ is coupled to the source via the G_3 term.

In theories with $c_{\rm GW}^2 \neq 1$, A in the coefficient of the linear term plays an important role. For $A \gg 1$, equation (175) reduces to

$$\begin{aligned}&2(\alpha_2 - c_{\rm GW}^2\alpha_1)Ax + \left[\nu + 4\alpha_1\alpha_2 - 2c_{\rm GW}^2\alpha_1^2\right]x^3 \simeq 0\\ &\Rightarrow \quad x^2 \simeq -\frac{2(\alpha_2 - c_{\rm GW}^2\alpha_1)}{\nu + 4\alpha_1\alpha_2 - 2c_{\rm GW}^2\alpha_1^2}A.\end{aligned} \tag{178}$$

This is in contrast to the screened solution on a static background (165), $x^3 \sim A$. Substituting the solution (178) to (171) and (172), one finds that

$$\Phi' = \Psi' = \frac{G_N\mathcal{M}}{r^2}, \quad 8\pi G_N = \left.\frac{1}{2[G_4 - 4X(G_{4X} + XG_{4XX})]}\right|_{\phi=\phi_0(t)}. \tag{179}$$

As seen from equations (177) and (179), apparently the standard gravitational law is recovered[6]. However, the effective Newton 'constant' on a cosmological background depends on time even inside the Vainshtein radius through the cosmological evolution of ϕ_0 [228]. Although it would be natural to think of a slow variation $|\dot{G}_N/G_N| = \mathcal{O}(1) \times H_0$, the observational bounds from Lunar Laser Ranging require a much slower variation, $|\dot{G}_N/G_N| < 0.02H_0$ [229]. This limit can be used to constrain cosmological scalar–tensor theories.

We have seen how Vainshtein screening operates around a (quasi-)static, spherically symmetric body. The Vainshtein mechanism away from this simplified setup has been investigated in [230–240].

5.3. Partial breaking of Vainshtein screening

In DHOST theories, the operation of the Vainshtein mechanism turns out to be very nontrivial. This can be seen as follows. Let us consider a simple DHOST theory whose Lagrangian is given by

$$\mathcal{L} = G_2 - G_3\Box\phi + G_4 R + G_{4X}\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] + A_3\left\{X\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] + \Box\phi\phi^\mu\phi^\nu\phi_{\mu\nu} - \nabla_\mu X\nabla^\mu X\right\}. \quad (180)$$

This theory belongs to the GLPV family (see equation (116)). Due to the new term in the second line, the effective Lagrangian for the Vainshtein mechanism under the quasi-static approximation is now given by [241] (see also [242, 243])

$$\begin{aligned}\mathcal{L}_{\rm eff} = a\Big\{&M^2\left[-c_{\rm GW}^2\Psi\partial^2\Psi + 2(1+\alpha_H)\Psi\partial^2\Phi\right] \\ &- \frac{\eta}{2}(\partial\varphi)^2 - 2M\left(\xi_1\Phi - 2\xi_2\Psi\right)X^{(1)}\Big\} \\ &+ \frac{\mu}{a\Lambda^3}\mathcal{L}_3^{\rm Gal} + \frac{\nu}{a^3\Lambda^6}\mathcal{L}_4^{\rm Gal} - \frac{2M}{a\Lambda^3}\left(\alpha_1\Phi - \alpha_2\Psi\right)X^{(2)} - a^3\Phi\delta\rho \\ &+ \frac{2aM^{3/2}}{\Lambda^{3/2}v}\alpha_H\dot{\Psi}\partial^2\varphi - \frac{2M}{a\Lambda^3 v^2}\alpha_H\partial_i\Psi\partial_j\varphi\partial_i\partial_j\varphi, \end{aligned} \quad (181)$$

where

$$M^2 := 2\left(G_4 - 2XG_{4X} - 2X^2A_3\right), \quad (182)$$

$$\frac{M\alpha_1}{\Lambda^3} := G_{4X} + 2XG_{4XX} + X(5A_3 + 2XA_{3X}), \quad (183)$$

$$\frac{M\alpha_2}{\Lambda^3} := G_{4X} + XA_3, \quad (184)$$

$$\frac{\nu}{\Lambda^6} := G_{4XX} + 2A_3 + XA_{3X}, \quad (185)$$

$$M^2\alpha_H := 4X^2A_3, \quad (186)$$

and we write $v := \dot{\phi}_0/(M^{1/2}\Lambda^{3/2})$ $(= \mathcal{O}(1))$. Explicit expressions of the other coefficients are not important. The two terms in the third line are essentially new contributions in this DHOST theory. Note that even in the quasi-static regime one cannot, in general, neglect the first term in the third line because

$$\frac{M^{3/2}}{\Lambda^{3/2}v}\alpha_H\dot{\Psi}\partial^2\varphi \sim \frac{M^{3/2}H_0}{\Lambda^{3/2}v}\alpha_H\Psi\partial^2\varphi \sim \frac{M\alpha_H}{v}\Psi X^{(1)}. \quad (187)$$

This term modifies the linear evolution equation for density perturbations. More specifically, the coefficient of the friction term ($\propto \dot{\delta}$) in the evolution equation for δ acquires an additional contribution other than the Hubble parameter [47, 48].

In the regime where the nonlinear terms are dominant, one obtains

$$(1+\alpha_H)y - c_{\rm GW}^2 z \simeq -\alpha_2 x^2 - \frac{\alpha_H}{v^2}(x^2 + rxx'), \quad (188)$$

$$(1+\alpha_H)z \simeq A + \alpha_1 x^2, \quad (189)$$

and

$$-2(\alpha_1 y - \alpha_2 z)x + \nu x^3 - \frac{\alpha_H}{v^2}(3xz + rxz') \simeq 0, \quad (190)$$

where for simplicity we ignored the cosmic expansion by taking $a = 1$. Equations (188)–(190) can be regarded as generalizations of equations (171)–(173). Using equations (188) and (189) one can express y and z in terms of x and x', and then eliminate y and z from (190). After doing so one would obtain a differential equation for x. However, in fact all the derivative terms are canceled out, yielding an algebraic equation for x. This is the consequence of the degeneracy of the system. The resultant algebraic equation is solved to give $x^2 = (\cdots)A + (\cdots)A'$. Substituting this to equations (188) and (189), we arrive at

$$y = 8\pi G_N M^2\left[A + \frac{\Upsilon_1}{4}\frac{(r^3A)''}{r}\right], \quad (191)$$

$$z = 8\pi G_N M^2\left[A - \frac{5\Upsilon_2}{4}\frac{(r^3A)'}{r^2}\right], \quad (192)$$

where we defined the effective Newton 'constant'[7]

$$8\pi G_N := \left[2G_4 - 4X(G_{4X} + XG_{4XX}) - 4X^2(5A_3 + 2XA_{3X})\right]^{-1}, \quad (193)$$

and the dimensionless parameters

$$\Upsilon_1 := -\frac{4X^2A_3^2}{G_4(G_{4XX} + 2A_3 + XA_{3X}) + G_{4X}(G_{4X} + XA_3)}, \quad (194)$$

$$\Upsilon_2 := -\frac{4XA_3(G_{4X} + 2XG_{4XX} + 5XA_3 + 2X^2A_{3X})}{5[G_4(G_{4XX} + 2A_3 + XA_{3X}) + G_{4X}(G_{4X} + XA_3)]}. \quad (195)$$

These two parameters characterize the deviation from the Horndeski theory. In terms of more familiar quantities, equations (191) and (192) are written as

$$\Phi' = G_N\left(\frac{\mathcal{M}}{r^2} + \frac{\Upsilon_1\mathcal{M}''}{4}\right), \quad (196)$$

[6] The Friedmann equation in the early time takes the form $3H^2 \simeq 8\pi G_{\rm cos}\rho$, where 'cosmological G' coincides with G in Newton's law: $G_{\rm cos} = G_N$. Note that in general this G_N is different from the effective gravitational coupling for gravitational waves, $G_{\rm GW} := (8\pi\mathcal{F}_T)^{-1}$. The difference can be constrained from the Hulse–Taylor pulsar [227].

[7] This also coincides with '$G_{\rm cos}$' in the Friedmann equation.

$$\Psi' = G_N\left(\frac{\mathcal{M}}{r^2} - \frac{5\Upsilon_2\mathcal{M}'}{4r}\right). \tag{197}$$

From equations (196) and (197) we see the following: (i) the Vainshtein mechanism works outside a source because $\mathcal{M} = \text{const}$ there; (ii) the Vainshtein mechanism breaks inside a source where $\mathcal{M}$ is no longer constant. This result implies that DHOST theories can be constrained by astronomical observations of stars, galaxies, and galaxy clusters [244–252]. This class of gravity modification can even be tested through the speed of sound in the atmosphere of the Earth [253].

We have thus seen that gravity is modified inside a source in a simple DHOST theory. Such partial breaking of Vainshtein screening occurs in more general DHOST theories as well. Of particular interest are theories satisfying $c_{\rm GW}^2 = 1$ (i.e. theories satisfying equations (124)–(126)). After some tedious calculations one ends up with [171, 254–256]

$$\Phi' = G_N\left(\frac{\mathcal{M}}{r^2} + \frac{\Upsilon_1\mathcal{M}''}{4}\right), \tag{198}$$

$$\Psi' = G_N\left(\frac{\mathcal{M}}{r^2} - \frac{5\Upsilon_2\mathcal{M}'}{4r} + \Upsilon_3\mathcal{M}''\right), \tag{199}$$

where

$$8\pi G_N := \left[2(f - Xf_X - 3X^2A_3)\right]^{-1}, \tag{200}$$

and

$$\Upsilon_1 := -\frac{(f_X - XA_3)^2}{A_3 f}, \quad \Upsilon_2 := \frac{8Xf_X}{5f}, \quad \Upsilon_3 := \frac{(f_X - XA_3)(f_X + XA_3)}{4A_3 f}. \tag{201}$$

The three parameters are not independent: $2\Upsilon_1^2 - 5\Upsilon_1\Upsilon_2 - 32\Upsilon_3^2 = 0$. Note that in deriving the above result we have implicitly assumed that $A_3 \neq 0$, which means that we need to be more careful when considering DHOST theories devoid of the decay of gravitational waves into ϕ [205]. The Vainshtein regime of this special class of DHOST theories has been investigated recently in [257, 258].

Going beyond the weak gravity regime, relativistic stars in DHOST theories have been studied in [259–262].

As explained in the previous section, class Ia DHOST theories can be mapped to the Horndeski theory via a disformal transformation. Therefore, DHOST theories with minimally coupled matter are equivalent to the Horndeski theory with disformally coupled matter. This is the reason why the behavior of gravity in DHOST theories is different from that in the Horndeski theory in the presence of matter.

6. Black holes in Horndeski theory and beyond

In general relativity, a black hole is characterized solely by its mass, angular momentum, and electric charge. This is the well-known no-hair theorem. In scalar–tensor theories, the scalar field would not be regular at the horizon in many cases unless it has a trivial profile. The no-hair theorem can thus be extended to cover a wider class of theories [263, 264], though it can certainly be evaded, e.g. by a non-minimal coupling to the Gauss–Bonnet term [265–268]. In light of the modern reformulation of the Horndeski theory, it has been argued that nontrivial profiles of the Galileon field are not allowed around static and spherically symmetric black holes [269]. The proof of [269] is based on the shift symmetry of the scalar field and several other assumptions. It is therefore intriguing to explore how one can circumvent the no-hair theorem in the context of the Horndeski/beyond Horndeski theories.

For example, by tuning the form of the Horndeski functions one can evade the no-hair theorem [270, 271]. Another possibility is relaxing the assumptions on the time independence of ϕ and/or its asymptotic behavior [272–276]. In particular, it is important to notice that in shift-symmetric scalar–tensor theories the metric can be static even if the scalar field is linearly dependent on time [277],

$$\mathrm{d}s^2 = -h(r)\mathrm{d}s^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right), \tag{202}$$

$$\phi = qt + \psi(r), \quad q = \text{const}, \tag{203}$$

because the field equations depend on ϕ through $\partial_\mu\phi$ due to the shift symmetry. Starting from the ansatz (202) and (203), various hairy black hole solutions have been obtained in scalar–tensor theories with the derivative coupling of the form $\sim G^{\mu\nu}\phi_\mu\phi_\nu$ in [275]. The same strategy was then used to derive hairy black hole solutions from more general Lagrangians in the Horndeski family [278–280], its bi-scalar extension [281], and the GLPV/DHOST theories [282–285] (see [286, 287] for a review). Some of these solutions have the Schwarzschild(-(A)dS) geometry dressed with nontrivial scalar field profiles, i.e. a stealth property.

As explained in the previous section, strong constraints have been imposed on scalar–tensor theories as alternatives to dark energy after GW170817. Implications of the limit $c_{\rm GW}^2 = 1$ for black holes in scalar–tensor theories have been discussed in [284, 288, 289].

Perturbations of black holes in scalar–tensor theories are also worth investigating for the same reasons as in the case of cosmological perturbations: one can judge the stability of a given black hole solution and give predictions for observations. As in general relativity, for a spherically symmetric background it is convenient to decompose metric perturbations into even parity (polar) and odd parity (axial) modes. The scalar field perturbations come into play only in the even parity sector. Since the Horndeski theory and its extensions preserve parity, the equations of motion for the even and odd modes are decoupled. Within the Horndeski theory, the quadratic actions and the stability conditions of the even and odd parity perturbations were derived for a general static and spherically symmetric background with a time-independent scalar field in [290, 291][8]. See also [294, 295]. General black hole perturbation theories covering a wider class of Lagrangians have been developed in [296, 297]. Odd parity perturbations and the stability of static and spherically symmetric solutions

[8] These papers contain typos, some of which were pointed out in [292, 293]. The reader is recommended to refer to the latest versions of arXiv:1202.4893 and arXiv:1402.6740.

with a linearly time-dependent scalar field are discussed in [298, 299], but their conclusions about instabilities have been questioned [300].

The perturbation analysis of spherically symmetric solutions in the Horndeski theory can be applied not only to black holes, but also to wormholes. The structure of the stability conditions for spherically symmetric solutions is analogous to that for cosmological solutions, which allows us to formulate the no-go theorem for stable wormholes in the Horndeski theory in a similar way to proving the no-go for non-singular cosmologies introduced in section 3.3 [133, 301–303]. Also in the wormhole case, theories beyond Horndeski admit stable solutions [293, 304].

7. Conclusion

In this review, we have discussed recent advances in the Horndeski theory [12] and its healthy extensions, i.e. degenerate higher-order scalar–tensor (DHOST) theories [156]. We have reviewed how the Horndeski theory was 'rediscovered' in its modern form in the course of developing the Galileon theories [22, 23]. This rediscovery has stimulated extensive research on physics beyond the cosmological standard model using the general framework of scalar–tensor theories. Along with renewed interest in the Horndeski theory, the border of Ostrogradsky-stable scalar–tensor theories has expanded recently to include more general DHOST theories. Among the DHOST theories, it is quite likely that only the Horndeski theory and its disformal relatives admit stable cosmological solutions and hence can potentially be viable [169, 175].

In light of GW170817, we have seen that some of the free functions in DHOST Lagrangians can be strongly constrained upon imposing $c_{\mathrm{GW}} = 1$ [195–202]. However, one must be careful about the range of the validity of modified gravity under consideration. The cutoff scale of modified gravity as an alternative to dark energy may be close to the energy scales observed at LIGO [209], and modified gravity in the early universe (i.e. at much higher energies) is free from the constraint $c_{\mathrm{GW}} \simeq 1$. Even if one imposes $c_{\mathrm{GW}} = 1$, there is still an interesting class of scalar–tensor theories in which the nonstandard behavior of gravity arises only inside matter [171, 254–256].

Having obtained a general framework of healthy scalar–tensor theories, it would be exciting to test gravity with cosmological and astrophysical observations as well as to explore novel models of the early universe. Now we are at the dawn of gravitational wave astrophysics and cosmology, and gravitational waves allow us to access physics at extremely high energies and in the strong-gravity regime. In view of this, we hope that the general framework presented in this review will prove more and more useful in exploring the fundamental nature of gravity. We also hope that generalizing gravity will result in gaining even deeper insights into theoretical aspects of gravity.

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