PAPER • OPEN ACCESS

Phase transitions in the fractional three-dimensional Navier–Stokes equations

To cite this article: Daniel W Boutros and John D Gibbon 2024 *Nonlinearity* **37** 045010

View the article online for updates and enhancements.

Phase transitions in the fractional three-dimensional Navier–Stokes equations

Daniel W Boutros[1],* and **John D Gibbon[2]**

[1] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, United Kingdom
[2] Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom

E-mail: dwb42@cam.ac.uk

Abstract

The fractional Navier–Stokes equations on a periodic domain $[0, L]^3$ differ from their conventional counterpart by the replacement of the $-\nu\Delta\boldsymbol{u}$ Laplacian term by $\nu_s A^s\boldsymbol{u}$, where $A = -\Delta$ is the Stokes operator and $\nu_s = \nu L^{2(s-1)}$ is the viscosity parameter. Four critical values of the exponent $s \geqslant 0$ have been identified where functional properties of solutions of the fractional Navier–Stokes equations change. These values are: $s = \frac{1}{3}$; $s = \frac{3}{4}$; $s = \frac{5}{6}$ and $s = \frac{5}{4}$. In particular: (i) for $s > \frac{1}{3}$ we prove an analogue of one of the Prodi–Serrin regularity criteria; (ii) for $s \geqslant \frac{3}{4}$ we find an equation of local energy balance and; (iii) for $s > \frac{5}{6}$ we find an infinite hierarchy of weak solution time averages. The existence of our analogue of the Prodi–Serrin criterion for $s > \frac{1}{3}$ suggests the sharpness of the construction using convex integration of Hölder continuous solutions with epochs of regularity in the range $0 < s < \frac{1}{3}$.

Keywords: fractional Navier–Stokes equations, regularity criteria, energy balance, phase transition, hypodissipation

Mathematics Subject Classification numbers: 35B33, 35B65, 35B60, 35B45, 35Q35, 35R11, 76D99

* Author to whom any correspondence should be addressed.

1. The fractional Navier–Stokes equations

We consider the incompressible fractional Navier–Stokes equations in a form based on the Stokes operator $A = -\Delta$

$$(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu_s A^s \boldsymbol{u} = -\nabla P, \qquad s \geqslant 0, \qquad (1.1)$$

together with $\operatorname{div}\boldsymbol{u} = 0$ and $\nu_s = \nu L^{2(s-1)}$, on a three-dimensional periodic domain $[0, L]^3$. The fractional Laplacian A^s has the spectral representation

$$A^s \boldsymbol{u}(\boldsymbol{x}, t) := \sum_{k \in \mathbb{Z}^3} |\boldsymbol{k}|^{2s} \widehat{\boldsymbol{u}}_k(t) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}), \qquad (1.2)$$

where $\widehat{\boldsymbol{u}}_k$ are the Fourier coefficients of $\boldsymbol{u}$. Instead of keeping s fixed at $s = 1$ and then studying the inviscid $\nu \to 0$ limit in the conventional way, we keep ν fixed and study properties of solutions of (1.1) in the limit $s \to 0$. Inspired by the Lions result [1, chapter 1, remark 6.11] (see also [2, section 8]), which shows that solutions of (1.1) are regular when $s \geqslant \frac{5}{4}$ (see also Tao [3] and Luo and Titi [4]), much work has concentrated on the hyper-viscous ($s > 1$) case [5–11]. However, it is our view that the hypo-viscous regime ($0 < s < 1$) is of equal if not greater interest: see [12] for work on the fractional Burgers equation. In the limit $s \to 0$ the question arises whether there are significant changes to the properties of solutions of (1.1) before reaching the limit of the damped Euler equations at $s = 0$

$$(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu_0 \boldsymbol{u} = -\nabla P, \qquad \nu_0 = \nu L^{-2}. \qquad (1.3)$$

Before summarising and discussing our main results, it is worth remarking on the fact that the fractional Navier–Stokes equations bear a close relation to the fractional diffusion equation

$$\partial_t u + \nu_s A^s u = 0, \qquad (1.4)$$

whose solutions are related to the theory of random walks. The language of Brownian motion, with its associated literature [13–17], has determined the nomenclature of the latter. For $s = 1$ the mean square displacement of a particle is linear with time: $\langle X^2 \rangle \sim t$. However, for the fractional diffusion equation[3] the relation $\langle X^2 \rangle \sim t^{1/s}$ indicates anomalous diffusion when $s \neq 1$. The case $s > 1$ commonly occurs in biological, fractal and porous media [18–24], whereas the $s < 1$ case occurs in turbulent plasmas and polymer transport [25, 26]. It is in this latter range where fat-tailed spectra and Lévy flights are observed in data.

A system is commonly considered to go through a phase transition when its properties undergo qualitative changes as a parameter passes through a critical value. The parameter in question is the exponent s of the fractional Laplacian. The fractional Navier–Stokes equations have many different kinds of solution whose properties may vary depending upon their regularity, their (non-)uniqueness, or the size of their singular set. We list some of them below:

1. Wild solutions originally associated with the 3D Euler equations and Onsager's conjecture [27–29].
2. Distributional solutions [4,30].
3. Suitable weak solutions which have partial regularity (Caffarelli *et al* [31]).
4. Weak solutions of Leray–Hopf type.
5. Strong solutions which possess both (local) existence and uniqueness.

---

[3] Somewhat confusingly, because of the $1/s$ exponent on t, the hyper-viscous case $s > 1$ corresponds to *sub*-diffusion in the theory of random walks while the hypo-viscous case $s < 1$ corresponds to *super*-diffusion.

Dependent on the setting, there may be some overlap among those listed above. Four critical values of s have been identified: $s = \frac{1}{3}$; $s = \frac{3}{4}$; $s = \frac{5}{6}$ and $s = \frac{5}{4}$. The changes to the qualitative properties of solutions at these points are summarised in section 1.3, together with references in the literature. These results lay the groundwork for future numerical simulations.

1.1. Notation and invariance properties

Throughout the paper the domain is taken to be the three-dimensional unit torus $\mathbb{T}^3$. For Sobolev norms of the solution we will use the following notation

$$H_{n,m} = \int_{\mathbb{T}^3} |\nabla^n \boldsymbol{u}|^{2m} dx \equiv \|\nabla^n \boldsymbol{u}\|_{2m}^{2m}. \tag{1.5}$$

For example, the square of the standard $\dot{H}^1$-norm is expressed as $H_{1,1}$ and n-derivatives in L^2 are expressed as $H_{n,1}$. To avoid confusion we remark that the superscript H^n refers to the Sobolev space whereas the subscripts $H_{n,m}$ refer to the norms defined in (1.5). Moreover fractional Sobolev norms for $m = 1$ are defined as follows

$$\int_{\mathbb{T}^3} |(-\Delta)^{s/2} \boldsymbol{u}|^2 dx \equiv \int_{\mathbb{T}^3} |A^{s/2} \boldsymbol{u}|^2 dx = H_{s,1}. \tag{1.6}$$

Further properties of the fractional Laplacian can be found in appendix A.

We remark at this point that the $3D$ fractional Navier–Stokes equations are invariant under the scaling transformation

$$\boldsymbol{x}' = \lambda^{-1} \boldsymbol{x}; \quad t' = \lambda^{-2s} t; \quad \boldsymbol{u} = \lambda^{1-2s} \boldsymbol{u}', \tag{1.7}$$

which reduces to the standard Navier–Stokes scaling when $s = 1$. It is also of interest to see how the properties of solutions across the hypo/hyper-viscous regimes are tied together through invariance properties, as in the standard Navier–Stokes equations [32–41]—see section 6. The technical material in references [42–47] has been used throughout the paper.

1.2. Leray–Hopf solutions of the fractional Navier–Stokes equations

We begin by introducing the weak formulation of the hypo-dissipative Navier–Stokes equations.

Definition. Let $\boldsymbol{u} \in L^\infty \left[(0,T); L^2(\mathbb{T}^3) \right] \cap L^2 \left[(0,T); H^s(\mathbb{T}^3) \right]$ and let $\boldsymbol{u}_0 \in L^2(\mathbb{T}^3)$ be the initial data. We say that $\boldsymbol{u}$ is a Leray–Hopf weak solution if it satisfies the following weak formulation

$$\int_0^T \int_{\mathbb{T}^3} \left[\boldsymbol{u} \partial_t \psi - \nu \left(A^{s/2} \boldsymbol{u} \right) \left(A^{s/2} \psi \right) + \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \psi + P \nabla \cdot \psi \right] dx dt = - \int_{\mathbb{T}^3} \boldsymbol{u}_0 \psi \left(\boldsymbol{x}, 0 \right) dx, \tag{1.8}$$

for all $\psi \in \mathcal{D} \left[\mathbb{T}^3 \times [0,T] \right]$. Moreover, for all $T \geqslant 0$ the solution satisfies the following energy inequality

$$\frac{1}{2} H_{0,1}(T) + \nu \int_0^T H_{s,1} dt \leqslant \frac{1}{2} H_{0,1}(0). \tag{1.9}$$

At this point we recall the standard existence result for Leray–Hopf solutions:

Theorem 1. *For all $s > 0$ and all initial data in $L^2(\mathbb{T}^3)$, there exists a global Leray–Hopf solution satisfying the weak formulation of the fractional Navier–Stokes equations.*

For a proof see appendix A in [48].

1.3. Summary of results

The task of this subsection is to summarise the various functional properties possessed by solutions of the fractional Navier–Stokes equations in different ranges of $s > 0$. These are laid out in the table below. Three of these results are new: namely an analogue of a result[4] of Prodi [49] and Serrin [50] for $s > \frac{1}{3}$; an equation of local energy balance for $s \geqslant \frac{3}{4}$; and an infinite hierarchy of time averages for $s > \frac{5}{6}$. Various theorems valid in different ranges of s are expressed in the rest of the subsection. Their proofs can be found in the following sections of the paper.

s	Functional properties
$0 < s < \frac{1}{3}$	Non-uniqueness of Leray–Hopf solutions [48, 52].
$\frac{1}{3} < s < 1$	An analogue of a Prodi–Serrin criterion [49, 50] involving $\int_0^{T^*} \lVert A^{s/2}\boldsymbol{u} \rVert_\infty^{\frac{2s}{3s-1}} dt$.
$s \geqslant \frac{3}{4}$	An equation of local energy balance for Leray–Hopf solutions.
$s > \frac{3}{4}$	A generalised Caffarelli–Kohn–Nirenberg result [11, 31, 53, 54].
$s > \frac{5}{6}$	An infinite hierarchy of Leray–Hopf weak solution time averages.
$0 < s < \frac{5}{4}$	Non-uniqueness of distributional solutions [4, 30].
$s \geqslant \frac{5}{4}$	Existence and uniqueness of solutions [1–3].

(1) The case $0 < s < \frac{1}{3}$: It has previously been noted in section 1.2 that for any $s > 0$, there exists a global Leray–Hopf weak solution. It has been shown by Colombo, De Lellis and De Rosa in [48] that these solutions are non-unique for $s < \frac{1}{5}$. This result was later improved in [52] to show the non-uniqueness if $s < \frac{1}{3}$. In the range $\frac{1}{3} \leqslant s < \frac{1}{2}$ non-uniqueness of weak solutions with Leray–Hopf regularity has been proved in [48], but the constructed solutions do not satisfy the energy inequality. Buckmaster and Vicol [30] have proved the non-uniqueness of distributional solutions of the Navier–Stokes equations (i.e. with $s = 1$) while the work of Luo and Titi [4] has extended this result to prove non-uniqueness of distributional solutions for any $s < \frac{5}{4}$. These results have all been proved using the method of convex integration.

(2) The case $s > \frac{1}{3}$: The following theorem expresses a result which is similar in spirit to one of the Prodi–Serrin regularity criteria for the 3D Navier–Stokes equations [49, 50] (see section 2 for the proof);

Theorem 2. *When $\frac{1}{3} < s < 1$ and for initial data $\boldsymbol{u}_0 \in H^2(\mathbb{T}^3)$, suppose there exists a strong solution of the fractional Navier–Stokes equations which loses regularity at the earliest time T^*, then*

$$\int_0^{T^*} \lVert A^{s/2}\boldsymbol{u} \rVert_\infty^{\frac{2s}{3s-1}} dt = \infty. \tag{1.10}$$

Conversely, for every $T > 0$, if $\int_0^T \lVert A^{s/2}\boldsymbol{u} \rVert_\infty^{\frac{2s}{3s-1}} dt < \infty$, then strong solutions of the fractional Navier–Stokes equations remain regular.

---

[4] In addition to the general regularity criteria on the velocity field for the three dimensional Navier–Stokes equations [49, 50], $\int_0^t \lVert \nabla \boldsymbol{u} \rVert_\infty d\tau$ is another sufficient regularity condition which is applicable in both two and three dimensions. This time integral also applies to the Euler equations. Beale *et al* [51] then showed how this result for the three dimensional Euler equations could be converted to control over $\int_0^t \lVert \boldsymbol{\omega} \rVert_\infty d\tau$ at the price of making the upper bound super-exponential in time. In this paper we consider our result in theorem 2 to be an analogue of that of Prodi and Serrin.

There are four things on which to remark. Firstly, the proof displayed in section 2 works only in the range $\frac{1}{3} < s < 1$. Secondly, when $s = 1$ we recover the standard result, namely $\int_0^T \|\nabla \boldsymbol{u}\|_\infty dt$. Thirdly, close to $s = \frac{1}{3}$, the fractional velocity gradient $A^{s/2}\boldsymbol{u}$ needs to be not only L^∞ in space but also nearly L^∞ in time. Fourthly, we remark that this is truly a (fractional) Navier–Stokes and not an Euler result, as the proof will show. In passing we remark that the integral in (1.10) is the only object that need be monitored for regularity purposes in a numerical simulation.

(3) **The case $s \geqslant \frac{3}{4}$:** Next we turn to the equation of local energy balance. It has been proved by Duchon and Robert [55] that Leray–Hopf solutions of the (standard) Navier–Stokes equations satisfy an equation of local energy balance with a 'defect term' (which physically describes the energy flux and arises due to a potential lack of smoothness of the solution). Under an additional regularity assumption, this result is also true for the Euler equations. Here, we extend Duchon and Robert's approach [55] to the fractional Navier–Stokes equations.

First we introduce some notation. Let $\varphi \in C_c^\infty \left[\mathbb{R}^3; \mathbb{R}\right]$ be a standard radial mollifier with the property that $\int_{\mathbb{R}^3} \varphi(\boldsymbol{x}) dx = 1$. We also introduce the notation

$$\varphi^\epsilon(\boldsymbol{x}) := \frac{1}{\epsilon^3} \varphi\left(\frac{\boldsymbol{x}}{\epsilon}\right).$$

In the case $s \geqslant \frac{3}{4}$, it is possible to establish an equation of local energy balance for Leray–Hopf solutions. This can be demonstrated in a Corollary to:

Theorem 3. *Let $\boldsymbol{u} \in L^3 \left[(0,T); L^3(\mathbb{T}^3)\right]$ be a Leray–Hopf weak solution of the fractional Navier–Stokes equations. Then the following equation of local energy balance holds for all $\psi \in \mathcal{D}\left[\mathbb{T}^3 \times (0,T)\right]$*

$$\int_0^T \int_{\mathbb{T}^3} \left[|\boldsymbol{u}|^2 \partial_t \psi - 2\nu \left(A^{s/2}\boldsymbol{u}\right) \cdot A^{s/2}(\boldsymbol{u}\psi) + 2P\boldsymbol{u} \cdot \nabla\psi - \frac{1}{2}D(\boldsymbol{u})\psi + |\boldsymbol{u}|^2 (\boldsymbol{u} \cdot \nabla\psi)\right] dxdt = 0, \quad (1.11)$$

where the defect term is given by

$$D(\boldsymbol{u})(\boldsymbol{x},t) := \frac{1}{2} \lim_{\epsilon \to 0} \int_{\mathbb{T}^3} \nabla\varphi_\epsilon(\boldsymbol{\xi}) \cdot \delta\boldsymbol{u}(\boldsymbol{\xi};\boldsymbol{x},t) |\delta\boldsymbol{u}(\boldsymbol{\xi};\boldsymbol{x},t)|^2 d\xi, \quad (1.12)$$

$$\delta\boldsymbol{u}(\boldsymbol{\xi};\boldsymbol{x},t) := \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t) - \boldsymbol{u}(\boldsymbol{x},t). \quad (1.13)$$

Moreover, the defect term is independent of the choice of mollifier.

Corollary 1. *The equation of local energy balance (1.11) holds automatically for Leray–Hopf solutions of the hypo-dissipative Navier–Stokes equations if $s \geqslant \frac{3}{4}$.*

The proof can be found in section 3.

(4) **The case $s > \frac{5}{6}$:** before stating the results for the regularity of Leray–Hopf solutions[5], let us begin with the definition

$$\delta_{n,s} := \frac{6s - 5}{2n + 4s - 5}. \quad (1.14)$$

Theorem 4. *Let $\frac{5}{6} < s < \frac{5}{4}$ and $1 \leqslant n < \infty$, and let $\boldsymbol{u}$ be a Leray–Hopf solution. Then $\boldsymbol{u}$ belongs to the following spaces*

$$\boldsymbol{u} \in L^{2\delta_{n,s}} \left[(0,T); H^n(\mathbb{T}^3)\right]. \quad (1.15)$$

---

[5] The origin of the exponent $s = \frac{5}{6}$ is as follows: it is elementary to show that the critical space for the fractional Navier–Stokes equations is $H^{5/2-2s}(\mathbb{T}^3)$. This coincides with $H^s(\mathbb{T}^3)$ (which is part of the Leray–Hopf regularity) when $s = \frac{5}{6}$.

Remark 5. We note that it is possible with a slightly different method to establish a hierarchy of weak solution time averages if $s \geqslant \frac{5}{4}$. However, as the fractional Navier–Stokes equations are known to be globally well-posed for this range of s such a result does not yield anything new and hence we do not present it here.

The proof can be found in section 4 and is based on the seminal but relatively unknown paper of Foias *et al* [36] in which theorem 4 was proved in the case $s = 1$. As part of the proof, we need a local well-posedness result as well as a weak-strong uniqueness result for the fractional Navier–Stokes equations. As these results in the required form seem to be absent from the literature, we establish them in section 5. Theorem 4 shows that there is an infinite hierarchy of finite time integrals (or averages), as advertised in the 5th line of the table in section 1.3. How this result ties in with the invariance properties given in (1.7) is left to section 6.

2. Proof of theorem 2

The statement of theorem 2 is based on the assumption that we start with a regular solution in $[0, T^*)$. Thus we are able to differentiate the (spatial) $H_{n,1}$-norms with respect to time. We begin with the standard ladder of Sobolev norms which can be obtained using standard energy estimates in an adaption of the proof of theorem 6.1 in [32]:

$$\frac{1}{2}\frac{d}{dt}H_{n,1} \leqslant -\nu_s H_{n+s,1} + c_{n,s}\|\nabla \boldsymbol{u}\|_\infty H_{n,1}. \tag{2.1}$$

Now we would like to adapt this estimate. $\|\nabla \boldsymbol{u}\|_\infty H_{n,1}$ and $\|\nabla^s \boldsymbol{u}\|_\infty H_{n+p,1}$ (where $p = \frac{1}{2}(1-s) \geqslant 0$) have the same dimensions; i.e. under the transformation (1.7) they satisfy the same scaling relation. Thus, we seek an inequality relation between them, which we prove in the next lemma.

Lemma 1. *Provided $0 < s < 1$ and $n > 2 + \frac{1}{2}s$, with $p = \frac{1}{2}(1-s)$, then the following inequality holds*

$$\|\nabla \boldsymbol{u}\|_\infty H_{n,1} \leqslant c_{n,s}\|A^{s/2}\boldsymbol{u}\|_\infty H_{n+p,1}. \tag{2.2}$$

Proof. We define $U := A^{s/2}\boldsymbol{u}$. We also fix r such that $s + \frac{1}{2} < r < \frac{3}{2}$, and by using Agmon's inequality we find

$$\|\nabla \boldsymbol{u}\|_\infty \leqslant \|\nabla \boldsymbol{u}\|_{\dot{H}^r}^a \|\nabla \boldsymbol{u}\|_{\dot{H}^{n+p-1}}^{1-a} = \|U\|_{\dot{H}^{r+1-s}}^a \|U\|_{\dot{H}^{n+p-s}}^{1-a}, \tag{2.3}$$

where

$$\frac{3}{2} = ar + (1-a)(n+p-1) \qquad \Longrightarrow \qquad a = \frac{n+p-\frac{5}{2}}{n+p-r-1}. \tag{2.4}$$

Then, by using the Gagliardo–Nirenberg–Sobolev interpolation inequality (see [43]) we find

$$\|U\|_{\dot{H}^{r+1-s}} \leqslant C\|U\|_\infty^b \|U\|_{\dot{H}^{n+p-s}}^{1-b}, \tag{2.5}$$

with the following relation between the exponents

$$\frac{1}{2} = \frac{1-b}{2} - \frac{(1-b)(n+p-s)-(r+1-s)}{3}. \tag{2.6}$$

This implies that

$$b\left(\frac{n+p-s}{3} - \frac{1}{2}\right) = \frac{n+p-r-1}{3}. \tag{2.7}$$

Again, by applying a Gagliardo–Nirenberg–Sobolev inequality we obtain

$$\|\nabla^n \boldsymbol{u}\|_2 = \|A^{(n-s)/2} U\|_2 = \|U\|_{\dot{H}^{n-s}} \leqslant C \|A^{(n+p-s)/2} U\|_2^{1-c} \|U\|_\infty^c \tag{2.8}$$

with the following relation between the exponents

$$\frac{1}{2} = \frac{1-c}{2} - \frac{(1-c)(n+p-s)-(n-s)}{3}. \tag{2.9}$$

This implies that

$$c\left(\frac{n+p-s}{3} - \frac{1}{2}\right) = \frac{p}{3}. \tag{2.10}$$

Combining these inequalities gives us

$$\|\nabla \boldsymbol{u}\|_\infty H_{n,1} \leqslant C \|U\|_\infty^{ab+2c} \|\boldsymbol{u}\|_{\dot{H}^{n+p}}^{3-ab-2c}. \tag{2.11}$$

The proof is completed if we can show that $ab + 2c = 1$, which is confirmed by

$$ab + 2c = \frac{n+p-\frac{5}{2}}{n+p-r-1} \cdot \frac{n+p-r-1}{n+p-s-\frac{3}{2}} + \frac{2p}{n+p-s-\frac{3}{2}}$$

$$= \frac{n+p-\frac{5}{2}+2p}{n+p-s-\frac{3}{2}} = 1. \tag{2.12}$$

$\square$

We are now ready to proceed with the proof of theorem 2:

Proof of theorem 2. By a standard interpolation inequality for homogeneous Sobolev spaces we have (for $\frac{1}{3} < s < 1$)

$$H_{n+p,1}^s \leqslant H_{n+s,1}^{(1-s)/2} H_{n,1}^{(3s-1)/2}. \tag{2.13}$$

Recalling that $p = \frac{1}{2}(1-s)$, one can check that

$$\frac{1}{2}(1-s)(n+s) + \frac{1}{2}n(3s-1) = (n+p)s.$$

Thus, for $\frac{1}{3} < s < 1$ and $2 + \frac{1}{2}s < n$, by using the ladder of Sobolev norms, as well as inequalities (2.2) and (2.13), we find

$$\frac{1}{2}\frac{d}{dt} H_{n,1} \leqslant -\nu_s H_{n+s,1} + c_{n,s} \|\nabla \boldsymbol{u}\|_\infty H_{n,1}$$

$$\leqslant -\nu_s H_{n+s,1} + c_{n,s} \|A^{s/2} \boldsymbol{u}\|_\infty H_{n+p,1}$$

$$\leqslant -\nu_s H_{n+s,1} + c_{n,s} \|A^{s/2} \boldsymbol{u}\|_\infty H_{n+s,1}^{(1-s)/2s} H_{n,1}^{(3s-1)/2s}$$

$$= -\nu_s H_{n+s,1} + \{\nu_s H_{n+s,1}\}^{(1-s)/2s} \left\{ c_{n,s} \nu_s^{-\frac{1-s}{3s-1}} \|A^{s/2} \boldsymbol{u}\|_\infty^{2s/(3s-1)} H_{n,1} \right\}^{(3s-1)/2s}$$

$$\leqslant -\nu_s H_{n+s,1} + \frac{(1-s)\nu_s}{2s} H_{n+s,1} + \left(\frac{3s-1}{2s}\right) c_{n,s} \nu_s^{-\frac{1-s}{3s-1}} \|A^{s/2} \boldsymbol{u}\|_\infty^{2s/(3s-1)} H_{n,1}$$

$$\leqslant -\nu_s \left(\frac{3s-1}{2s}\right) H_{n+s,1} + \left(\frac{3s-1}{2s}\right) c_{n,s} \nu_s^{-\frac{1-s}{3s-1}} \|A^{s/2} \boldsymbol{u}\|_\infty^{2s/(3s-1)} H_{n,1}. \tag{2.14}$$

In the penultimate line we have used Young's inequality. Note that the constant $c_{n,s}$ may change from line to line. The last line shows why this is a Navier–Stokes and not an Euler result, because of the necessary use of the dissipation term at the last step. Then, by removing the negative $H_{n+s,1}$-term and applying Gronwall's inequality we can write

$$H_{n,1}(T) \leqslant c_{n,s} H_{n,1}(0) \exp\left\{ \nu_s^{-\frac{1-s}{3s-1}} \int_0^T \|A^{s/2}\boldsymbol{u}\|_\infty^{\frac{2s}{3s-1}} dt \right\} \qquad \text{for} \qquad \frac{1}{3} < s < 1. \qquad (2.15)$$

The proof is now finished by contradiction. Let us assume that $\int_0^{T^*} \|A^{s/2}\boldsymbol{u}\|_\infty^{\frac{2s}{3s-1}} dt$ is finite. Then $H_{n,1}(T^*)$ is finite, which contradicts the supposition that regularity is lost at T^*. Thus the opposite must be true, i.e. the integral must be infinite if regularity is lost at T^*. $\qquad\qquad\square$

3. Proof of theorem 3

Now we will show that for $s \geqslant \frac{3}{4}$ the Leray–Hopf solutions satisfy an equation of local energy balance. In order to prove theorem 3 the following identity is necessary

$$\int_{\mathbb{T}^3} (A^s f)\, g\, dx = \int_{\mathbb{T}^3} \left(A^{s/2} f\right)\left(A^{s/2} g\right) dx. \qquad (3.1)$$

The proof is similar to that for (A.1) by using the spectral characterisation of the fractional Laplacian as well as the Plancherel identity. First, however, we prove the following Lemma:

Lemma 2. *Let $\boldsymbol{u}$ be a Leray–Hopf weak solution of the fractional Navier–Stokes equations. The weak formulation (1.8) still holds for $\psi \in W_0^{1,1}\left[(0,T);L^2(\mathbb{T}^3)\right] \cap L^1\left[(0,T);H^3(\mathbb{T}^3)\right]$.*

Proof. Let us take an arbitrary $\psi \in W_0^{1,1}\left[(0,T);L^2(\mathbb{T}^3)\right] \cap L^1\left[(0,T);H^3(\mathbb{T}^3)\right]$, then there exists a sequence $\{\psi_n\} \subset \mathcal{D}\left[\mathbb{T}^3 \times (0,T)\right]$ such that $\psi_n \to \psi$ in $W_0^{1,1}\left[(0,T);L^2(\mathbb{T}^3) \cap L^1\left[0,T);H^3(\mathbb{T}^3)\right]\right.$. First we observe that for any ψ_n equation (1.8) holds because $\psi_n \in \mathcal{D}\left[\mathbb{T}^3 \times (0,T)\right]$.

We know that $\boldsymbol{u}\partial_t \psi_n \to \boldsymbol{u}\partial_t \psi$ in $L^1\left[(0,T);L^1(\mathbb{T}^3)\right]$ and therefore

$$\int_0^T \int_{\mathbb{T}^3} \boldsymbol{u}\partial_t \psi_n\, dxdt \xrightarrow{n\to\infty} \int_0^T \int_{\mathbb{T}^3} \boldsymbol{u}\partial_t \psi\, dxdt. \qquad (3.2)$$

Similarly, we know that $(A^{s/2}\boldsymbol{u})(A^{s/2}\psi_n) \to (A^{s/2}\boldsymbol{u})(A^{s/2}\psi)$, $\boldsymbol{u} \otimes \boldsymbol{u} : \nabla \psi_n \to \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \psi$ and $P\nabla \cdot \psi_n \to P\nabla \cdot \psi$, where all the limits converge in $L^1\left[\mathbb{T}^3 \times (0,T)\right]$. Therefore we have

$$\int_0^T \int_{\mathbb{T}^3} \left[-\nu\left(A^{s/2}\boldsymbol{u}\right)\left(A^{s/2}\psi_n\right) + \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \psi_n + P\nabla \cdot \psi_n\right] dxdt$$
$$\to \int_0^T \int_{\mathbb{T}^3} \left[-\nu\left(A^{s/2}\boldsymbol{u}\right)\left(A^{s/2}\psi\right) + \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \psi + P\nabla \cdot \psi\right] dxdt. \qquad (3.3)$$

We conclude that the weak formulation holds for all $\psi \in W_0^{1,1}\left[(0,T);L^2(\mathbb{T}^3)\right] \cap L^1\left[(0,T);H^3(\mathbb{T}^3)\right]$. $\qquad\qquad\square$

Now we prove the following lemma:

Lemma 3. *Let $s \geqslant \frac{3}{4}$ and let $\boldsymbol{u}$ be a Leray–Hopf weak solution of the fractional Navier–Stokes equations. Then $\boldsymbol{u} \in L^3\left[\mathbb{T}^3 \times (0,T)\right]$ and $P \in L^{3/2}\left[\mathbb{T}^3 \times (0,T)\right]$.*

Proof. The following $3D$ interpolation inequality is useful

$$\|f\|_{L^p} \leqslant C\|f\|_{L^q}^{\theta}\|f\|_{H^s}^{1-\theta}, \quad \frac{1}{p} = \frac{\theta}{q} + (1-\theta)\left(\frac{1}{2} - \frac{s}{3}\right). \tag{3.4}$$

From which we find

$$\|\boldsymbol{u}\|_{L^3} \leqslant CH_{0,1}^{(2s-1)/4s}H_{s,1}^{1/4s}. \tag{3.5}$$

We recall that $\boldsymbol{u} \in L^{\infty}\left[(0,T);L^2(\mathbb{T}^3)\right]$ and hence the time integral of any power of the L^2 norm is finite. However since $\boldsymbol{u} \in L^2\left[(0,T);H^s(\mathbb{T}^3)\right]$, in order for $\boldsymbol{u} \in L^3\left[\mathbb{T}^3 \times (0,T)\right]$ we require

$$\frac{3}{4s} \leqslant 1 \qquad \Longrightarrow \qquad s \geqslant \frac{3}{4}. \tag{3.6}$$

The pressure satisfies the following equation (in the sense of distributions)

$$-\Delta P = (\nabla \otimes \nabla) : (\boldsymbol{u} \otimes \boldsymbol{u}), \tag{3.7}$$

and since $\boldsymbol{u} \in L^3\left[\mathbb{T}^3 \times (0,T)\right]$, it follows by the boundedness of the Riesz transform (see appendix B in [34]) that $P \in L^{3/2}\left[\mathbb{T}^3 \times (0,T)\right]$, which is what we needed to show. $\qquad\square$

Proof of theorem 3: We mollify the hypo-dissipative Navier–Stokes equations, multiply by $\psi\boldsymbol{u}$ and integrate in time and space to obtain

$$\int_0^T \int_{\mathbb{T}^3} \psi\boldsymbol{u} \cdot \left[\partial_t\boldsymbol{u}^{\epsilon} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u})^{\epsilon} + \nu A^s\boldsymbol{u}^{\epsilon} + \nabla P^{\epsilon}\right]dxdt = 0. \tag{3.8}$$

We first observe that $\boldsymbol{u}^{\epsilon} \in L^{\infty}\left[(0,T);C^{\infty}(\mathbb{T}^3)\right]$. From mollifying the equation we find that

$$\partial_t\boldsymbol{u}^{\epsilon} \in L^2\left[(0,T);C^{\infty}(\mathbb{T}^3)\right] \tag{3.9}$$

as $\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u})^{\epsilon} + \nu A^s\boldsymbol{u}^{\epsilon} + \nabla P^{\epsilon}$ lies in this space. Hence $\boldsymbol{u}^{\epsilon} \in H^1\left[(0,T);C^{\infty}(\mathbb{T}^3)\right]$. Therefore, we can apply lemma 2 and take $\boldsymbol{u}^{\epsilon}\psi$ as a test function in the weak formulation (1.8). Subtracting equation (3.8) gives us

$$\int_0^T \int_{\mathbb{T}^3} \left[\boldsymbol{u} \cdot \partial_t\left(\boldsymbol{u}^{\epsilon}\psi\right) - \psi\boldsymbol{u} \cdot \partial_t\boldsymbol{u}^{\epsilon} - \nu\left(A^{s/2}\boldsymbol{u}\right) \cdot A^{s/2}\left(\boldsymbol{u}^{\epsilon}\psi\right) - \nu A^{s/2}\left(\boldsymbol{u}\psi\right)\left(A^{s/2}\boldsymbol{u}^{\epsilon}\right) + P\nabla \cdot \left(\boldsymbol{u}^{\epsilon}\psi\right)\right.$$
$$\left. - \psi\boldsymbol{u} \cdot \nabla P^{\epsilon} + \boldsymbol{u} \otimes \boldsymbol{u} : \nabla\left(\psi\boldsymbol{u}^{\epsilon}\right) - \boldsymbol{u}\psi \cdot \left(\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u})^{\epsilon}\right)\right]dxdt = 0. \tag{3.10}$$

Next we introduce a mollified defect term $D_{\epsilon}(\boldsymbol{u})$. Noting that φ^{ϵ} is a smooth mollifier, $D_{\epsilon}(\boldsymbol{u})$ becomes

$$D_{\epsilon}(\boldsymbol{u})(\boldsymbol{x},t) := \int_{\mathbb{R}^3} \nabla\varphi^{\epsilon} \cdot \delta\boldsymbol{u}(\boldsymbol{\xi};\boldsymbol{x},t)|\delta\boldsymbol{u}(\boldsymbol{\xi};\boldsymbol{x},t)|^2 d\xi$$
$$= -\nabla \cdot \left(|\boldsymbol{u}|^2\boldsymbol{u}\right)^{\epsilon} + \boldsymbol{u} \cdot \nabla\left(|\boldsymbol{u}|^2\right)^{\epsilon} + 2\boldsymbol{u} \otimes \nabla : (\boldsymbol{u} \otimes \boldsymbol{u})^{\epsilon} - 2\boldsymbol{u} \otimes \boldsymbol{u} : \nabla\boldsymbol{u}^{\epsilon}, \tag{3.11}$$

with $\delta\boldsymbol{u}$ defined as in (1.13). We observe that $D_{\epsilon}(\boldsymbol{u})$ is well-defined for any $\epsilon > 0$ because of the assumption that $\boldsymbol{u} \in L^3\left[\mathbb{T}^3 \times (0,T)\right]$. Equation (3.10) can also be rewritten as follows

$$\int_0^T \int_{\mathbb{T}^3} \left[\boldsymbol{u} \cdot \boldsymbol{u}^{\epsilon}\partial_t\psi - \nu\left(A^{s/2}\boldsymbol{u}\right) \cdot A^{s/2}\left(\boldsymbol{u}^{\epsilon}\psi\right) - \nu A^{s/2}\left(\boldsymbol{u}\psi\right)\left(A^{s/2}\boldsymbol{u}^{\epsilon}\right) + \left(\boldsymbol{u}^{\epsilon}P + \boldsymbol{u}P^{\epsilon}\right) \cdot \nabla\psi\right.$$
$$\left. - \frac{1}{2}D_{\epsilon}(\boldsymbol{u})(\boldsymbol{x},t)\psi - \frac{1}{2}\psi\nabla \cdot \left(|\boldsymbol{u}|^2\boldsymbol{u}\right)^{\epsilon} + \frac{1}{2}\psi\boldsymbol{u} \cdot \nabla\left(|\boldsymbol{u}|^2\right)^{\epsilon} + (\boldsymbol{u} \cdot \boldsymbol{u}^{\epsilon})\boldsymbol{u} \cdot \nabla\psi\right]dxdt = 0, \tag{3.12}$$

where we have used the incompressiblity when rewriting the pressure terms. As $\epsilon \to 0$, we observe that we have the following convergence in $L^1 \left[\mathbb{T}^3 \times (0,T) \right]$

$$\boldsymbol{u} \cdot \boldsymbol{u}^\epsilon \partial_t \psi - \nu \left(A^{s/2} \boldsymbol{u} \right) \cdot A^{s/2} \left(\boldsymbol{u}^\epsilon \psi \right) - \nu A^{s/2} \left(\boldsymbol{u} \psi \right) \left(A^{s/2} \boldsymbol{u}^\epsilon \right) + \left(\boldsymbol{u}^\epsilon P + \boldsymbol{u} P^\epsilon \right) \cdot \nabla \psi$$

$$\xrightarrow{\epsilon \to 0} |\boldsymbol{u}|^2 \partial_t \psi - 2\nu \left(A^{s/2} \boldsymbol{u} \right) \cdot A^{s/2} \left(\boldsymbol{u} \psi \right) + 2P \boldsymbol{u} \cdot \nabla \psi.$$

In addition we have

$$\int_0^T \int_V \left(\boldsymbol{u} \cdot \boldsymbol{u}^\epsilon \right) \boldsymbol{u} \cdot \nabla \psi \, dx dt \xrightarrow{\epsilon \to 0} \int_T \int_V |\boldsymbol{u}|^2 \boldsymbol{u} \cdot \nabla \psi \, dx dt, \tag{3.13}$$

as well as (by integrating by parts)

$$\int_0^T \int_V \left[-\frac{1}{2} \psi \nabla \cdot \left(|\boldsymbol{u}|^2 \boldsymbol{u} \right)^\epsilon + \frac{1}{2} \psi \boldsymbol{u} \cdot \nabla \left(|\boldsymbol{u}|^2 \right)^\epsilon \right] dx dt \xrightarrow{\epsilon \to 0} 0. \tag{3.14}$$

We can now write the following equation for the defect term

$$\frac{1}{2} D_\epsilon (\boldsymbol{u}) = -\partial_t (\boldsymbol{u} \cdot \boldsymbol{u}^\epsilon) - \nu A^s \boldsymbol{u} \cdot \boldsymbol{u}^\epsilon - \nu A^s \boldsymbol{u}^\epsilon \cdot \boldsymbol{u} - \nabla \cdot (\boldsymbol{u}^\epsilon P + \boldsymbol{u} P^\epsilon)$$

$$+ \frac{1}{2} \nabla \cdot \left[\left(|\boldsymbol{u}|^2 \boldsymbol{u} \right)^\epsilon - \boldsymbol{u} \left(|\boldsymbol{u}|^2 \right)^\epsilon \right] - \nabla \cdot ((\boldsymbol{u} \cdot \boldsymbol{u}^\epsilon) \boldsymbol{u}). \tag{3.15}$$

We note that $A^s \boldsymbol{u} \in L^2 \left[(0,T); H^{-s}(\mathbb{T}^3) \right]$, then by using the para-differential calculus (see [44]), it follows that $A^s \boldsymbol{u} \cdot \boldsymbol{u} \psi \in L^1 \left[(0,T); W^{-s-b,1}(\mathbb{T}^3) \right]$ for some small $b > 0$. By examining equation (3.15) we conclude that the right-hand side lies in $W^{-1,1} \left[(0,T); W^{-1-b,1}(\mathbb{T}^3) \right]$ and the limit as $\epsilon \to 0$ is independent of the choice of mollifier φ_ϵ. Therefore $D(\boldsymbol{u}) := \lim_{\epsilon \to 0} D_\epsilon(\boldsymbol{u})$ exists as an element in $W^{-1,1} \left[(0,T); W^{-(1+b),1}(\mathbb{T}^3) \right]$ and is also independent of the choice of mollifier. Alternatively, this can be seen from the following equation

$$\frac{1}{2} \int_0^T \int_{\mathbb{T}^3} D_\epsilon (\boldsymbol{u}) (\boldsymbol{x}, t) \psi \, dx dt = \int_0^T \int_{\mathbb{T}^3} \left[\boldsymbol{u} \cdot \boldsymbol{u}^\epsilon \partial_t \psi - \nu \left(A^{s/2} \boldsymbol{u} \right) \cdot A^{s/2} \left(\boldsymbol{u}^\epsilon \psi \right) \right.$$

$$- \nu A^{s/2} \left(\boldsymbol{u} \psi \right) \left(A^{s/2} \boldsymbol{u}^\epsilon \right) + \left(\boldsymbol{u}^\epsilon P + \boldsymbol{u} P^\epsilon \right) \cdot \nabla \psi - \frac{1}{2} \psi \nabla \cdot \left(|\boldsymbol{u}|^2 \boldsymbol{u} \right)^\epsilon$$

$$\left. + \frac{1}{2} \psi \boldsymbol{u} \cdot \nabla \left(|\boldsymbol{u}|^2 \right)^\epsilon + (\boldsymbol{u} \cdot \boldsymbol{u}^\epsilon) \boldsymbol{u} \cdot \nabla \psi \right] dx dt. \tag{3.16}$$

We conclude that in the limit $\epsilon \to 0$, we obtain the equation of local energy balance

$$\int_0^T \int_{\mathbb{T}^3} \left[|\boldsymbol{u}|^2 \partial_t \psi - 2\nu \left(A^{s/2} \boldsymbol{u} \right) \cdot A^{s/2} \left(\boldsymbol{u} \psi \right) + 2P \boldsymbol{u} \cdot \nabla \psi - \frac{1}{2} D(\boldsymbol{u}) \psi + |\boldsymbol{u}|^2 \boldsymbol{u} \cdot \nabla \psi \right] dx dt = 0, \tag{3.17}$$

as in (1.11).

Proof of corollary 1. By lemma 3 we find that $\boldsymbol{u} \in L^3 \left[\mathbb{T}^3 \times (0,T) \right]$ if $s \geqslant \frac{3}{4}$. Then the result follows by theorem 3. □

Remark 6. If $0 < s < \frac{3}{4}$, one needs to make the separate regularity assumption $\boldsymbol{u} \in L^3\left[\mathbb{T}^3 \times (0, T)\right]$, in order to prove that the Leray–Hopf solution satisfies an equation of local energy balance.

We now prove a sufficient condition for the defect term $D(\boldsymbol{u})$ to be zero (i.e. for the energy equality to hold), which is similar to the condition from Duchon and Robert [55]. In the next theorem we use Besov spaces $B_{p,q}^s(\mathbb{T}^3)$, which are defined in appendix A.

Proposition 7. *Let $\boldsymbol{u} \in L^3\left[(0, T); B_{3,\infty}^\alpha(\mathbb{T}^3)\right]$ with $\alpha > \frac{1}{3}$ be a Leray–Hopf weak solution of the fractional Navier–Stokes equations, then the defect term $D(\boldsymbol{u}) = 0$ in $L^1\left[\mathbb{T}^3 \times (0, T)\right]$. This implies that equation (1.11) is an energy balance; i.e. the following holds*

$$\int_0^T \int_{\mathbb{T}^3} \left[|\boldsymbol{u}|^2 \partial_t \psi - 2\nu \left(A^{s/2}\boldsymbol{u}\right) \cdot A^{s/2}\left(\boldsymbol{u}\psi\right) + 2P\boldsymbol{u} \cdot \nabla\psi + |\boldsymbol{u}|^2\boldsymbol{u} \cdot \nabla\psi\right] dxdt = 0. \quad (3.18)$$

Proof. We make the following estimate

$$\int_0^T \int_{\mathbb{T}^3} |D_\epsilon(\boldsymbol{u})| \, dxdt \leqslant \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\nabla\varphi^\epsilon(\xi)| |\delta\boldsymbol{u}|^3 \, d\xi \, dxdt$$

$$\leqslant \int_0^T \|\boldsymbol{u}\|_{B_{3,\infty}^\alpha}^3 \, dt \int_{\mathbb{R}^3} |\nabla\varphi^\epsilon(\xi)| |\xi|^{3\alpha} \, d\xi$$

$$= \int_0^T \|\boldsymbol{u}\|_{B_{3,\infty}^\alpha}^3 \, dt \int_{\mathbb{R}^3} |\nabla\varphi(z)| |z| |\epsilon z|^{3\alpha - 1} \, dz, \quad (3.19)$$

where in the last line we have made the change of variable $\xi = \epsilon z$. By the dominated convergence theorem it follows that $D_\epsilon(\boldsymbol{u}) \xrightarrow{\epsilon \to 0} 0$ in $L^1\left[\mathbb{T}^3 \times (0, T)\right]$. $\qquad\square$

The results are self-consistent as we can recover the energy equality originally found in [1].

Proposition 8. *Let $\boldsymbol{u}$ be a Leray–Hopf solution of the fractional Navier–Stokes equations with $s > \frac{5}{4}$, then $D(\boldsymbol{u}) = 0$ and the energy equality holds.*

Proof. We first observe that $W^{\alpha,3}(\mathbb{T}^3) \subset B_{3,\infty}^\alpha(\mathbb{T}^3)$. By again relying on the Gagliardo–Nirenberg–Sobolev inequality (as stated in [43]) we find that (for $\alpha + \frac{1}{2} < s$)

$$\|\boldsymbol{u}\|_{W^{\alpha,3}} \leqslant \|\boldsymbol{u}\|_{L^2}^a \|\boldsymbol{u}\|_{H^s}^{1-a}, \quad (3.20)$$

where we have the following relation between the exponents

$$\frac{1}{3} = \frac{a}{2} + \frac{1-a}{2} - \frac{(1-a)s - \alpha}{3} \quad \implies \quad a = \frac{2s - 1 - 2\alpha}{2s}. \quad (3.21)$$

Therefore we find the following inequality

$$\|\boldsymbol{u}\|_{W^{\alpha,3}} \leqslant \|\boldsymbol{u}\|_{L^2}^{\frac{2s-1-2\alpha}{2s}} \|\boldsymbol{u}\|_{H^s}^{\frac{1+2\alpha}{2s}}.$$

For $\boldsymbol{u}$ to be in $L^3\left[(0, T); W^{\alpha,3}(\mathbb{T}^3)\right]$, we need

$$\frac{1 + 2\alpha}{2s} \leqslant \frac{2}{3} \quad \implies \quad \frac{3}{4}(1 + 2\alpha) \leqslant s.$$

Because we can take $\alpha > \frac{1}{3}$ arbitrarily close to $\frac{1}{3}$, this gives the condition

$$s > \frac{5}{4}. \tag{3.22}$$

Then by theorem 3 and proposition 7 the result follows. $\qquad\square$

4. Proof of theorem 4

The proof of theorem 4 will be split into several parts and follows the method of Foias *et al* [36] who originally proved the $s = 1$ case. We introduce the following notation.

$$\zeta_s = \frac{2s}{3s-1}, \qquad \beta = \frac{3}{2(n-s)}, \qquad \rho_1 = 1 + \frac{1}{2}\beta\zeta_s, \qquad \rho_2 = \frac{1}{2}\zeta_s(1-\beta). \tag{4.1}$$

First we will establish a set of *a priori* estimates.

Proposition 9. *Let* **u** *be a smooth solution of the fractional Navier–Stokes equations with* $s > \frac{5}{6}$*. Then the following differential inequalities hold:*

$$\frac{1}{2}\frac{d}{dt}H_{n,1} \leqslant -\nu_s\zeta_s^{-1}H_{n+s,1} + c_{n,s}\zeta_s^{-1}\nu_s^{1-\zeta_s}H_{n,1}^{\rho_1}H_{s,1}^{\rho_2}, \tag{4.2}$$

$$\frac{1}{2}\frac{d}{dt}H_{n,1} \leqslant -\left(\frac{6s-5}{4n}\right)\nu_s H_{n+s,1} + \left(\frac{6s-5}{4n}\right)c_{n,s}\nu_s^{\frac{6s-5-4n}{6s-5}}H_{s,1}^{1+\frac{2}{6s-5}n}, \tag{4.3}$$

where for estimate (4.2) $n > s + \frac{3}{2}$ *and* $n > 2 + \frac{1}{2}s$*, and for estimate (4.3)* $n \geqslant 1$*.*

Proof. We define $w = A^{s/2}\boldsymbol{u}$ and let $\beta = \frac{3}{2(n-s)}$. We have

$$\|w\|_\infty \leqslant c\,\|A^{(n-s)/2}w\|_2^\beta\|w\|_2^{1-\beta}, \tag{4.4}$$

which can be rewritten as

$$\|A^{s/2}\boldsymbol{u}\|_\infty \leqslant c\,H_{n,1}^{\frac{1}{2}\beta}H_{s,1}^{\frac{1}{2}(1-\beta)}. \tag{4.5}$$

By using (4.5) in (2.14) we have

$$\frac{1}{2}\frac{d}{dt}H_{n,1} \leqslant -\nu_s\left(\frac{3s-1}{2s}\right)H_{n+s,1} + \frac{(3s-1)c_{n,s}}{2s}\nu_s^{-\frac{1-s}{3s-1}}\|A^{s/2}\boldsymbol{u}\|_\infty^{2s/(3s-1)}H_{n,1}$$

$$\leqslant -\nu_s\zeta_s^{-1}H_{n+s,1} + c_{n,s}\zeta_s^{-1}\nu_s^{1-\zeta_s}H_{n,1}^{\rho_1}H_{s,1}^{\rho_2}, \tag{4.6}$$

having used the definition $\zeta_s = \frac{2s}{3s-1}$. This proves estimate (4.2).

In order to prove the second inequality, we recall the following interpolation inequality

$$H_{n,1} \leqslant H_{s,1}^{\frac{s}{n}}H_{n+s,1}^{\frac{n-s}{n}}.$$

Inserting this inequality into (4.6), we find that

$$\frac{1}{2}\frac{d}{dt}H_{n,1} \leqslant -\nu_s\zeta_s^{-1}H_{n+s,1} + c_{n,s}\zeta_s^{-1}\nu_s^{1-\zeta_s}H_{n+s,1}^{\frac{\rho_1(n-s)}{n}}H_{s,1}^{\rho_2+\rho_1\frac{s}{n}}.$$

Then by applying Young's inequality we find (where $\chi_{n,s} := [(1-\rho_1)n + \rho_1 s]/n = [s - \frac{3}{4}\zeta_s]/n$)

$$\frac{1}{2}\frac{d}{dt}H_{n,1} \leqslant -\nu_s\zeta_s^{-1}H_{n+s,1} + \left(\nu_s\zeta_s^{-1}H_{n+s,1}\right)^{\frac{\rho_1(n-s)}{n}} c_{n,s}\zeta_s^{-1+\frac{\rho_1(n-s)}{n}}\nu_s^{1-\zeta_s-\frac{\rho_1(n-s)}{n}}H_{s,1}^{\rho_2+\rho_1\frac{s}{n}}$$

$$\leqslant -\chi_{n,s}\nu_s\zeta_s^{-1}H_{n+s,1} + \chi_{n,s}\left(c_{n,s}\zeta_s^{-1+\frac{\rho_1(n-s)}{n}}\nu_s^{1-\zeta_s-\frac{\rho_1(n-s)}{n}}H_{s,1}^{\rho_2+\rho_1\frac{s}{n}}\right)^{\frac{n}{(1-\rho_1)n+\rho_1 s}}$$

$$\leqslant -\frac{s-\frac{3}{4}\zeta_s}{n}\nu_s\zeta_s^{-1}H_{n+s,1} + \frac{s-\frac{3}{4}\zeta_s}{n}\zeta_s^{-1}\nu_s\left(c_{n,s}\nu_s^{-\zeta_s}H_{s,1}^{\frac{1}{n}\left(s+\frac{1}{2}\zeta_s n-\frac{3}{4}\zeta_s\right)}\right)^{\frac{n}{s-\frac{3}{4}\zeta_s}}$$

$$\leqslant -\left(\frac{6s-5}{4n}\right)\nu_s H_{n+s,1} + \left(\frac{6s-5}{4n}\right)c_{n,s}\nu_s^{1-\frac{n\zeta_s}{s-\frac{3}{4}\zeta_s}}H_{s,1}^{1+\frac{\zeta_s}{2\left(s-\frac{3}{4}\zeta_s\right)}n}$$

$$\leqslant -\left(\frac{6s-5}{4n}\right)\nu_s H_{n+s,1} + \left(\frac{6s-5}{4n}\right)c_{n,s}\nu_s^{\frac{6s-5-4n}{6s-5}}H_{s,1}^{1+\frac{2}{6s-5}n}.$$

This completes the proof of estimate (4.3) for the case $n > s + \frac{3}{2}$. Now we consider the cases $n = 1, 2$ separately. If $n = 1$ we have

$$\frac{1}{2}\frac{d}{dt}H_{1,1} \leqslant -\nu_s H_{1+s,1} + c_{n,s}\|\boldsymbol{u}\|_{W^{1,3}}^3$$

$$\leqslant -\nu_s H_{1+s,1} + c_{n,s}H_{s,1}^{\frac{3}{2}\left(s-\frac{1}{2}\right)}H_{1+s,1}^{\frac{3}{2}\left(\frac{3}{2}-s\right)}$$

$$\leqslant -\nu_s\left(\frac{3}{2}s-\frac{5}{4}\right)H_{1+s,1} + \left(\frac{3}{2}s-\frac{5}{4}\right)c_{n,s}\nu_s^{-\frac{\frac{3}{2}-s}{s-\frac{5}{6}}}H_{s,1}^{\frac{s-\frac{1}{2}}{s-\frac{5}{6}}},$$

where we have used a Gagliardo–Nirenberg interpolation inequality in the second line, and Young's inequality in the third line. This proves estimate (4.3) in the case $n = 1$. For $n = 2$ we have

$$\frac{1}{2}\frac{d}{dt}H_{2,1} \leqslant -\nu_s H_{2+s,1} + c_{n,s}\|\nabla\boldsymbol{u}\|_\infty H_{2,1}$$

$$\leqslant -\nu_s H_{2+s,1} + c_{n,s}H_{1+s,1}^{\frac{1}{2}\left(s-\frac{1}{2}\right)}H_{2+s,1}^{\frac{1}{2}\left(\frac{3}{2}-s\right)}H_{s,1}^{\frac{1}{2}s}H_{2+s,1}^{\frac{1}{2}(2-s)}$$

$$\leqslant -\nu_s H_{2+s,1} + c_{n,s}H_{s,1}^{\frac{3}{4}s-\frac{1}{8}}H_{2+s,1}^{\frac{13}{8}-\frac{3}{4}s}$$

$$\leqslant -\nu_s\left(\frac{6s-5}{8}\right)H_{2+s,1} + c_{n,s}\left(\frac{6s-5}{8}\right)\nu_s^{\frac{6s-13}{6s-5}}H_{s,1}^{\frac{6s-1}{6s-5}},$$

which concludes the proof of estimate (4.3). $\qquad\square$

Proposition 10. *Let $\boldsymbol{u}_0 \in H^n(\mathbb{T}^3)$ for $n \geqslant 1$. Then there exists a unique local-in-time solution $\boldsymbol{u} \in L^\infty\left[(0,T); H^n(\mathbb{T}^3)\right] \cap L^2\left[(0,T); H^{n+s}(\mathbb{T}^3)\right]$ for all $T < t_1(\boldsymbol{u}_0)$ where the existence time $t_1(\boldsymbol{u}_0)$ depends on $\boldsymbol{u}_0$ and ν, but is independent of n.*

Proof. The case $n = 1$ is shown in theorem 12 in section 5. To prove the case $n \geqslant 2$ we introduce the following perturbed problem (for some $\epsilon > 0$)

$$\partial_t\boldsymbol{u}_\epsilon + \nu A^s\boldsymbol{u}_\epsilon + \epsilon A^{5/4}\boldsymbol{u}_\epsilon + \boldsymbol{u}_\epsilon \cdot \nabla\boldsymbol{u}_\epsilon + \nabla P_\epsilon = 0,$$

where the subscripts of $\boldsymbol{u}$ and P denote a solution of the problem for a given choice of $\epsilon > 0$. By the results in [1] we know that there exists a unique smooth solution $\boldsymbol{u}_\epsilon$ to the problem for

any choice $\epsilon > 0$. Moreover, $\boldsymbol{u}_\epsilon$ (which is smooth) satisfies the following rigorous estimates adapted from proposition 9

$$\frac{1}{2}\frac{d}{dt}H_{n,1} \leqslant -\nu_s\zeta_s^{-1}H_{n+s,1} - \epsilon H_{n+5/4,1} + c_{n,s}\zeta_s^{-1}\nu_s^{1-\zeta_s}H_{n,1}^{\rho_1}H_{s,1}^{\rho_2}, \tag{4.7}$$

$$\frac{1}{2}\frac{d}{dt}H_{n,1} \leqslant -\left(\frac{6s-5}{4n}\right)\nu_sH_{n+s,1} - \epsilon H_{n+5/4,1} + \left(\frac{6s-5}{4n}\right)c_{n,s}\nu_s^{\frac{6s-5-4n}{6s-5}}H_{s,1}^{1+\frac{2}{6s-5}n}. \tag{4.8}$$

It follows from these inequalities that there exists a time $t_n(\boldsymbol{u}_0)$ such that ess $\sup_{t\in[0,T]}H_{n,1} + \int_0^T H_{n+s,1}dt$ is controlled uniformly in ϵ for any $T < t_n(\boldsymbol{u}_0)$. Therefore we can extract a weak-$*$ converging subsequence (which we also call $\{\boldsymbol{u}_\epsilon\}$) converging to a solution

$$\boldsymbol{u} \in L^\infty\left[(0,t_n(\boldsymbol{u}_0);H^n(\mathbb{T}^3)\right] \cap L^2\left[(0,t_n(\boldsymbol{u}_0));H^{n+s}(\mathbb{T}^3)\right]. \tag{4.9}$$

It follows that $\boldsymbol{u}$ must be the unique local strong solution, whose existence and uniqueness was established in theorem 12. Moreover, $\boldsymbol{u}$ satisfies the following estimate

$$\text{ess } \sup_{t\in[0,T]}H_{n,1} + \nu\int_0^T H_{n+s,1}dt \lesssim \int_0^T H_{s,1}^{1+\frac{2}{6s-5}n}dt. \tag{4.10}$$

In fact, this implies that $t_n(\boldsymbol{u}_0) = t_1(\boldsymbol{u}_0)$ for any $n \geqslant 1$. This is because for $T < t_1(\boldsymbol{u}_0)$ we have $\boldsymbol{u} \in L^\infty\left[(0,T);H^s(\mathbb{T}^3)\right]$. This means that for any $t < t_n(\boldsymbol{u}_0)$ we have $H_{n,1}$ is uniformly bounded in time up to $t_n(\boldsymbol{u}_0)$. Then by the local existence result that has just been proved, we can extend the solution beyond $t_n(\boldsymbol{u}_0)$. This process can be reiterated up to $t_1(\boldsymbol{u}_0)$. Therefore $t_n(\boldsymbol{u}_0) = t_1(\boldsymbol{u}_0)$. □

By following the method of Foias *et al* [36], we will next show that if $s > \frac{5}{6}$, the set of singular times of a Leray–Hopf weak solution has zero Lebesgue measure. We first recall the idea of a regular time, the set of regular times $\mathcal{R}_n$ for some $n \geqslant s$ for a given Leray–Hopf solution $\boldsymbol{u}$ is defined as follows

$$\mathcal{R}_n := \left\{t \in \mathbb{R}_+ \,|\, \exists \epsilon > 0 \text{ such that } \boldsymbol{u} \in C\left[(t-\epsilon, t+\epsilon); H^n\left(\mathbb{T}^3\right)\right]\right\}. \tag{4.11}$$

We define the set of singular times as follows

$$\mathcal{S}_n := \left\{t \in \mathbb{R}_+ \,|\, \boldsymbol{u}\left(\cdot, t\right) \notin H^n\left(\mathbb{T}^3\right)\right\} \tag{4.12}$$

and then prove the following result about the Lebesgue measure of the regular times.

Proposition 11. *Let $\boldsymbol{u}$ be a Leray–Hopf solution and $n \in \mathbb{N}$, then $\boldsymbol{u}$ is $H^n(\mathbb{T}^3)$ regular for an open subset of $(0,\infty)$, such that $\mathbb{R}_+\backslash\mathcal{R}_n$ has zero Lebesgue measure.*

Proof. First we derive an *a priori* estimate. Suppose $\boldsymbol{u}$ is a smooth solution of the fractional Navier–Stokes equations, then by taking the $L^2(\mathbb{T}^3)$ inner product with $A^s\boldsymbol{u}$ and using several interpolation inequalities, we find that

$$\begin{aligned}\frac{d}{dt}H_{s,1} + \nu_sH_{2s,1} &\leqslant \left\|\left[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right]\cdot A^s\boldsymbol{u}\right\|_{L^1} \leqslant \|\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{L^2}H_{2s,1}^{1/2}\\ &\leqslant H_{s,1}^{\frac{4s-3}{4s}}H_{2s,1}^{\frac{3-2s}{4s}}H_{s,1}^{\frac{2s-1}{2s}}H_{2s,1}^{\frac{1-s}{2s}}H_{2s,1}^{1/2} = H_{s,1}^{\frac{8s-5}{4s}}H_{2s,1}^{\frac{5-2s}{4s}},\end{aligned} \tag{4.13}$$

we require

$$\frac{5-2s}{4s} < 1 \quad \Longrightarrow \quad s > \frac{5}{6}. \tag{4.14}$$

Therefore we are justified in using Young's inequality to derive the following inequality

$$\frac{1}{2}\frac{d}{dt}H_{s,1} + \nu_s\left(\frac{6s-5}{4s}\right)H_{2s,1} \leqslant \left(\frac{6s-5}{4s}\right)\nu_s^{\frac{2s-5}{6s-5}}H_{s,1}^{\frac{8s-5}{6s-5}}. \tag{4.15}$$

In order to derive estimate (4.15) for the case $s > 1$, we observe that

$$\begin{aligned}
\frac{d}{dt}H_{s,1} + \nu_s H_{2s,1} &\leqslant \left\| [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot A^s\boldsymbol{u} \right\|_{L^1} \leqslant \|\boldsymbol{u}\|_{L^{\frac{3}{s-1}}}\|\nabla\boldsymbol{u}\|_{L^{\frac{6}{5-2s}}}H_{2s,1}^{1/2} \\
&\leqslant H_{s,1}^{\frac{6s-5}{4s}}H_{2s,1}^{\frac{5-4s}{4s}}H_{s,1}^{1/2}H_{2s,1}^{1/2} = H_{s,1}^{\frac{8s-5}{4s}}H_{2s,1}^{\frac{5-2s}{4s}}. \tag{4.16}
\end{aligned}$$

Then by applying Young's inequality we find estimate (4.15).

For $m \geqslant 2$ we derive the following *a priori* estimate

$$\frac{1}{2}\frac{d}{dt}H_{ms,1} + \nu_s H_{(m+1)s,1} \leqslant \left\| [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot A^{ms}\boldsymbol{u} \right\|_{L^1} \leqslant H_{(m+1)s,1}^{1/2}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{\dot{H}^{(m-1)s}}. \tag{4.17}$$

Then we derive the inequality (by using the para-differential calculus, see appendix A and [44] for details)

$$\begin{aligned}
\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{\dot{H}^{(m-1)s}} &\leqslant \|\boldsymbol{u}\|_{B_{\infty,2}^{(m-1)s}}H_{(m-1)s+1,1}^{1/2} \\
&\leqslant \|\boldsymbol{u}\|_{B_{2,2}^{(m-1)s+3/2}}H_{(m-1)s+1,1}^{1/2} \\
&\leqslant H_{ms,1}^{\frac{4s-3}{4s}}H_{(m+1)s,1}^{\frac{3-2s}{4s}}H_{ms,1}^{\frac{2s-1}{2s}}H_{(m+1)s,1}^{\frac{1-s}{2s}} \\
&= H_{ms,1}^{\frac{8s-5}{4s}}H_{(m+1)s,1}^{\frac{5-4s}{4s}}. \tag{4.18}
\end{aligned}$$

Therefore we can conclude that

$$\frac{1}{2}\frac{d}{dt}H_{ms,1} + \nu_s H_{(m+1)s,1} \leqslant H_{ms,1}^{\frac{8s-5}{4s}}H_{(m+1)s,1}^{\frac{5-2s}{4s}}, \tag{4.19}$$

which implies

$$\frac{1}{2}\frac{d}{dt}H_{ms,1} + \nu_s\left(\frac{6s-5}{4s}\right)H_{(m+1)s,1} \leqslant \left(\frac{6s-5}{4s}\right)\nu_s^{\frac{2s-5}{6s-5}}H_{ms,1}^{\frac{8s-5}{6s-5}}. \tag{4.20}$$

Now we will show that $\mathcal{R}_n$ has full measure by induction. We first observe that by the energy inequality we have

$$\text{ess sup}_{t\in[0,\infty)}H_{0,1} + 2\nu_s\int_0^\infty H_{s,1}dt \leqslant \|\boldsymbol{u}_0\|_2^2. \tag{4.21}$$

This means that $H_{s,1}$ must be finite for almost all times and hence $\mathcal{R}_s$ has full measure (as the number of endpoints of disjoint intervals is countable). Now we proceed by induction and suppose we know that the sets $\mathcal{R}_{ms}$ have full measure for $1 \leqslant m \leqslant n$.

We consider an $H^{ns}(\mathbb{T}^3)$ regularity interval (t_l, t_r). By using the *a priori* estimate (4.20) for $m = n$ and an adaption of the proof of proposition 10 there exists a strong solution coinciding with the weak solution on this time interval (by weak-strong uniqueness as stated in section 5). For any $[t_0, t_1] \subset (t_l, t_r)$ this strong solution satisfies

$$\text{ess sup}_{t\in[t_0,t_1]}H_{ns,1} + 2\nu_s\left(\frac{6s-5}{4s}\right)\int_{t_0}^{t_1}H_{(n+1)s,1}dt \leqslant \frac{1}{2}H_{ns,1}(t_0) + \left(\frac{6s-5}{4s}\right)\nu_s^{\frac{2s-5}{6s-5}}\int_{t_0}^{t_1}H_{ns,1}^{\frac{8s-5}{6s-5}}dt.$$

It follows that $H_{(n+1)s,1}$ is finite for almost all times in (t_l, t_r). As this is true for any regularity interval, we conclude that $\mathcal{R}_{(n+1)s}$ has full measure. Therefore the result follows by induction.
$\square$

Now we are ready to prove theorem 4.

Proof of theorem 4. For any $n \geqslant 1$ there is a countable number of regularity intervals for the $H^n(\mathbb{T}^3)$ norm. In this proof we will work with integrals on the time interval $[0, T]$, which should be split into an (infinite) sum over the regularity intervals, which we will (often) not write down explicitly. We first derive a new energy estimate. We observe that for $m \geqslant 2$ (where the second term in the second line vanishes by integrating by parts)

$$\frac{1}{2}\frac{d}{dt}H_{ms,1} + \nu_s H_{(m+1)s,1} = -\int_{\mathbb{T}^3} [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] \cdot A^{ms}\boldsymbol{u}\,dx$$
$$= -\int_{\mathbb{T}^3} \left[A^{ms/2}[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] - \left[(\boldsymbol{u} \cdot \nabla)A^{ms/2}\boldsymbol{u}\right] \right] \cdot A^{ms/2}\boldsymbol{u}\,dx.$$

Then by applying lemma 4 in appendix B we find that

$$\frac{1}{2}\frac{d}{dt}H_{ms,1} + \nu_s H_{(m+1)s,1} \leqslant \left\| A^{ms/2}[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] - \left[(\boldsymbol{u} \cdot \nabla)A^{ms/2}\boldsymbol{u}\right] \right\|_{L^2} H_{ms,1}^{1/2}$$
$$\leqslant H_{1,1}^{\frac{2(m+1)s-5}{4(m+1)s-4}} H_{(m+1)s,1}^{\frac{3}{4(m+1)s-4}} H_{ms,1} \leqslant H_{s,1}^{\frac{2(m+1)s-5}{4ms}} H_{(m+1)s,1}^{\frac{(2(m+1)s-5)(1-s)}{(4(m+1)s-4)ms}} H_{(m+1)s,1}^{\frac{3}{4(m+1)s-4}} H_{ms,1}$$
$$\leqslant H_{s,1}^{\frac{2(m+1)s-5}{4ms}} H_{(m+1)s,1}^{\frac{5-2s}{4ms}} H_{ms,1} \leqslant H_{s,1}^{\frac{2(m+2)s-5}{4ms}} H_{(m+1)s,1}^{\frac{2(m-2)s+5}{4ms}} H_{ms,1}^{1/2}$$
$$\leqslant \frac{2(m-2)s+5}{4ms} \nu_s H_{(m+1)s,1} + c_{m,s} H_{s,1} H_{ms,1}^{\frac{2ms}{2(m+2)s-5}}.$$

We note that if $s > 1$ some of the intermediate steps given above are not needed, as lemma 4 can be applied directly and it results in the same final inequality. Rearranging the obtained inequality then yields

$$\frac{1}{2}\frac{d}{dt}H_{ms,1} + \frac{2(m+2)s-5}{4ms}\nu_s H_{(m+1)s,1} \leqslant c_{m,s} H_{s,1} H_{ms,1}^{\frac{2ms}{2(m+2)s-5}}. \tag{4.22}$$

We have now proved this estimate for $m \geqslant 2$, but we recall that the case $m = 1$ was already given in equation (4.15). Therefore we can use this estimate for all $m \geqslant 1$. We observe that this *a priori* estimate is rigorous inside a regularity interval, and we will sum over regularity intervals at a later stage.

Then by dividing in equation (4.22) by $(1+H_{ms,1})^{\frac{2ms}{2(m+2)s-5}}$ and integrating in time (where (t_l, t_r) is a single regularity interval) we find that (for any $\epsilon > 0$)

$$\frac{1}{2}\int_{t_l+\epsilon}^{t_r-\epsilon} \frac{\frac{d}{dt}H_{ms,1}}{(1+H_{ms,1})^{\frac{2ms}{2(m+2)s-5}}}\,dt + \frac{2(m+2)s-5}{4ms}\nu_s \int_{t_l+\epsilon}^{t_r-\epsilon} \frac{H_{(m+1)s,1}}{(1+H_{ms,1})^{\frac{2ms}{2(m+2)s-5}}}\,dt \leqslant c_{m,s}\int_{t_l+\epsilon}^{t_r-\epsilon} H_{s,1}\,dt.$$

We evaluate the first integral as follows

$$\frac{1}{2}\int_{t_l+\epsilon}^{t_r-\epsilon} \frac{\frac{d}{dt}H_{ms,1}}{(1+H_{ms,1})^{\frac{2ms}{2(m+2)s-5}}}\,dt = \frac{2(m+2)s-5}{10-8s}\left[\frac{1}{(1+H_{ms,1}(t_l+\epsilon))^{\frac{5-4s}{2(m+2)s-5}}} \right.$$
$$\left. - \frac{1}{(1+H_{ms,1}(t_r-\epsilon))^{\frac{5-4s}{2(m+2)s-5}}} \right].$$

Combining these two expressions then gives

$$\frac{2\,(m+2)\,s-5}{4ms}\nu_s\int_{t_l+\epsilon}^{t_r-\epsilon}\frac{H_{(m+1)s,1}}{\left(1+H_{ms,1}\right)^{\frac{2ms}{2(m+2)s-5}}}\,dt \leqslant c_{m,s}\int_{t_l+\epsilon}^{t_r-\epsilon}H_{s,1}\,dt + \left(\frac{2\,(m+2)\,s-5}{10-8s}\right)$$
$$\times \frac{1}{\left(1+H_{ms,1}\left(t_r-\epsilon\right)\right)^{\frac{5-4s}{2(m+2)s-5}}}.$$

Then taking the limit $\epsilon \to 0$ then yields (recalling that $\lim_{t\to t_r}H_{ms}(t)=\infty$)

$$\frac{2\,(m+2)\,s-5}{4ms}\nu_s\int_{t_l}^{t_r}\frac{H_{(m+1)s,1}}{\left(1+H_{ms,1}\right)^{\frac{2ms}{2(m+2)s-5}}}\,dt \leqslant c_{m,s}\int_{t_l}^{t_r}H_{s,1}\,dt. \qquad (4.23)$$

Summing over the (at most) countable number of regularity intervals then results in the following bound

$$\frac{2\,(m+2)\,s-5}{4ms}\nu_s\int_{0}^{T}\frac{H_{(m+1)s,1}}{\left(1+H_{ms,1}\right)^{\frac{2ms}{2(m+2)s-5}}}\,dt \leqslant c_{m,s}\int_{0}^{T}H_{s,1}\,dt < \infty, \qquad (4.24)$$

due to the energy inequality. We observe here that the estimation of integral in time of the term involving $\frac{d}{dt}H_{ms,1}$ which was given above, does not carry over verbatim to the case $s \geqslant \frac{5}{4}$. It is possible to employ a different estimate here. However as mentioned before, we will not do so here. This is because if $s \geqslant \frac{5}{4}$ the equations are known to be globally well-posed and hence a bound on a hierarchy of time averages does not yield anything new.

Now we will proceed to prove the regularity bounds by induction. From the energy inequality we know that $H_{s,1} \in L^1(0,T)$. Now suppose that $H_{ms,1} \in L^{\gamma_m}(0,T)$, where $0 < \gamma_m \leqslant 1$ is a for now undetermined constant. Then one finds that

$$\int_{0}^{T}H_{(m+1)s,1}^{\gamma_{m+1}}\,dt = \int_{0}^{T}\frac{H_{(m+1)s,1}^{\gamma_{m+1}}}{\left(1+H_{ms,1}\right)^{\frac{2ms\gamma_{m+1}}{2(m+2)s-5}}}\left(1+H_{ms,1}\right)^{\frac{2ms\gamma_{m+1}}{2(m+2)s-5}}\,dt$$
$$\leqslant \left(\int_{0}^{T}\frac{H_{(m+1)s,1}}{\left(1+H_{ms,1}\right)^{\frac{2ms}{2(m+2)s-5}}}\,dt\right)^{\gamma_{m+1}}\left(\int_{0}^{T}\left(1+H_{ms,1}\right)^{\frac{\gamma_{m+1}}{1-\gamma_{m+1}}\frac{2ms}{2(m+2)s-5}}\,dt\right)^{1-\gamma_{m+1}},$$

where in the second line we have used Hölder's inequality with $p=\frac{1}{\gamma_{m+1}}$ and $p'=\frac{1}{1-\gamma_{m+1}}$. The first integral on the second line is bounded due to inequality (4.24). In order to be able to use the inductive hypothesis (i.e. the fact that $H_{ms,1} \in L^{\gamma_m}(0,T)$), the following inequality needs to hold

$$\frac{\gamma_{m+1}}{1-\gamma_{m+1}}\frac{2ms}{2\,(m+2)\,s-5} \leqslant \gamma_m. \qquad (4.25)$$

Combined with $\gamma_1=1$, we find that

$$\gamma_m = \frac{6s-5}{2\,(m+2)\,s-5}. \qquad (4.26)$$

Thus by induction we have shown that $H_{ms,1} \in L^{\gamma_m}(0,T)$ with the γ_m given in equation (4.26). Finally, by using an interpolation inequality we obtain that (for some for now undetermined $0 < \delta_{n,s} < 1$)

$$\int_0^T H_{n,1}^{\delta_{n,s}} \, dt \leqslant \int_0^T H_{s,1}^{\frac{\delta_{n,s}((2n+1)s-n)}{2ns}} H_{(2n+1)s,1}^{\frac{\delta_{n,s}(n-s)}{2ns}} \, dt \leqslant \left(\int_0^T H_{s,1} \, dt \right)^{\frac{\delta_{n,s}((2n+1)s-n)}{2ns}}$$
$$\times \left(\int_0^T H_{(2n+1)s,1}^{\frac{\delta_{n,s}(n-s)}{2ns-\delta_{n,s}((2n+1)s-n)}} \, dt \right)^{\frac{2ns-\delta_{n,s}((2n+1)s-n)}{2ns}}.$$

The third integral on the first line is finite due to the energy inequality. The integral on the second line is bounded if (because of the bounds we just established)

$$\frac{\delta_{n,s}(n-s)}{2ns - \delta_{n,s}((2n+1)s-n)} \leqslant \gamma_{2n+1}.$$

Then we can compute the constants $\delta_{n,s}$ to be

$$\delta_{n,s} = \frac{6s-5}{2n+4s-5}, \tag{4.27}$$

which agrees with the definition in (1.14). We note that this expression reduces to $\delta_{n,1} = \frac{1}{2n-1}$ when $s = 1$, which agrees with the result of Foias *et al* [36]. Thus we have proved the regularity stated in theorem 4.

$\square$

5. Local well-posedness of the fractional Navier–Stokes equations

In this section we provide a self-contained proof of the local well-posedness of the fractional Navier–Stokes equations, as well as a weak-strong uniqueness result. These results were used in the proofs in the previous sections and appear to be absent in the literature: see [47, 52] for proofs of related local well-posedness results.

Theorem 12. *Consider the fractional Navier–Stokes equation* (1.1) *with s as the power of the fractional Laplacian. We consider three cases:*

- *If* $s > \frac{5}{6}$ *and* $\boldsymbol{u}_0 \in H^1(\mathbb{T}^3)$*, then there exists a unique local strong solution* $\boldsymbol{u} \in L^\infty\left[(0,T); H^1(\mathbb{T}^3)\right] \cap L^2\left[(0,T); H^{1+s}\right]$.
- *If* $\frac{1}{3} < s \leqslant \frac{5}{6}$ *and* $\boldsymbol{u}_0 \in H^2(\mathbb{T}^3)$*, then there is a unique local strong solution of the fractional Navier–Stokes equations with regularity* $L^\infty\left[(0,T); H^2(\mathbb{T}^3)\right] \cap L^2\left[(0,T); H^{2+s}\right]$.
- *For* $0 < s \leqslant \frac{1}{3}$ *and initial data* $\boldsymbol{u}_0 \in H^3(\mathbb{T}^3)$*, there exists a unique local strong solution in* $L^\infty\left[(0,T); H^3(\mathbb{T}^3)\right] \cap L^2\left[(0,T); H^{3+s}\right]$.

Proof. We will not deal with the case $0 < s \leqslant \frac{1}{3}$, which is given in [52, theorem 3.4]. In order to prove the other two cases, we first apply the Galerkin projection P_N to the equations

$$\partial_t \boldsymbol{u}^N + \nu A^s \boldsymbol{u}^N + P_N\left((\boldsymbol{u}^N \cdot \nabla)\boldsymbol{u}^N\right) = 0. \tag{5.1}$$

For every finite N, we know that there exists a unique smooth solution $\boldsymbol{u}^N$ to these equations. If $s > \frac{5}{6}$, the Galerkin approximations will satisfy estimate (4.3) where we take $n = 1$. This means

that there is a time $t_1(\boldsymbol{u}_0)$ such that there exists a sub-sequence of $\{\boldsymbol{u}^N\}$ converging weak-* in $L^\infty\left[(0,T);H^1(\mathbb{T}^3)\right]$ and weakly in $L^2\left[(0,T);H^{1+s}(\mathbb{T}^3)\right]$ to a strong solution $\boldsymbol{u}$.

For the case $\frac{1}{3} < s \leqslant \frac{5}{6}$, by performing a standard energy estimate one finds

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^N\|_{H^2}^2 \leqslant -\nu\|\boldsymbol{u}^N\|_{H^{2+s}}^2 + c_{n,s}\|\Delta\boldsymbol{u}^N\|_{L^{5/2}}^2\|\nabla\boldsymbol{u}^N\|_{L^5}. \tag{5.2}$$

We then recall the following interpolation inequality

$$\|\Delta\boldsymbol{u}^N\|_{L^{5/2}} \leqslant c\|\Delta\boldsymbol{u}^N\|_{L^2}^{1-3/(10s)}\|\boldsymbol{u}^N\|_{H^{2+s}}^{3/(10s)}. \tag{5.3}$$

By using Young's inequality we find

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^N\|_{H^2}^2 \leqslant -\nu\|\boldsymbol{u}^N\|_{H^{2+s}}^2 + c_{n,s}\|\Delta\boldsymbol{u}^N\|_{L^2}^{3-3/(5s)}\|\boldsymbol{u}^N\|_{H^{2+s}}^{3/(5s)} \tag{5.4}$$

$$\leqslant -\left(\frac{10s-3}{10s}\right)\nu\|\boldsymbol{u}^N\|_{H^{2+s}}^2 + c_{n,s}\left(\frac{10s-3}{10s}\right)\nu^{-3/(10s-3)}\|\boldsymbol{u}^N\|_{H^2}^{2(15s-3)/(10s-3)}. \tag{5.5}$$

As previously observed, one can extract a subsequence of the Galerkin sequence which converges to the strong solution. The uniqueness in all the considered ranges of s can be proved by standard methods.

Finally, we would like to remark that this result could also have been proved by adding a hyperviscous term $\epsilon A^{5/4}\boldsymbol{u}$ to the equations and then passing to a subsequence of strong solutions in the limit $\epsilon \to 0$, as demonstrated in proposition 10. $\qquad\square$

Remark 13. As already noted before, the critical space is $H^{5/2-2s}(\mathbb{T}^3)$. We observe that it is possible to adapt the proof of local existence of strong solutions to these spaces, as opposed to the integer Sobolev spaces that were used in theorem 12. However, this is not needed for our purposes.

Now we state and prove a weak-strong uniqueness result for the fractional Navier–Stokes equations, which again seems to be absent from the literature.

Theorem 14. *Let $\boldsymbol{u}_S$ be a strong solution of the fractional Navier–Stokes equations on $[0,T]$ and let $\boldsymbol{u}_W$ be a Leray–Hopf weak solution on the same time interval with the same initial data $\boldsymbol{u}_0$. Then $\boldsymbol{u}_W \equiv \boldsymbol{u}_S$ on $[0,T]$.*

Proof. By using $\boldsymbol{u}_S$ as a test function in the weak formulation that is obeyed by $\boldsymbol{u}_W$, we find that

$$\int_0^T\int_{\mathbb{T}^3}\left[\boldsymbol{u}_W\partial_t\boldsymbol{u}_S - \nu\left(A^{s/2}\boldsymbol{u}_W\right)\left(A^{s/2}\boldsymbol{u}_S\right) + \boldsymbol{u}_W\otimes\boldsymbol{u}_W:\nabla\boldsymbol{u}_S\right]dxdt$$
$$= -\int_{\mathbb{T}^3}\boldsymbol{u}_0^2\,dx + \int_{\mathbb{T}^3}\boldsymbol{u}_W(\boldsymbol{x},T)\boldsymbol{u}_S(\boldsymbol{x},T)\,dx. \tag{5.6}$$

Since the strong solution satisfies the equation in an L^2-sense, taking the $L^2(\mathbb{T}^3)$ inner product with $\boldsymbol{u}_W$ yields

$$\int_0^T\int_{\mathbb{T}^3}\left[-\boldsymbol{u}_W\partial_t\boldsymbol{u}_S - \nu\left(A^{s/2}\boldsymbol{u}_W\right)\left(A^{s/2}\boldsymbol{u}_S\right) - \boldsymbol{u}_S\otimes\boldsymbol{u}_W:\nabla\boldsymbol{u}_S\right]dxdt = 0. \tag{5.7}$$

Adding these two equations gives that

$$
\int_0^T \int_{\mathbb{T}^3} \left[-2\nu \left(A^{s/2} \boldsymbol{u}_W \right) \left(A^{s/2} \boldsymbol{u}_S \right) - \boldsymbol{u}_S \otimes \boldsymbol{u}_W : \nabla \boldsymbol{u}_S + \boldsymbol{u}_W \otimes \boldsymbol{u}_W : \nabla \boldsymbol{u}_S \right] dxdt
$$

$$
= -\int_{\mathbb{T}^3} |\boldsymbol{u}_0|^2 \, dx + \int_{\mathbb{T}^3} \boldsymbol{u}_W \left(\boldsymbol{x}, T \right) \boldsymbol{u}_S \left(\boldsymbol{x}, T \right) dx. \tag{5.8}
$$

We now introduce the notation $\boldsymbol{v} := \boldsymbol{u}_W - \boldsymbol{u}_S$, which allows to rewrite the equation above as follows

$$
\int_0^T \int_{\mathbb{T}^3} \left[-\nu |A^{s/2} \boldsymbol{u}_W|^2 - \nu |A^{s/2} \boldsymbol{u}_S|^2 + \nu |A^{s/2} \boldsymbol{v}|^2 + \boldsymbol{v} \otimes \boldsymbol{v} : \nabla \boldsymbol{u}_S \right] dxdt
$$

$$
= -\int_{\mathbb{T}^3} |\boldsymbol{u}_0|^2 \, dx + \frac{1}{2} \int_{\mathbb{T}^3} \left[|\boldsymbol{u}_W \left(\boldsymbol{x}, T \right)|^2 + |\boldsymbol{u}_S \left(\boldsymbol{x}, T \right)|^2 - |\boldsymbol{v} \left(\boldsymbol{x}, T \right)|^2 \right] dx. \tag{5.9}
$$

We can rearrange this expression as follows

$$
\frac{1}{2} \int_{\mathbb{T}^3} |\boldsymbol{v} \left(\boldsymbol{x}, T \right)|^2 \, dx + \int_0^T \int_{\mathbb{T}^3} \left[\nu |A^{s/2} \boldsymbol{v}|^2 + \boldsymbol{v} \otimes \boldsymbol{v} : \nabla \boldsymbol{u}_S \right] dxdt = \frac{1}{2} \int_{\mathbb{T}^3} \left[|\boldsymbol{u}_W \left(\boldsymbol{x}, T \right)|^2 - |\boldsymbol{u}_0|^2 \right] dx
$$

$$
+ \nu \int_0^T \int_{\mathbb{T}^3} |A^{s/2} \boldsymbol{u}_W|^2 \, dxdt + \frac{1}{2} \int_{\mathbb{T}^3} \left[|\boldsymbol{u}_S \left(\boldsymbol{x}, T \right)|^2 - |\boldsymbol{u}_0|^2 \right] dx + \nu \int_0^T \int_{\mathbb{T}^3} |A^{s/2} \boldsymbol{u}_S|^2 \, dxdt \leqslant 0, \tag{5.10}
$$

where the inequality follows from the energy equality for strong solutions and the energy inequality for Leray–Hopf weak solutions. We then obtain the following estimate

$$
\frac{1}{2} \int_{\mathbb{T}^3} |\boldsymbol{v} \left(\boldsymbol{x}, T \right)|^2 \, dx + \int_0^T \int_{\mathbb{T}^3} \nu |A^{s/2} \boldsymbol{v}|^2 \, dxdt \leqslant - \int_0^T \int_{\mathbb{T}^3} \boldsymbol{v} \otimes \boldsymbol{v} : \nabla \boldsymbol{u}_S \, dxdt
$$

$$
\leqslant \int_0^T \|\nabla \boldsymbol{u}_S\|_{L^{3/s}} \|\boldsymbol{v} \left(\cdot, t \right)\|_{L^{6/(3-2s)}} \|\boldsymbol{v} \left(\cdot, t \right)\|_{L^2} \, dt \leqslant \int_0^T \|\boldsymbol{u}_S\|_{H^{5/2-s}} \|\boldsymbol{v} \left(\cdot, t \right)\|_{\dot{H}^s} \|\boldsymbol{v} \left(\cdot, t \right)\|_{L^2} \, dt. \tag{5.11}
$$

Then by applying Young's inequality we find that

$$
\frac{1}{2} \int_{\mathbb{T}^3} |\boldsymbol{v} \left(\boldsymbol{x}, T \right)|^2 \, dx + \frac{1}{2} \nu \int_0^T \int_{\mathbb{T}^3} |A^{s/2} \boldsymbol{v}|^2 \, dxdt \leqslant \frac{1}{2} \nu^{-1} \int_0^T \|\boldsymbol{u}_S\|_{H^{5/2-s}}^2 \|\boldsymbol{v} \left(\cdot, t \right)\|_{L^2}^2 \, dt. \tag{5.12}
$$

Since $\boldsymbol{v}(\cdot, 0) = 0$, it follows from Gronwall's inequality that $\boldsymbol{v} \equiv 0$ on $\mathbb{T}^3 \times [0, T]$. $\qquad \square$

6. Summary and concluding remarks

The different functional properties of solutions of the three-dimensional fractional Navier–Stokes equations have been considered across five ranges of the exponent s, which are divided by four significant critical points: $s = \frac{1}{3}$; $s = \frac{3}{4}$; $s = \frac{5}{6}$ and $s = \frac{5}{4}$. Their existence suggests that solutions undergo a set of phase transitions at these points. Several explanatory remarks are in order.

1. In the range $0 < s < \frac{1}{3}$, the non-uniqueness of Leray–Hopf solutions has already been demonstrated in [48, 52] using convex integration methods. In addition, Bulut *et al* [56] have used these techniques to show the non-uniqueness of weak solutions with epochs of regularity; i.e. solutions of which the non-smoothness is limited to a set of bounded Hausdorff dimension. In particular, the result in [56] states that there are infinitely many weak solutions of the fractional Navier–Stokes equations for $s < \frac{1}{3}$ with regularity $C_t^0 C_x^s$. These can be chosen to coincide with the local strong solution for a short initial time interval. Our analogue of the Prodi–Serrin regularity criterion (theorem 2) shows that an initially strong solution with control of the $L_t^\infty C_x^s$ norm for $s > \frac{1}{3}$ will stay smooth. Therefore a non-uniqueness result of the type in [56] cannot be expected to hold for $s > \frac{1}{3}$. This suggests that the results from convex integration schemes which construct Hölder continuous solutions are sharp with regard to the value of s ($s < \frac{1}{3}$), at least from the epochs of regularity perspective.

2. What of the critical point $s = \frac{3}{4}$? We have observed that if $s \geqslant \frac{3}{4}$ then Leray–Hopf solutions satisfy an equation of local energy balance (theorem 3). Moreover, when $s > \frac{3}{4}$ there exists a suitable weak solution satisfying a partial regularity result, as proved in [54]. An improvement of the latter result was made in [53]. As noted in [53, p 10], the origin of the exponent $s = \frac{3}{4}$ comes from the requirement that a weak solution be an $L^3 \left[\mathbb{T}^3 \times (0, T) \right]$ function. This regularity is needed as part of the definition of a suitable weak solution, and in particular for the interpretation of the local energy inequality. As mentioned in remark 6, the equation of local energy balance can be established for a Leray–Hopf solution that lies in $L^3 \left[\mathbb{T}^3 \times (0, T) \right]$. Similar to the proof of the partial regularity result in [53], this regularity is needed to bound the cubic term $|\boldsymbol{u}|^2 \boldsymbol{u}$ in the local energy balance. This degree of regularity only follows from the Leray–Hopf regularity for $s \geqslant \frac{3}{4}$, as computed in lemma 3. Both theorem 3 together with the partial regularity results from [53, 54] have similar regularity requirements, so it is natural that this imposes the same lower bound on s. Some further discussion on the connection between the equation of local energy balance and the suitability of a weak solution is provided in [35, section 6.2].

3. We could argue loosely that in the range $0 \leqslant s < \frac{1}{3}$ the properties of the fractional Navier–Stokes equations correspond more to those of the Euler equations, while in the range $\frac{3}{4} \leqslant s < \frac{5}{6}$ they correspond more to the CKN-type suitable weak solutions of the Navier–Stokes equations [11, 31, 54] which satisfy partial regularity results. In the range $s > \frac{5}{6}$ their behaviour is of the standard Leray–Hopf type associated with $s = 1$ Navier–Stokes equations. Full regularity is reached at (most at) $s = \frac{5}{4}$.

4. Finally, we wish to make a clarification with respect to the standard Leray–Hopf results expressed in theorem 4 for the case $s > \frac{5}{6}$. For the standard ($s = 1$) three-dimensional Navier–Stokes equations, it has been shown in [40, 41] that there exists an infinite hierarchy of bounded time averages

$$\left\langle \| \nabla^n \boldsymbol{u} \|_{2m}^{\alpha_{n,m}} \right\rangle_T < \infty, \tag{6.1}$$

where the $\alpha_{n,m}$ are defined by

$$\alpha_{n,m} = \frac{2m}{2m(n+1) - 3} \tag{6.2}$$

and where $\langle \cdot \rangle_T$ is a time average up to time T. The $\alpha_{n,m}$ appear as a direct result of the scaling property of the norms under the invariance properties expressed in (1.7)

$$\| \nabla^n \boldsymbol{u} \|_{2m} = \lambda^{-1/\alpha_{n,m}} \| \nabla'^n \boldsymbol{u}' \|_{2m}. \tag{6.3}$$

The question arises whether the result in (6.1) is consistent with (1.15), which says that

$$\boldsymbol{u} \in L^{2\delta_{n,s}}\left[(0,T); H^n\left(\mathbb{T}^3\right)\right]. \tag{6.4}$$

Recall that $\delta_{n,s}$ has been defined in (1.14). To address this question we note that the equivalent of $\alpha_{n,m}$ for the fractional Navier–Stokes equations is

$$\alpha_{n,m,s} = \frac{2m}{2m\left(n + 2s - 1\right) - 3}. \tag{6.5}$$

A straightforward application of interpolation inequalities to the result of theorem 4 shows that the equivalent of (6.1) is

$$\left\langle \|\nabla^n \boldsymbol{u}\|_{2m}^{(6s-5)\alpha_{n,m,s}} \right\rangle_T < \infty. \tag{6.6}$$

The $6s - 5$ is a necessary factor to make (6.6) at $n = s$ and $m = 1$ into $\langle H_{s,1} \rangle_T$ which, from the energy inequality, is bounded from above. Then we write

$$\left[(6s - 5)\,\alpha_{n,m,s}\right]_{m=1} = \frac{6s - 5}{2n + 4s - 5} = \delta_{n,s}, \tag{6.7}$$

as in (1.15). Thus, we see that theorem 4 is closely related to the invariance properties of the fractional Navier–Stokes equations.

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

Appendix A. Properties of the fractional Laplacian

In this appendix we recall some basic properties of the fractional Laplacian. By using the Fourier representation (1.2) as well as the Plancherel identity, one can prove the following identity (for $f, g \in H^{2s}(\mathbb{T}^3)$

$$\int_{\mathbb{T}^3} \left(A^s f\right) g \, dx = \int_{\mathbb{T}^3} f\left(A^s g\right) \, dx. \tag{A.1}$$

We also observe that for any $s \in \mathbb{R}$ and $f \in H^s(\mathbb{T}^3)$ it holds that

$$\|f\|_{\dot{H}^s} = \|A^{s/2} f\|_2, \tag{A.2}$$

which can be easily seen from the Fourier representation. In the case $p \neq 2$, we have to rely on Littlewood–Paley theory (see [44] for more details).

First we introduce a dyadic partition of unity $\{\rho_j\}_{j=1}^{\infty}$ which is given by

$$\rho_0(x) = \rho(x), \quad \rho_j(x) = \rho\left(2^{-j}x\right) \quad \text{for} \quad j = 1, 2, \ldots, \tag{A.3}$$

with $\rho_{-1}(x) = 1 - \sum_{j=0}^{\infty} \rho_j(x)$. Then for $f \in \mathcal{S}'(\mathbb{T}^3)$ we can define the Littlewood–Paley blocks as follows (for $\xi \in \mathbb{Z}^3$)

$$\widehat{\Delta_j f}(\xi) = \rho_j(\xi)\widehat{f}(\xi), \quad j = -1, 0, \ldots. \tag{A.4}$$

Then for $q < \infty$ we introduce the Besov norm as follows

$$\|f\|_{B^s_{p,q}} := \|\Delta_{-1}f\|_{L^p} + \left(\sum_{j=0}^{\infty} 2^{sjq}\|\Delta_j f\|_{L^p}^q\right)^{1/q}, \tag{A.5}$$

and if $q = \infty$ the norm is given by

$$\|f\|_{B^s_{p,\infty}} := \|\Delta_{-1}f\|_{L^p} + \sup_{j \geqslant 0}\left(2^{sj}\|\Delta_j f\|_{L^p}\right). \tag{A.6}$$

In [45, equation (A.3)] the following inequality is stated (where $1 \leqslant p \leqslant \infty$, $j \geqslant 0$ and $s \in \mathbb{R}$)

$$\|\Delta_j A^s f\|_{L^p} \sim 2^{js}\|\Delta_j f\|_{L^p}. \tag{A.7}$$

Therefore if $\int_{\mathbb{T}^3} f \, dx = 0$, we know that $\Delta_{-1}f = 0$ (by a suitable choice of a dyadic partition of unity). This means that for mean-free functions $f \in B^t_{p,q}(\mathbb{T}^3)$ by estimate (A.7) it follows that (for $1 \leqslant p, q \leqslant \infty$ and $s, t \in \mathbb{R}$)

$$\|A^s f\|_{B^{t-s}_{p,q}} \sim \|f\|_{B^t_{p,q}}. \tag{A.8}$$

Now we recall that $W^{s,p}(\mathbb{T}^3) = B^s_{p,p}(\mathbb{T}^3)$ (see [46, equation (3.5)]) for $s \in \mathbb{R} \backslash \mathbb{Z}$ and $p \in [1, \infty]$, therefore the estimate (A.8) also holds for (fractional) Sobolev spaces if $t - s, s \notin \mathbb{Z}$.

Finally, we state a few inequalities from para-differential calculus (the full details of which can be found in [44]). Let $1 \leqslant p, p_1, p_2, q, q_1, q_2 \leqslant \infty$ and $\alpha > 0 > \beta$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then the following inequalities hold:

- If $\alpha + \beta = 0$, $1 = \frac{1}{q_1} + \frac{1}{q_2}$, $f \in B^{\alpha}_{p_1,q_1}(\mathbb{T}^3)$ and $g \in B^{\beta}_{p_2,q_2}(\mathbb{T}^3)$, then

$$\|fg\|_{B^{\beta}_{p,q_2}} \lesssim \|f\|_{B^{\alpha}_{p_1,q_1}} \|g\|_{B^{\beta}_{p_2,q_2}}. \tag{A.9}$$

- If $f \in B^{\alpha}_{p_1,q}(\mathbb{T}^3)$ and $g \in B^{\alpha}_{p_2,q}(\mathbb{T}^3)$, then

$$\|fg\|_{B^{\alpha}_{p,q}} \lesssim \|f\|_{B^{\alpha}_{p_1,q}} \|g\|_{B^{\alpha}_{p_2,q}}. \tag{A.10}$$

Appendix B. A commutator estimate for the fractional Laplacian

In this appendix, we will prove the following commutator estimate for a general vector field **u** with Sobolev regularity. This commutator estimate is used in the proof of the energy estimates that are needed to establish the hierarchy of weak solution time averages.

Lemma 4. *Let $1 < s_1$ and $1 \leqslant s_2 < \frac{5}{2} < s_3$, and let $0 < \theta < 1$ be such that $\theta s_2 + (1 - \theta)s_3 = \frac{5}{2}$. Moreover, let $\boldsymbol{u} \in H^{s_1}(\mathbb{T}^3) \cap H^{s_3}(\mathbb{T}^3)$ be mean-free, then the following commutator estimate holds*

$$\left\| A^{s_1/2}\left[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] - (\boldsymbol{u} \cdot \nabla)\left[A^{s_1/2}\boldsymbol{u}\right] \right\|_{L^2} \lesssim \|\boldsymbol{u}\|_{H^{s_2}}^{\theta}\|\boldsymbol{u}\|_{H^{s_3}}^{1-\theta}\|\boldsymbol{u}\|_{H^{s_1}}. \tag{B.1}$$

Commutator estimates of this type were first established in the inhomogeneous case in [57] and in the homogeneous case in [58]. Lemma 4 is based on the adaptation of a result in (and the method from) [59, theorem 1.2].

Proof of lemma 4. We first take the Fourier transform of $A^{s_1/2}\left[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] - (\boldsymbol{u} \cdot \nabla)\left[A^{s_1/2}\boldsymbol{u}\right]$, which is given by (for $k \in \mathbb{Z}^3$)

$$\mathcal{F}\left[A^{s_1/2}\left[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] - (\boldsymbol{u} \cdot \nabla)\left[A^{s_1/2}\boldsymbol{u}\right]\right](k) = \sum_{m \in \mathbb{Z}^3}\sum_{i=1}^{3}\left[(|k|^{s_1} - |k-m|^{s_1})\widehat{\boldsymbol{u}}_i(m)(k-m)_i\,\widehat{\boldsymbol{u}}(k-m)\right].$$

By using the Parseval identity, to prove the estimate it is sufficient to estimate this quantity in $l^2(\mathbb{Z}^3)$. In order to prove the result, we separately consider the cases $|m| < \frac{|k|}{2}$ and $|m| \geqslant \frac{|k|}{2}$. If $|m| < \frac{|k|}{2}$, the following inequality holds (which was established in [59])

$$\left||k|^{s_1} - |k-m|^{s_1}\right| \leqslant c|k-m|^{s_1-1}|m|. \tag{B.2}$$

Note that in [59] they prove this inequality for $k, m \in \mathbb{R}^3$ (as they consider the commutator estimate when the domain is the whole space), and therefore it also holds in our case. By using inequality (B.2) we find that

$$\sum_{m \in \mathbb{Z}^3, |m| < \frac{|k|}{2}}\sum_{i=1}^{3}\left[(|k|^{s_1} - |k-m|^{s_1})\widehat{\boldsymbol{u}}_i(m)(k-m)_i\,\widehat{\boldsymbol{u}}(k-m)\right]$$

$$\leqslant c\sum_{m \in \mathbb{Z}^3}\left[|k-m|^{s_1}|m||\widehat{\boldsymbol{u}}(m)||\widehat{\boldsymbol{u}}(k-m)|\right] = c\left(|\cdot||\widehat{\boldsymbol{u}}|\right) * \left(|\cdot|^{s_1}|\widehat{\boldsymbol{u}}|\right). \tag{B.3}$$

To handle the case $|m| \geqslant \frac{|k|}{2}$, we recall the following inequality from [59] (which holds if $|m| \geqslant \frac{|k|}{2}$)

$$\left||k|^{s_1} - |k-m|^{s_1}\right| \leqslant c|m|^{s_1}. \tag{B.4}$$

Using estimate (B.4) then gives

$$\sum_{m \in \mathbb{Z}^3, |m| \geqslant \frac{|k|}{2}}\sum_{i=1}^{3}\left[(|k|^{s_1} - |k-m|^{s_1})\widehat{\boldsymbol{u}}_i(m)(k-m)_i\,\widehat{\boldsymbol{u}}(k-m)\right]$$

$$\leqslant c\sum_{m \in \mathbb{Z}^3}\left[|k-m||m|^{s_1}|\widehat{\boldsymbol{u}}(m)||\widehat{\boldsymbol{u}}(k-m)|\right] = c\left(|\cdot||\widehat{\boldsymbol{u}}|\right) * \left(|\cdot|^{s_1}|\widehat{\boldsymbol{u}}|\right). \tag{B.5}$$

Then by using inequalities (B.3) and (B.5) as well as the Young convolution inequality, we find

$$\left\| A^{s_1/2} \left[(\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \right] - (\boldsymbol{u} \cdot \nabla) \left[A^{s_1/2} \boldsymbol{u} \right] \right\|_{L^2}^2 \leqslant c \sum_{k \in \mathbb{Z}^3} \left| \left(|\cdot| |\widehat{\boldsymbol{u}}| \right) * \left(|\cdot|^{s_1} |\widehat{\boldsymbol{u}}| \right) (k) \right|^2$$

$$\leqslant c \left(\sum_{k \in \mathbb{Z}^3} |\widehat{\nabla \boldsymbol{u}}| (k) \right)^2 \sum_{k \in \mathbb{Z}^3} |k|^{2s_1} |\widehat{\boldsymbol{u}}(k)|^2.$$

Then by following the steps of the proof of Agmon's inequality in [34, p 415–6] (see also for example [60]), one finds that

$$\sum_{k \in \mathbb{Z}^3} |\widehat{\nabla \boldsymbol{u}}| (k) \leqslant \|\boldsymbol{u}\|_{H^{s_2}}^{\theta} \|\boldsymbol{u}\|_{H^{s_3}}^{1-\theta}.$$

Combining these inequalities then allows one to conclude that

$$\left\| A^{s_1/2} \left[(\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \right] - (\boldsymbol{u} \cdot \nabla) \left[A^{s_1/2} \boldsymbol{u} \right] \right\|_{L^2}^2 \leqslant c \|\boldsymbol{u}\|_{H^{s_2}}^{2\theta} \|\boldsymbol{u}\|_{H^{s_3}}^{2-2\theta} \|\boldsymbol{u}\|_{H^{s_1}}^2.$$

$\square$

Remark 15. The restriction $s_1 > 1$ is not significant. The case $0 < s_1 < 1$ was considered in [58]. The case $s_1 = 1$ follows from a trivial adaption of the proof given above as we have

$$\left| |k| - |k - m| \right| \leqslant |m|,$$

by the reverse triangle inequality.

Remark 16. We note that in [61, remark 1.14] the following estimate was proven (for $s_1 > 0$ and $1 < p < \infty$)

$$\left\| A^{s_1/2} (fg) - f A^{s_1/2} g \right\|_{L^p} \lesssim \|\nabla f\|_{L^\infty} \|A^{(s_1-1)/2} g\|_{L^p} + \|g\|_{L^\infty} \|A^{s_1/2} f\|_{L^p} + \|f\|_{\dot{B}_{\infty,1}^1} \|g\|_{\dot{B}_{p,\infty}^{s_1-1}}. \tag{B.6}$$

However, in this paper we will only use the estimate from lemma 4.

ORCID iDs

Daniel W Boutros ⓘ https://orcid.org/0000-0003-1921-2102
John D Gibbon ⓘ https://orcid.org/0000-0002-6459-7112

References

[1] Lions J L 1969 *Quelques Méthodes de Résolution des Problèmes aux Limites Non-linéaires* vol 1 (Dunod)
[2] Lions J L 1959 Quelques résultats d'existence dans des équations aux dérivées partielles non linéaires *Bull. Soc. Math. France* **87** 245–73
[3] Tao T 2009 Global regularity for a logarithmically supercritical hyper-dissipative Navier–Stokes equation *Anal. PDE* **3** 361–6
[4] Luo T and Titi E S 2020 Non-uniqueness of weak solutions of hyperviscous Navier–Stokes equations – on the sharpness of the J.-L. Lions exponent *Calc. Var. PDE* **59** 1–15
[5] Frisch U, Kurien S, Pandit R, Pauls W, Ray S S, Wirth A and Zhu J-Z 2008 Hyperviscosity, Galerkin truncation and bottlenecks in turbulence *Phys. Rev. Lett.* **101** 144501
[6] Avrin J 2003 Singular initial data and uniform global bounds for the hyper-viscous Navier–Stokes equation with periodic boundary conditions *J. Differ. Equ.* **190** 330–51

[7] Katz N H and Pavlović N 2002 A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier–Stokes equation with hyper-dissipation *Geom. Funct. Anal.* **12** 355–79

[8] Zhang L 2004 On the modified Navier–Stokes equations in n-dimensional spaces *Bull. Instit. Math. Acad. Sin.* **32** 185–94

[9] Bardos C, Penel P, Frisch U and Sulem P L 1979 Modified dissipativity for a non-linear evolution equation arising in turbulence *Arch. Ration. Mech. Anal.* **71** 237–56

[10] Dong B-Q and Song J 2012 Global regularity and asymptotic behavior of modified Navier–Stokes equations with fractional dissipation *Discrete Contin. Dyn. Syst.* A **32** 157–79

[11] Colombo M, De Lellis C and Massaccesi A 2020 The generalized Caffarelli-Kohn-Nirenberg theorem for the hyperdissipative Navier–Stokes system *Commun. Pure Appl. Math.* **73** 609–63

[12] Kiselev A, Nazarov F and Shterenberg R 2008 Blow up and regularity for fractal Burgers equation *Dyn. PDE* **5** 211–40

[13] Mandelbrot B B and Van Ness J W 1968 Fractional Brownian motions, fractional noises and applications *SIAM Rev.* **10** 422–37

[14] Metzler R and Klafter J 2000 The random walk's guide to anomalous diffusion: a fractional dynamics approach *Phys. Rep.* **339** 1–77

[15] Metzler R and Klafter J 2004 The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics *J. Phys. A: Math. Gen.* **37** R161–208

[16] Vazquez J L 2017 The mathematical theories of diffusion. Nonlinear and fractional diffusion *Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions (Springer Lecture Notes in Mathematics (C.I.M.E. Subseries))* (Springer)

[17] Henry B I, Langlands T A M and Straka P 2009 *An Introduction to Fractional Diffusion* (World Scientific)

[18] Henry B I, Langlands T A M and Wearne S L 2006 Anomalous diffusion with linear reaction dynamics: from continuous time random walks to fractional reaction-diffusion equations *Phys. Rev.* E **74** 031116

[19] Angstmann C N, Henry B I and McGann A V 2017 Generalized fractional diffusion equations for sub-diffusion in arbitrarily growing domains *Phys. Rev.* E **96** 042153

[20] O'Shaughnessy B and Procaccia I 1985 Diffusion on fractals *Phys. Rev.* A **32** 3073–83

[21] Crampin E J, Gaffney E A and Maini P K 1999 Reaction and diffusion on growing domains: scenarios for robust pattern formation *Bull. Math. Biol.* **61** 1093–120

[22] Woolley T E, Baker R E, Gaffney E A and Maini P K 2011 Stochastic reaction and diffusion on growing domains: understanding the breakdown of robust pattern formation *Phys. Rev.* E **84** 046216

[23] Sokolov I M, Schmidt M G W and Sagues F 2006 Reaction-sub-diffusion equations *Phys. Rev.* E **73** 031102

[24] Saxton M J 2007 A biological interpretation of transient anomalous sub-diffusion. I Qualitative model *Biophys. J.* **92** 1178–91

[25] del-Castillo-Negrete D, Carreras B A and Lynch V E 2005 Non-diffusive transport in plasma turbulence: a fractional diffusion approach *Phys. Rev. Lett.* **94** 065003

[26] del-Castillo-Negrete D 2010 Non-diffusive, non-local transport in fluids and plasmas *Nonlinear Process. Geophys.* **17** 795–807

[27] De Lellis C and Szekelyhidi L 2009 The Euler equations as a differential inclusion *Ann. Math.* **170** 1417–36

[28] Constantin P, E W and Titi E S 1994 Onsager's conjecture on the energy conservation for solutions of Euler's equation *Commun. Math. Phys.* **165** 207–9

[29] Isett P 2018 A proof of Onsager's conjecture *Ann. Math.* **188** 1–93

[30] Buckmaster T and Vicol V 2019 Non-uniqueness of weak solutions to the Navier–Stokes equation *Ann. Math.* **189** 101–44

[31] Caffarelli L, Kohn R and Nirenberg L 1982 Partial regularity of suitable weak solutions of the Navier–Stokes equations *Commun. Pure Appl. Math.* **35** 771–831

[32] Doering C and Gibbon J D 1995 *Applied Analysis of the Navier–Stokes Equations* (CUP)

[33] Foias C, Manley O, Rosa R and Temam R 2001 *Navier–Stokes Equations and Turbulence* (CUP)

[34] Robinson J C, Rodrigo J L and Sadowski W 2016 *The Three-dimensional Navier–Stokes Equations: Classical Theory* (*Cambridge Studies in Advanced Mathematics*) (CUP)

[35] Bedrossian J and Vicol V 2022 *The Mathematical Analysis of the Incompressible Euler and Navier–Stokes Equations: An Introduction* vol 225 (American Mathematical Society)

[36] Foias C, Guillopé C and Temam R 1981 New a priori estimates for Navier–Stokes equations in dimension 3 *Commun. Part. Differ. Equ.* **6** 329–59

[37] Donzis D, Kerr R M, Gupta A, Gibbon J D, Pandit R and Vincenzi D 2013 Vorticity moments in four numerical simulations of the 3D Navier–Stokes equations *J. Fluid Mech.* **732** 316–31

[38] Gibbon J D, Donzis D, Kerr R M, Gupta A, Pandit R and Vincenzi D 2014 Regimes of nonlinear depletion and regularity in the 3D Navier–Stokes equations *Nonlinearity* **27** 1–19

[39] Frisch U 1995 *Turbulence: The Legacy of A. N. Kolmogorov* (CUP)

[40] Gibbon J D 2019 Weak and strong solutions of the 3D Navier–Stokes equations and their relation to a chessboard of convergent inverse length scales *J. Nonlinear Sci.* **29** 215–28

[41] Gibbon J D 2020 Intermittency, cascades and thin sets in three-dimensional Navier–Stokes turbulence *Europhys. Lett.* **131** 64001

[42] McCormick D S, Robinson J C and Rodrigo J L 2013 Generalised Gagliardo-Nirenberg inequalities using weak Lebesgue spaces and BMO *Milan J. Math.* **81** 265–89

[43] Brezis H and Mironescu P 2019 Where Sobolev interacts with Gagliardo-Nirenberg *J. Funct. Anal.* **277** 2839–64

[44] Bahouri H, Chemin J Y and Danchin R 2011 *Fourier Analysis and Nonlinear Partial Differential Equations* (Springer)

[45] Tao T 2006 *Nonlinear Dispersive Equations: Local and Global Analysis* (American Mathematical Society)

[46] Amann H 2000 Compact embeddings of vector valued Sobolev and Besov spaces *Glas. Mat.* **35** 161–77

[47] Wu J 2003 Generalized MHD equations *J. Differ. Equ.* **195** 284–312

[48] Colombo M, De Lellis C and De Rosa L 2018 Ill-posedness of Leray solutions for the hypo-dissipative Navier–Stokes Equations *Commun. Math. Phys.* **362** 659–88

[49] Prodi G 1959 Un teorema di unicità per le equazioni di Navier–Stokes *Ann. Mat. Pura Appl.* **48** 173–82

[50] Serrin J 1962 On the interior regularity of weak solutions of the Navier–Stokes equations *Arch. Ration. Mech. Anal.* **9** 187–91

[51] Beale J T, Kato T and Majda A J 1984 Remarks on the breakdown of smooth solutions for the 3*D* Euler equations *Commun. Math. Phys.* **94** 61–66

[52] De Rosa L 2019 Infinitely many Leray-Hopf solutions for the fractional Navier–Stokes equations *Commun. PDEs* **44** 335–65

[53] Kwon H and Ożański W S 2022 Local regularity of weak solutions of the hypodissipative Navier–Stokes equations *J. Funct. Anal.* **282** 1–77

[54] Tang L and Yu Y 2015 Partial regularity of suitable weak solutions to the fractional Navier–Stokes equations *Commun. Math. Phys.* **334** 1455–82

[55] Duchon J and Robert R 2000 Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations *Nonlinearity* **13** 249–55

[56] Bulut A, Huynh M K and Palasek S 2022 Epochs of regularity for wild Hölder-continuous solutions of the Hypodissipative Navier–Stokes System (arXiv:2201.05600)

[57] Kato T and Ponce G 1988 Commutator estimates and the Euler and Navier–Stokes equations *Commun. Pure Appl. Math.* **41** 891–907

[58] Kenig C E, Ponce G and Vega L 1993 Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle *Commun. Pure Appl. Math.* **46** 527–620

[59] Fefferman C L, McCormick D S, Robinson J C and Rodrigo J L 2014 Higher order commutator estimates and local existence for the non-resistive MHD equations and related models *J. Funct. Anal.* **267** 1035–56

[60] Liflyand E, Samko S and Trigub R 2012 The Wiener algebra of absolutely convergent Fourier integrals: an overview *Anal. Math. Phys.* **2** 1–68

[61] Bourgain J and Li D 2014 On an endpoint Kato-Ponce inequality *Differ. Integral Equ.* **27** 1037–72