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Simple quantitative examples illustrating how the centrifugal and Coriolis forces 'rescue' Newton's second law in rotating frames

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Abstract

This paper is prepared for those who have already learned the physics of circular motions and the centripetal force and are curious about the centrifugal and Coriolis forces. Instead of deriving them or exploring the theory, we aim to bring forward several quantitative examples to illustrate what conceptually they are, along with evidencing the notion that 'Newton's second law in rotating reference frames holds only when the centrifugal and Coriolis forces are in action'. All our examples involve mostly circular motions, and, most importantly, our approach to dealing with the frame transformation is without using the equations and purely on a qualitative argument basis.

Keywords: Coriolis force, centrifugal force, rotating frame of reference, inertial and noninertial frames, validity of Newton's second law of motion, fictitious pseudo-force, circular motion centripetal force

1. Introduction

Students may be perplexed as in classrooms, they learned the 'centripetal force', but, in our daily lives, the term 'centrifugal force' is more often heard instead, e.g. clothes are dried by centrifugal force in a washing machine; car passengers are pushed outwards by centrifugal force when the car is turning a corner. Quite commonly, learners of uniform circular motion have difficulty grasping and applying its physics, especially when confronted with the often-heard but physics-strange-to-them 'centrifugal force' [1, 2].

The confusion emerges and lasts; probably this is attributed to the fact that they may not have the opportunity to learn the centrifugal force systematically in schools. But answering the question, 'What is centrifugal force?' seriously is indeed not an easy task because it inevitably involves other difficult concepts, namely, fictitious force, rotating frames of reference, perhaps as well as the Coriolis force, and the invalidity of Newton's second law in rotating frames. Despite this, we do believe an effective teaching strategy can allow our students to acquire some of the basic, true understanding of the subject. A successful teaching strategy, on the other hand, relies on many factors, including the support of the available teaching resources.

On the topic of centrifugal and Coriolis forces, college introductory textbooks [3] usually give qualitative expositions; mechanics textbooks from the intermediate level [4, 5] deliver broad and quantitative discussions; journal papers have explored the topic in various aspects of different levels, including revisiting or introducing new thoughts [6–12], interesting experiments [13–18], concise derivations [19] and practical applications [20]. Nevertheless, we hope that we can contribute a little as well by suggesting several quantitative cases that students may find helpful. Although in each of our examples, the above-mentioned difficult concepts are all involved, we are confident that they are still 'friendly' to most pre-college physics students. At least the last two of our cases, to the best of our knowledge, are not available elsewhere.

We are going to discuss four cases in the order of their difficulties. In each of them, we first define a motion, mostly circular motion, on the ground (inertial frame). Then, its corresponding motion in our defined rotating reference frame is achieved by applying some geometry and physics arguments. Finally, real and the centrifugal and Coriolis forces are applied to the transformed motion to check if Newton's second law holds. Our criterion for selecting these examples is that the motion after the transformation must be comparatively simple too. All motions discussed here are restricted to 2D.

Some basic knowledge relating to our later discussions is revised first, and then the rotating reference frame employed by us is defined. For readers' reference, the formal mathematical frame transformations corresponding to our last two cases are supplemented in the appendix.

2. Basic knowledge [3–5]

It is important that the *centripetal* force is not one on the list: weight, friction, contact force, tension, spring force, elastic force, gravitational force, Coulomb force, magnetic force, etc. It is only the name specially given to the radial net force required to produce the circular motion of an object. The magnitude is equal to $m\omega^2 r$, where m is the mass of the object, ω is the angular speed of the circular motion, and r is the radius of the circular motion. This force is directed towards the center of the circular motion.

The *centrifugal* and *Coriolis* forces are fictitious in Newtonian mechanics. They only appear in rotating frames of reference, in which Newton's second law is thus 'rescued' by forming a net force complying with the law by adding the vectors of these two fictitious forces and any real force(s) (the above list) acting on the object. This idea will be further elaborated through our examples. The magnitude of the centrifugal force on a particle of mass m is $m\Omega^2 r'$, where Ω is the angular speed of the rotating frame and r' is the perpendicular distance between m and the axis of rotation of the rotating frame, while its direction is always outward away from that axis. People may know that the Coriolis force plays a key role in the formation of cyclones, but its significance is much more than that. When the velocity of m seen from the rotating frame, v', is confined on a plane perpendicular to the axis of rotation of the rotating frame, the magnitude of the Coriolis force is $2m\Omega v'$. The direction of the Coriolis force can be found conveniently from its vector form: $-2m\vec{\Omega} \times \vec{v'}$.

Note that the centripetal and centrifugal forces refer to two different centers, which are the center of the circular motion and the center of rotation of the rotating frame, respectively. They are, in principle, not the same.

In the following paragraphs, the tangential anticlockwise and radially inward directions are taken as positive. Under this sign convention, the centrifugal force is $-m\Omega^2 r'$; of the Coriolis force, the radial component is $-2m\Omega v'_t$ and the tangential component is $2m\Omega v'_r$, where v'_t and v'_r are the tangential and radial components of v', respectively.

3. Our rotating reference frame (RF)

As shown in figure 1, our rotating reference frame (RF) can be thought of as a plane parallel to the horizontal ground surface (inertial frame), and its coordinate origin (O′) aligns vertically with that of the ground (O). At time $t = 0$, its two axes (x', y') are instantaneously parallel to those of the ground (x, y), respectively. As seen from above, RF rotates anticlockwise with the axis of rotation passing through O′ vertically at the constant angular rate Ω.

To help visualise how a transformed motion is produced, one can imagine that m shines a vertical laser beam on RF. As m moves on the ground, the laser spot on RF depicts exactly what motion an observer on RF will see. The movement of m observed from the ground and that from RF are connected by a set of equations (see appendix), which may, however, somehow obscure the physics. Below, we select some easy cases, exemplifying how the two fictitious forces work.

4. Case 1

On the ground, a particle of mass m is at rest and placed at a distance r from O. No net force is acting on it. As seen from RF, m rotates backwards, so it moves around a circle centred at O′, with a radius $r' = r$ and angular speed $= -\Omega$, i.e. clockwise. Now, m undergoes a circular motion without a net force, contradicting Newton's second law of motion. The introduction of the two fictitious forces is to 'save' the law. Here, the centrifugal force is $-m\Omega^2 r'$, and since $v' = -\Omega r'$, the Coriolis force is $-2m\Omega(-\Omega r') = 2m\Omega^2 r'$ (radially inward). Adding these two fictitious forces gives $m\Omega^2 r'$, which is exactly the necessary centripetal force for the circular motion seen from RF. The validity of Newton's second law is thus restored.

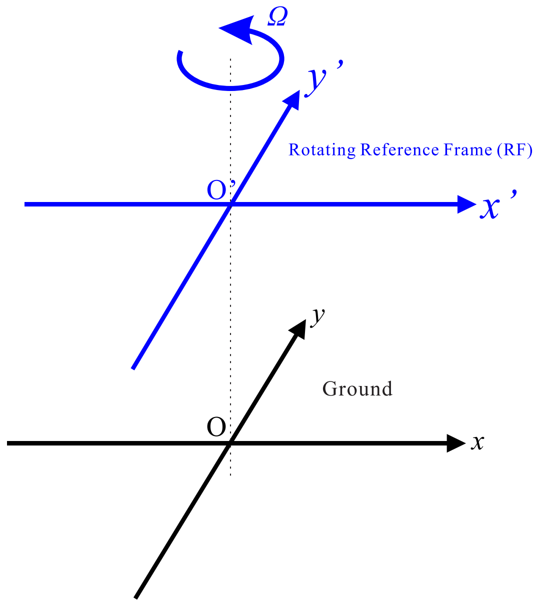

Figure 1. The rotating frame of reference (RF) employed in this paper is so defined. RF's origin (O′) sits atop that of the ground (O). At time $t = 0$, the two coordinate systems are parallel with each other. RF rotates anticlockwise with the axis of rotation passing through O′ vertically at the constant rate Ω, as seen from above.

5. Case 2

Like Case 1, but this time m itself also rotates on the ground about O with an angular speed ω. Therefore, a real net force of value $m\omega^2 r$, acting as the centripetal force, is acting on m.

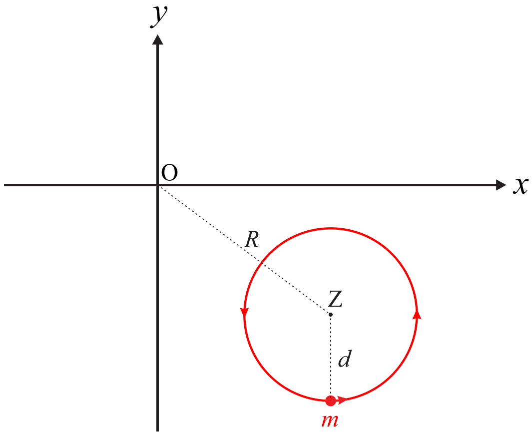

Figure 2. The circle on which m moves constantly on the ground at the rate Ω is centred at Z and has a radius of d. At $t = 0$, m is at the point of the smallest ordinate on the circle and starts to move.

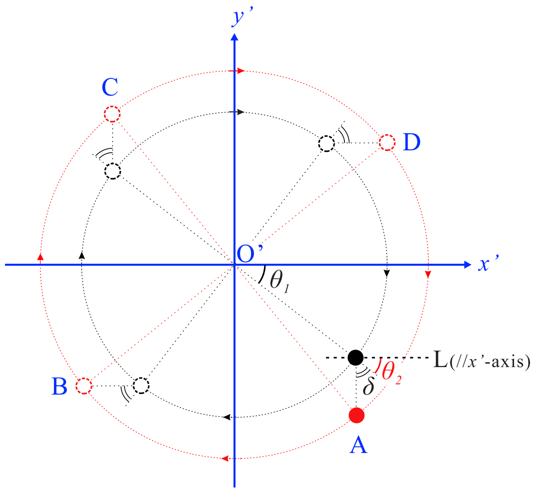

Figure 3. Two objects, one in black and one in red, rotate together clockwise about O′ at the same rate. As they go from A to B, to C, and then to D, the red object is at the bottom, left side, top, and right side of the black object, respectively, showing that the former rotates about the latter once in one revolution of the two objects. Note that only the orientation, not the shape of the triangle formed by the origin and the two objects, changes during the revolution, so the angle δ is a constant, but the angle θ_2 varies exactly in the same manner as θ_1.

As seen from RF, m rotates at $\omega - \Omega$ about O′, with the radius $r' = r$. The centrifugal force is $-m\Omega^2 r'$, and the Coriolis force is $-2m\Omega v' = -2m\Omega(\omega - \Omega) r'$. The net sum of all the real and fictitious forces is $m\omega^2 r - m\Omega^2 r' - 2m\Omega(\omega - \Omega) r' = m(\omega - \Omega)^2 r'$. This result is exactly what we expected.

6. Case 3

This one is a bit complicated, but qualitative arguments are still adequate. As shown in figure 2, a particle m rotates about an arbitrary point Z on the ground with an arbitrary radius d and at the rate Ω, the same rotation rate as that of RF. Assume, without loss of generality, that m is at the point of the smallest ordinate on the circle when it starts to move at $t = 0$. What is the subsequent motion of m on RF? Let us explain step-by-step.

One may intuitively think that, since the center Z rotates backwards as seen from RF, the observed motion of m on RF would be like that: according to the problem, m rotates at Ω about Z, which at the same time rotates backwards (clockwise at $-\Omega$) about O′. However, this picture is only partly correct since a critical effect has not yet been considered.

What is it? Consider the two objects, one in black and one in red, shown in figure 3. They rotate together about the origin clockwise at the same rate but at different radii (not necessarily collinear with the center). It is interesting to note that, during the period of one revolution, the red one rotates clockwise about the black one once, as evidenced by comparing their positions at the four positions, A-D, marked in the figure. Below is a more formal proof.

The line L in figure 3 passes through the black object and is parallel with the x'-axis, hence $\theta_1 = \theta_2$. Since the angle δ is constant, the rate of change of θ_1 and that of $\theta_2 + \delta$ must be the same, implying that the red object rotates about the black one at the same rate as the latter revolves about O′.

Also, it is correct to say that the black one rotates around the red one. We see that, when any two objects rotate together about the same point at the same rate, one will automatically rotate about the other at the same rate as their common revolution. For our later reference, we call this the co-rotation effect.

Why is this effect so important? The reason is obvious: as seen from RF, all things on the ground, including Z and m, rotate backwards about O′ at the same rate $-\Omega$. Hence, it is legitimate to apply

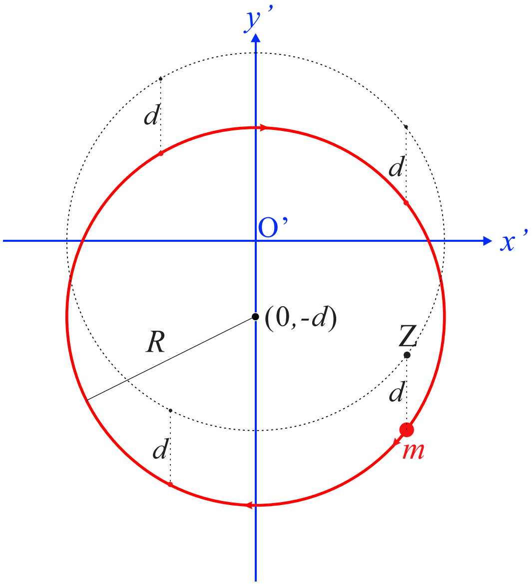

Figure 4. The red circle is the trajectory of m, seen from RF, corresponding to the ground motion shown in figure 2. The black circle is the trajectory of Z. These two circles are displaced the same as the initial displacement of m from Z.

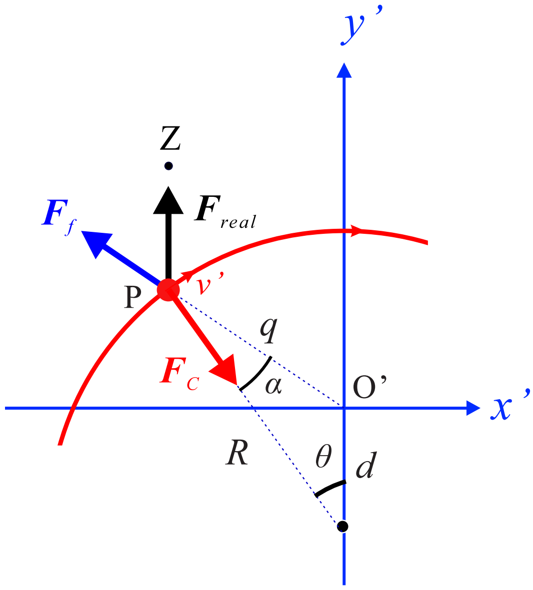

Figure 5. The centrifugal (F_f) and Coriolis (F_C) forces, together with the real force (F_{real}), produce the necessary centripetal force to make m turn around on the circle at point P.

the co-rotation effect to Z and m. Therefore, on RF, m is to perform two rotations about Z simultaneously, one at $-\Omega$ due to the co-rotation effect and, according to the problem, an additional one at Ω. These two exactly opposite rotations completely cancel each other out, thus resulting in m being, relative to Z, stationary and fixed at its initial position all the time.

Hence, the trajectories of Z and m on RF are simply circles of the same radius R (the distance between Z and O), while the center of the latter is displaced from that of the former by the same amount as that of m displaced from Z, as shown in figure 4.

To generalise, when an object is at the position (e, f) and starts to rotate at $t = 0$ on the ground at the same rotating rate as that of RF, Ω, about the center $Z(a, b)$, then the transformed motion on RF will be a circular motion of radius $R = \sqrt{a^2 + b^2}$, center $(e - a, f - b)$, and rotating rate $-\Omega$.

Next, we examine how Newton's second law is satisfied. Figure 5 shows the forces on m at the chosen point P on the transformed motion, corresponding to the ground motion shown in figure 2.

F_{real} is the real net force. It is the centripetal force in the ground motion, so its value is $m\Omega^2 d$. Note that at P, Z appears directly above m (see figure 4), so F_{real} directs upwards.

F_f is the centrifugal force. Its value is $m\Omega^2 q$, where q is the distance of m from O′, the center of rotation of RF. Its direction is outward, away from O′.

F_C is the Coriolis force. Its value is $2m\Omega^2 R$, where R is the distance of m from its center of rotation $(0, -d)$. Here, F_C points radially inward along R.

With the aid of the two angles defined in figure 5, the sums of the radial (along R) and tangential components of the three forces are $F_C - F_f\cos\alpha - F_{\text{real}}\cos\theta = 2m\Omega^2 R - m\Omega^2(q\cos\alpha + d\cos\theta) = m\Omega^2 R$ and $F_f\sin\alpha - F_{\text{real}}\sin\theta = m\Omega^2(q\sin\alpha - d\sin\theta) = 0$, respectively. Once again, Newton's second law is 'rescued'.

Readers can check the net force at places other than P on the circle or when m has a different initial position relative to Z.

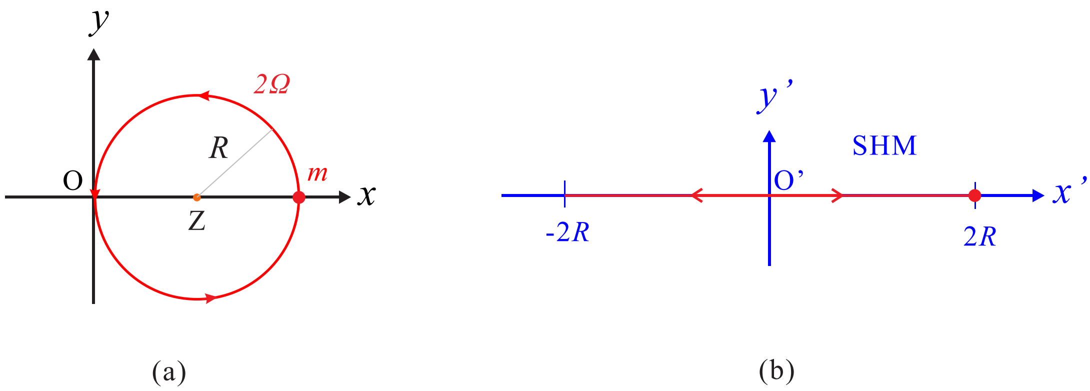

Figure 6. (a) A ground motion is shown when m starts to rotate anticlockwise at twice the rotating frequency of RF's. As shown, the circle touches the origin at its farthest left end, and m is situated at its farthest right end at $t = 0$. (b) The resulting RF motion is a simple harmonic motion.

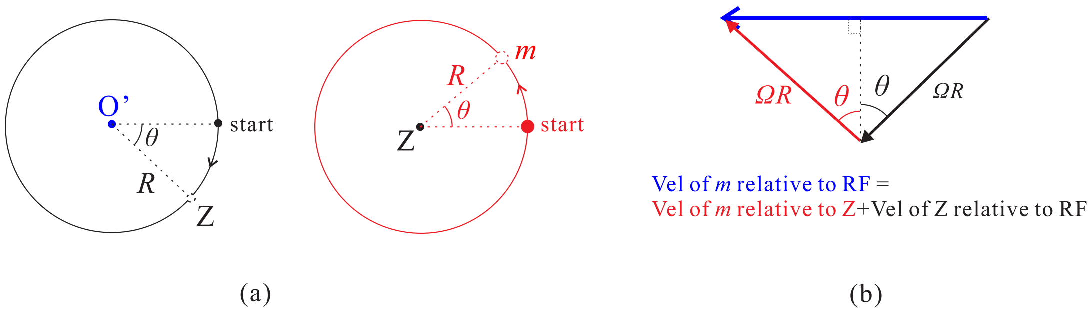

Figure 7. (a) As seen from RF, Z rotates backwards. So, relative to RF, Z rotates at $-\Omega$. Relative to Z, m rotates at Ω (2Ω according to the problem, but one Ω is deducted because of the co-rotation effect). (b) The magnitudes of the black and red vectors are both ΩR. The resultant vector (the velocity of m on RF) remains horizontal as the angle θ changes.

7. Case 4

The particle m in figure 6(a) moves around the circle shown anticlockwise at *twice* the rate Ω about Z. The circle touches the origin O at its farthest left end. An RF observer will see m executing a simple harmonic motion (SHM) centred at O′ on the x'-axis of amplitude $2R$ and angular frequency Ω.

As seen from RF, Z rotates backwards. So, relative to RF, Z moves on a circle of radius $O'Z = R$ at the rate $-\Omega$. At the same time, relative to Z, m carries on moving on its locus, which is a circle of radius R, at the rate Ω (because of the co-rotation effect, one Ω is deducted). These two circular motions are of the same radius R and, because of the initial position of m (see figure 6(a)), both start at their farthest right ends with an opposite sense of rotation of the same magnitude Ω. When m starts its motion, the angles turned in these two circular motions are always equal and labeled as θ in figure 7. The vector addition of the instantaneous velocities of these two circular motions, giving the motion of m relative to RF, is always horizontal (figure 7(b)). As we know, the axis projection of a circular motion is an SHM. Hence, relative to RF, m exhibits an SHM on the x'-axis.

Translate the two circular motions in figure 7(a) to overlap themselves with Z coinciding with each other; the instantaneous position of m on the x'-axis is then located, as shown in figure 8 when m is moving towards O′. In figure 8, the distance between m and O′ is x', and the velocity of m is $v' = 2\Omega R \sin\theta$ (see figure 7(b)). Hence, the three forces are direct, as shown in the figure. The magnitudes of F_{real} (the real net

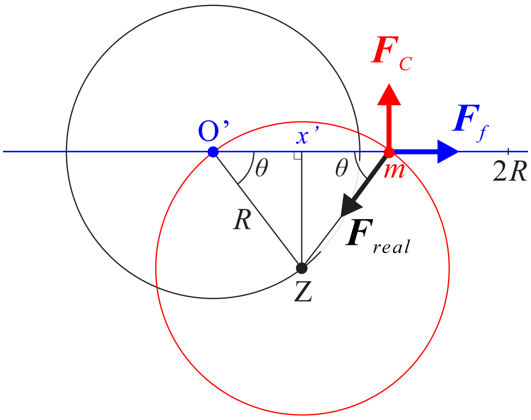

Figure 8. When m leaves the right extremity and is going to O$'$, the three forces with their directions are shown: F_{real} (the real force), F_f (the centrifugal force), and F_C (the Coriolis force). The two circles and angle θ are the same as those defined in figure 7.

force), F_f (the centrifugal force), and F_C (the Coriolis force) are $m(2\Omega)^2R = 4m\Omega^2R$, $m\Omega^2x'$, and $2m\Omega v' = 4m\Omega^2R\sin\theta$, respectively. The vertical component of F_{real}, i.e. $F_{\mathrm{real}}\sin\theta$, cancels F_C, while the horizontal component of F_{real} and F_f are added: $4m\Omega^2R\cos\theta - m\Omega^2x' = 4m\Omega^2(x'/2) - m\Omega^2x'$, giving the net force $m\Omega^2x'$ (towards O$'$). Hence, the acceleration of m is $a = -\Omega^2x'$, where x' is now interpreted as the displacement of m from O$'$. The SHM equation is thus derived.

In addition, one can, based on this case, easily prove the general one: if the ground circular motion passes through the origin and is at the rate 2Ω, irrespective of what the initial position of the particle is, the transformed motion on RF is an SHM oscillating on a tilted axis.

8. Discussions

Sometimes, teachers in college first-year or even pre-college classes will touch on the Coriolis force by introducing some relevant phenomena [3–5, 9, 12, 13], or doing some interesting experiments [13–18], allowing students themselves to 'see' or feel the force. But, to our knowledge, there are very few teaching resources at that level discussing the rotating frame transformation itself because this used to be considered very mathematical, less basic physics-related, and hence inappropriate for beginners. We hope we can make a slight change in this regard. Our examples, especially the last two, seem not so straightforward indeed, but the transformations employed rely only on some comparatively simple and understandable arguments, evading the equations at all. Besides, one advantage of our approach is that, in a single problem, the motion on the ground, the transformation, the transformed motion, and the testing of Newton's second law are all dealt with at once in a logical order, giving students a whole picture of what a rotating frame is and why fictitious forces are necessary for the transformed motion. Of course, the major limitation of this argument-based transformation is that it becomes exceedingly difficult or futile when the ground motion is a little bit more complicated, say, a circular motion with a rate of 2.5Ω, and then the corresponding transformed motion is a pattern that may be quite impossible to analyse without mathematics.

Nevertheless, our examples could serve as a preparatory exercise for those ready for explorations of practical applications of the Coriolis force, such as the more advanced problems in geophysics [20].

The examples put forward here can be used in reverse. They can be used to prove the necessity of the Coriolis force and derive its value if we first assume that Newton's second law remains valid in the presence of it and the centrifugal force.

We would be glad if our examples could give students an impressive glimpse of the uses of the two fictitious forces, helping them to take a step forward towards answering the question, 'What are centrifugal and Coriolis forces?'.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

The formal mathematical derivations of some of the transformations that appeared in this paper are briefed. The coordinates (x, y) and (x', y') are related by

$$x' = x\cos(\Omega t) + y\sin(\Omega t) \text{ and} \tag{1}$$

$$y' = -x\sin(\Omega t) + y\cos(\Omega t). \tag{2}$$

In figure 2, the ground rotation is $x = a + d\sin(\Omega t)$ and $y = b - d\cos(\Omega t)$, where a and b are the abscissa and ordinate of the center Z, respectively. By using equations (1) and (2), one gets $x' = R\cos(-\Omega t + \varphi)$ and $y' = -d + R\sin(-\Omega t + \varphi)$, where $R = \sqrt{a^2 + b^2}$ and $\varphi = \tan^{-1}(b/a)$, being consistent with the red circle shown in figure 4. When $x = R[1 + \cos(2\Omega t)]$ and $y = R\sin(2\Omega t)$, corresponding to the ground rotation shown in figure 6(a), are put into equations (1) and (2), one will get $x' = 2R\cos(\Omega t)$ and $y' = 0$, an SHM on the x'-axis.

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