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A generalized time-domain constitutive finite element approach for viscoelastic materials

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Abstract

Despite the existence of time domain finite element formulations for viscoelastic materials, there are still substantial ways to improve the analysis. To the authors' knowledge, the formulation of the problem is always done with respect to a single constitutive relation and so limits the implementer to a single scheme with which to model relaxation. Furthermore, all current constitutive relations involve the finding of fitting parameters for an analytical function, which is a sufficiently painful process to warrant the study of best fitting procedures to this day. In contrast, this effort is the first full derivation of the two dimensional problem from fundamental principles. It is also the first generalization of the problem, which frees users to select constitutive relations without re-derivation or re-expression of the problem. This approach is also the first approach to the problem that could lead to the elimination of constitutive relations for representing relaxation in viscoelastic materials. Following, the full derivation, several common constitutive relations are outlined with analysis of how they may best be implemented in the generalized form. Several

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expressions for viscoelastic terms are also provided given linear, quadratic, and exponential interpolation assumptions.

Keywords: viscoelasticity, constitutive relations, finite elements

1. Introduction

The stress in a linear viscoelastic material generally depends on its strain history. A unique property of this material behavior is that it allows for energy losses in the form of heat generation during excitation. For this reason, viscoelastic materials are commonly used for vibration damping. Furthermore, the class of materials that exhibit viscoelastic behavior is substantial and includes many common materials, including rubber, fibrous material, cork, epoxies, and concretes [1]. It is even speculated that viscoelasticity was the key to rich tones in the early design of wooden violins during the 17th century [1]. Today, the incorporation of viscoelastic behavior is key to correctly computing the dynamic behavior of many systems because they contain viscoelastic components.

The basis for a finite element approach to viscoelasticity should start with a description of stress or strain, where a viscoelastic material varies from a perfectly elastic one. Flugge [2] and Lee [3] give early presentations of the relaxation stress integral used today. After this first step, there is a lack of complete and published works that develop the viscoelastic time domain problem. In contrast, the literature on viscoelastic constitutive relations and applications is immense. Indeed, in the past two years there have been several review papers on constitutive relations, which are either general or specific to a research area such as biomechanics [4, 5]. Papers expanding these modeling frameworks are also very common to the current day [6, 7]. Further review also shows that the focus in the literature is on solving frequency-domain problems. However, review of the viscoelastic theory used in three common widely used finite element codes, SIERRA [8], ANSYS [9], and COMSOL [10], shows that the time-domain formulation is extremely relevant.

The most complete analysis of the time-domain finite element problem known to the authors is found in the unpublished technical reports of Katona in 1974 [11] and in 1978 [12], which start from the relaxation integral and formulate the problem using a Generalized Maxwell Model. This implies two realities. Firstly, there is a missing connection between the widely available literature of viscoelastic finite elements. Secondly, the real operation of modern finite element tools that work on the foundational problem and formulation has taken a back seat to more specific work on model applications.

Constitutive relations are perhaps the most quintessential tool in the analysis of viscoelastic behavior, with the capacity to incorporate material behavior change into various modeling approaches. Common constitutive relations include the generalized Maxwell model, Kelvin–Voight model, Golla–Hughes–McTavish model and the fractional calculus model [8, 12–14]. All of the constitutive relations are similar in that they fit an analytic function to experimentally measured data by adjusting parameters. The problem of fitting parameters to these models to match real-world materials is a substantial and laborious one, with effort taking place still on finding the best fit methodologies [15]. Furthermore, matching complex behavior curves can require hundreds of series terms [16]. The number of publications using these models in the current day is substantial. Table 1 shows the number of search results on Google Scholar between 1 January 2023 and 8 June 2023 for articles that reference the exact phrase shown, each a common constitutive relation. A close examination of recently published articles reveals many interested in the modeling of transient behavior in viscoelastic finite elements. Many

Table 1. Publications referencing viscoelastic constitutive relations from the beginning of 2023 as of 8 June 2023.

Constitutive relation	Number of results
'Generalized Maxwell Model'	333
'Kelvin-Voigt Model'	518
'GHM Model'	20
'Fractional Derivative Model'	234
Total	1105

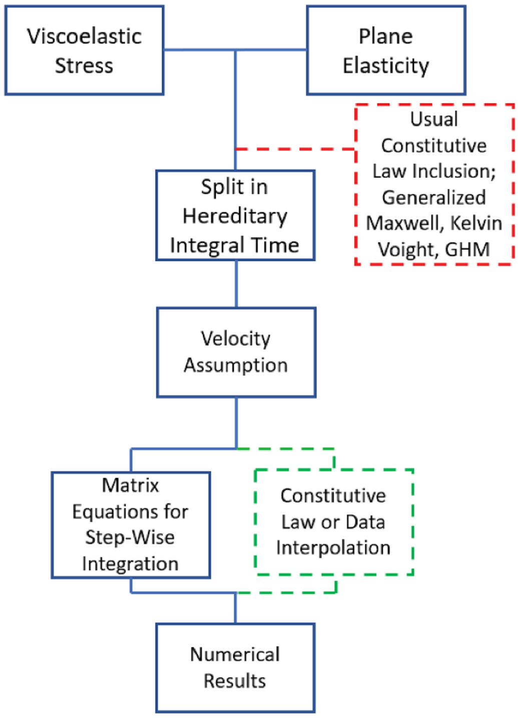

Figure 1. Outline of the generalized numerical approach.

employ these common constitutive relations and often use a step-wise integration scheme, such as the Newmark-Beta method [17–21].

All current finite element formulations choose a single constitutive relation and formulate the problem in the context of that model. However, this approach limits each formulation to applicability to both a single domain (usually frequency) and a single constitutive relation. The emphasis of this effort is to, for the first time, completely derive the time-domain approach to the problem that is numerically consistent with the Newmark-Beta method used in commercial software, as well as generalize it, so that multiple constitutive relations could be used interchangeably. Figure 1, shows the path of deriving a time-domain finite element approach from basic principles.

Where significantdevelopment in viscoelastic finite element formulation has taken place in recent years has been in the development of space-time Galerkin methods. These approaches are noteworthy since, they enable some beneficial behavior in solving these problems, such as parallel computing [22]. Others handle time domain viscoelasticity, but are formulated for wave propagation [23]. Still others incorporate discontinuous Galerkin schemes, which allow for a range of numerical benefits particularly in regards to convergence rates [24]. This work does not fault those efforts. Many of their results are of interest and have been implemented in some codes. However, a careful consideration of available documentation for commercial finite element packages [8, 9, 25], reveals that space-time formulations are not the primary formulation used by commercial packages when handling structural analysis of viscoelastic materials. The primary framework commonly used for time-domain structural studies of linear viscoelastic materials currently most resembles the work by Katona from 1974 [11] and 1978 [12] and the aim of this effort will be to improve upon that formulation because it is preferred by common finite element tools. This formulation is also unique in outlining how alternative representations of relaxation function, such as interpolation could be easily implemented in those codes.

Usually a constitutive relation is introduced very early in the formulation and terms of that relation are allowed to flow throughout the problem. By deriving the problem from first principles without incorporating a constitutive relation, analysts are still able to fit the same constitutive relations into the problem, but can now also experiment with other possibilities, such as direct data implementation discussed here. Implementation of relaxation data more directly may improve upon the state of the art in various ways. First, by easing analyst input, as fitting terms is no longer required. Second, by improving calculation speed, if the formulation can eliminate the cost of using exponential Prony terms. Third, by increasing accuracy in representing relaxation, if an interpolation scheme proves more accurate than an analytic fit across the entire function.

In line with figure 1, this formulation will start with the standard description of viscoelastic stress behavior and widely available work on 2D elasticity [26]. The approach will then move on to the splitting of the time integral, consistent with step-wise integration, and separate terms through integration using a velocity assumption consistent with current commercial codes [8]. Conveniently, the final result of this analysis is a traditional mass, stiffness, and damping system, with additional forcing terms. After that, the common constitutive relations are outlined, with discussion of how they can be incorporated into the generalized form of the problem and a series of interpolation schemes are considered that would be consistent with direct data implementation.

2. Mathematical background

2.1. Three-dimensional constitutive relations

The most general relationship between stress and stain for a linear viscoelastic solid is [3, 27],

$$\sigma(t) = \epsilon(t)\,Y(0) + \int_{0+}^{t} \epsilon(\tau)\,\frac{\mathrm{d}Y(t-\tau)}{\mathrm{d}(t-\tau)}\mathrm{d}\tau. \tag{1}$$

In equation (1), $\sigma(t)$ is stress, $\epsilon(t)$ is strain, and $Y(t)$ is the relaxation function. Following Cook [28], some additional analysis will be developed to further the problem [28]. The elements of stress and strain for a 3D case are

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{yz} & \tau_{zx} \end{bmatrix}^T \tag{2}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_x & \epsilon_y & \epsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}^T \tag{3}$$

where the statement of stress and strain are shown to only contain the standard set of shear and normal components. Throughout this work, bold font will be used to denote terms that represent matrices or vectors.

If thermal effects are ignored a standard relationship between stress and strain in the elastic case can be used as

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\epsilon}, \tag{4}$$

$$\mathbf{E} = G\mathbf{E}_G + B\mathbf{E}_B. \tag{5}$$

Furthermore, for forming an element stiffness matrix later in this effort it is worth splitting $\mathbf{E}$ into shear and bulk terms using

$$\mathbf{E}_G = \begin{bmatrix} 4/3 & -2/3 & -2/3 & 0 & 0 & 0 \\ -2/3 & 4/3 & -2/3 & 0 & 0 & 0 \\ -2/3 & -2/3 & -4/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$\text{and } \mathbf{E}_B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

where, as might be expected, G is shear modulus and B is bulk modulus.

Using equation (1), to generalize equations (4) and (5) to the viscoelastic case gives

$$\boldsymbol{\sigma}(t) = [G(0)\mathbf{E}_G + B(0)\mathbf{E}_B]\boldsymbol{\epsilon}(t) + \mathbf{E}_G\int_{0+}^{t} G'(t-\tau)\boldsymbol{\epsilon}(\tau)\,\mathrm{d}\tau + \mathbf{E}_B\int_{0+}^{t} B'(t-\tau)\boldsymbol{\epsilon}(\tau)\,\mathrm{d}\tau \tag{8}$$

where $G(t)$ is the shear relaxation function and $B(t)$ is the bulk relaxation function. Derivatives are

$$G'(t-\tau) = \frac{\mathrm{d}G(t-\tau)}{\mathrm{d}(t-\tau)} \text{ and } B'(t-\tau) = \frac{\mathrm{d}B(t-\tau)}{\mathrm{d}(t-\tau)} \tag{9}$$

2.2. Equation of motion for 2D elasticity

While it was worthwhile to present the more general three dimensional case, the focus of this effort will now shift to a two dimensional problem with the understanding that it could also be formulated similarly for one or three dimensions. The problem of two-dimensional (2D) elasticity is well understood and independent of the current development in this paper concerning viscoelasticity. However, for formulation of a generalized 2D problem utilizing viscoelasticity, it is best to include this analysis because it provides the definition of many of the relevant terms in the work that follows, including the mass matrix, $\mathbf{M}$, and the forcing terms, $\mathbf{f}$ and $\mathbf{Q}$. The following is therefore reiterated here from Reddy, 1994 [26] and stated in a way relevant to the problem of viscoelasticity for a layer of material with constant thickness.

The principle of virtual displacement in matrix form for a 2D analysis is

$$0 = h\int_{\Omega}\left\{\begin{array}{c}\delta\epsilon_x\\ \delta\epsilon_y\\ \delta\gamma_{xy}\end{array}\right\}^T\left\{\begin{array}{c}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{array}\right\} + \rho\left\{\begin{array}{c}\delta u\\ \delta_v\end{array}\right\}^T\left\{\begin{array}{c}\ddot{u}\\ \ddot{v}\end{array}\right\}\mathrm{d}x\,\mathrm{d}y$$

$$-h\int_{\Omega}\left\{\begin{array}{c}\delta_u\\ \delta_v\end{array}\right\}^T\left\{\begin{array}{c}f_x\\ f_y\end{array}\right\}\mathrm{d}x\,\mathrm{d}y - h\oint_{\Gamma}\left\{\begin{array}{c}\delta_u\\ \delta_v\end{array}\right\}^T\left\{\begin{array}{c}t_x\\ t_y\end{array}\right\}\mathrm{d}s \tag{10}$$

where h is the thickness of the 2D layer in the z direction, ρ is traditional material density, δ is the variational operator, σ_i are stress terms excluding z, and τ_{xy} is the only shear stress term. The dependent variables are displacements u and v in the directions of x and y, respectively.

The formulation will utilize typical finite element approximations of displacement over the domain as

$$u(x,y,t) = \sum_{j=1}^{n} u_j(t)\,\psi_j(x,y) \tag{11}$$

$$v(x,y,t) = \sum_{j=1}^{n} v_j(t)\,\psi_j(x,y) \tag{12}$$

where u_j and v_j are nodal displacements and the $\psi_j(x,y)$ are chosen basis functions. In condensed matrix form, equations (11) and (12) can be written as

$$\mathbf{d} = \left\{\begin{array}{c}u\\ v\end{array}\right\} = \mathbf{\Psi}\mathbf{\Delta} \tag{13}$$

giving a consolidated nodal displacement matrix $\mathbf{\Delta}$. The matrix forms of $\mathbf{\Psi}$ and $\mathbf{\Delta}$ are

$$\mathbf{\Psi} = \left[\begin{array}{cccccc}\psi_1(x,y) & 0 & \cdots & \psi_n(x,y) & 0 & \\ 0 & \psi_1(x,y) & 0 & \cdots & 0 & \psi_n(x,y)\end{array}\right] \tag{14}$$

$$\mathbf{\Delta} = \{\{\begin{array}{ccccccc}u_1(t) & v_1(t) & u_2(t) & v_2(t) & \cdots & u_n(t) & v_n(t)\end{array}\}\}^T. \tag{15}$$

The vector $\mathbf{d}$ holds nodal displacments u and v. It then follows that differentiation with respect to time will affect only the displacement vector $\mathbf{\Delta}$. Therefore, the second derivative of $\mathbf{d}$, the vector of u and v, can be given as

$$\ddot{\mathbf{d}} = \mathbf{\Psi}(x,y)\,\ddot{\mathbf{\Delta}}(t). \tag{16}$$

Using the operators that have just been defined in equations (11)–(16), two representations of strain are possible,

$$\boldsymbol{\epsilon} = \mathbf{T}\mathbf{d} \tag{17}$$

$$\boldsymbol{\epsilon} = \mathbf{B}\mathbf{\Delta}. \tag{18}$$

Equation (17) relates strain in a 2D structure to the matrix form of displacement $\boldsymbol{d}$, by introducing the operator $\mathbf{T}$ and $\mathbf{B}$, which are given as

$$\mathbf{T} = \left[\begin{array}{cc}\partial/\partial x & 0\\ 0 & \partial/\partial y\\ \partial/\partial y & \partial/\partial x\end{array}\right] \tag{19}$$

$$\mathbf{B} = \mathbf{T}\mathbf{\Psi}. \tag{20}$$

Using these terms and definitions, the vectors in equation (10) can be expressed as follows

$$\left\{ \begin{array}{c} \delta u \\ \delta v \end{array} \right\} = \boldsymbol{\Psi}\delta\boldsymbol{\Delta} \tag{21}$$

$$\{\delta\epsilon\} = \mathbf{B}\delta\boldsymbol{\Delta} \tag{22}$$

which in turn can be substituted into equation (10), to yield

$$\begin{aligned} 0 = h\int_\Omega (\delta\boldsymbol{\Delta})^T \left(\mathbf{B}^T\boldsymbol{\sigma} + \rho\boldsymbol{\Psi}^T\boldsymbol{\Psi}\ddot{\boldsymbol{\Delta}} + \rho\boldsymbol{\Psi}^T\boldsymbol{\Psi}\ddot{\boldsymbol{\Delta}}\right) \mathrm{d}x\,\mathrm{d}y \\ - h\int_\Omega (\delta\boldsymbol{\Delta})^T\boldsymbol{\Psi}^T \left\{ \begin{array}{c} f_x \\ f_y \end{array} \right\} \mathrm{d}x\,\mathrm{d}y - h\oint_\Gamma (\delta\boldsymbol{\Delta})^T\boldsymbol{\Psi}^T \left\{ \begin{array}{c} t_x \\ t_y \end{array} \right\} \mathrm{d}s \end{aligned} \tag{23}$$

The ultimate goal of this exercise is to work towards an $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ system consistent with step-wise integration. To this end, operators can be defined as follows

$$\mathbf{A} = h\int_\Omega \mathbf{B}^T\boldsymbol{\sigma}\,\mathrm{d}x\,\mathrm{d}y \tag{24}$$

$$\mathbf{M} = h\rho\int_\Omega \boldsymbol{\Psi}^T\boldsymbol{\Psi}\,\mathrm{d}x\,\mathrm{d}y \tag{25}$$

$$\mathbf{f} = h\int_\Omega \boldsymbol{\Psi}^T \left\{ \begin{array}{c} f_x \\ f_y \end{array} \right\} \mathrm{d}x\,\mathrm{d}y \tag{26}$$

$$\mathbf{Q} = h\oint_\Gamma \boldsymbol{\Psi}^T \left\{ \begin{array}{c} t_x \\ t_y \end{array} \right\} \mathrm{d}s. \tag{27}$$

In equations (24)–(27), $\mathbf{M}$ is the mass matrix, $\mathbf{f}$ contains force contributions acting on nodes in the interior of the structure, and $\mathbf{Q}$ contains traction forces acting over the surface. Using these operators, equation (23) can be reduced to the expression in equation (28),

$$\mathbf{A} + \mathbf{M}\ddot{\boldsymbol{\Delta}} = \mathbf{f} + \mathbf{Q}, \tag{28}$$

In the elastic case, the vector $\mathbf{A}$ would ultimately become $\mathbf{K}\boldsymbol{\Delta}$, by substitution with an elastic definition of $\boldsymbol{\sigma}$. In the case of viscoelasticity, this vector is more complicated and a detailed analysis of the integral in $\mathbf{A}$ will follow.

3. Mathematical development

Section 2 established the equations of motion for 2D elasticity and the three-dimensional constitutive relation, which has already been approached in previous work. Section 3 presents a new analysis that yields matrix equations for 2D viscoelasticity.

3.1. 2D bulk and shear modulus matrices

To implement a formulation in 2D, the matrices $\mathbf{E}_G$ and $\mathbf{E}_B$ for shear and bulk modulus respectively will need to be reduced for 2D plane strain. Removing the rows and columns of the three dimensional matrices corresponding to stresses in the z dimension or shear stresses with z components yields the 3×3 2D matrices

$$\mathbf{E}_G = \begin{bmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 4/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{E}_B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{29}$$

3.2. Analysis of the integral

As can be seen by the form of equation (8), when the viscoelastic stress equation developed in section 2 is substituted into equation (24), the resulting term will contain integration with respect to both space and time. The process of treating this integral is considerably more challenging than the elastic case. At this point in the derivation, all others previously published would already have introduced a constitutive relation. In this case, the form of relaxation will be left general with the expectation of producing a result that is not specific to one constitutive relation.

For ease of consideration and expression, the integral $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_G + \mathbf{A}_G \tag{30}$$

where $\mathbf{A}_0$ contains terms considering the relaxation modulus G only at time zero, $\mathbf{A}_G$ contains those elements of $\mathbf{A}$ related to shear modulus which varies in time and $\mathbf{A}_B$ similarly contains the elements of $\mathbf{A}$ related to bulk modulus varying in time. These terms are given as

$$\mathbf{A}_0 = h\int_{\Omega} \mathbf{B}^T \left[G(0)\,\mathbf{E}_G + B(0)\,\mathbf{E}_B\right] \mathbf{B}\boldsymbol{\Delta}(t)\,\mathrm{d}x\,\mathrm{d}y \tag{31}$$

$$\mathbf{A}_G = h\int_{\Omega} \mathbf{B}^T\mathbf{E}_G\mathbf{B}\left(\int_{0+}^{t} G'(t-\tau)\,\boldsymbol{\Delta}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}x\,\mathrm{d}y \tag{32}$$

$$\mathbf{A}_B = h\int_{\Omega} \mathbf{B}^T\mathbf{E}_B\mathbf{B}\left(\int_{0+}^{t} B'(t-\tau)\,\boldsymbol{\Delta}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}x\,\mathrm{d}y. \tag{33}$$

These definitions utilize only those terms defined earlier in this work. However, for ease of operation $\mathbf{J}_G$ and $\mathbf{J}_B$ are introduced as

$$\mathbf{J}_G = \int_{\Omega} \mathbf{B}^T\mathbf{E}_G\mathbf{B}\,\mathrm{d}x\,\mathrm{d}y \tag{34}$$

$$\mathbf{J}_B = \int_{\Omega} \mathbf{B}^T\mathbf{E}_B\mathbf{B}\,\mathrm{d}x\,\mathrm{d}y. \tag{35}$$

In equations (34) and (35), $\mathbf{E}_G$ is defined in this work for a 2D case and $\mathbf{B}$ is entirely dependent on element geometry and material parameters, so calculation of $\mathbf{J}_g$ and $\mathbf{J}_B$ would take place at the point of implementation with an understanding of the configuration of the problem. Using these new definitions, equations (31)–(33) are simplified as

$$\mathbf{A}_0 = h\left[G(0)\,\mathbf{J}_G\boldsymbol{\Delta}(t) + B(0)\,\mathbf{J}_B\boldsymbol{\Delta}(t)\right] \tag{36}$$

$$\mathbf{A}_G = h\mathbf{J}_G\int_{0+}^{t} G'(t-\tau)\,\boldsymbol{\Delta}(\tau)\,\mathrm{d}\tau \tag{37}$$

$$\mathbf{A}_B = h\mathbf{J}_B\int_{0+}^{t} B'(t-\tau)\,\boldsymbol{\Delta}(\tau)\,\mathrm{d}\tau. \tag{38}$$

While G and B will remain arbitrary, it is still necessary to evaluate the integral and to do so a form of displacement will need to be introduced that is consistent with a step-wise integration scheme. Other works on this subject that focused on specific constitutive relations have similarly utilized a form of linear velocity [8, 12]. This form is

$$\dot{\boldsymbol{\Delta}}(t) = \dot{\boldsymbol{\Delta}}_i + \left(\frac{t-t_i}{\Delta t}\right)\left(\dot{\boldsymbol{\Delta}}_{i+1} - \dot{\boldsymbol{\Delta}}_i\right) \text{ for } t_i \leqslant t \leqslant t_{i+1}. \tag{39}$$

Equation (39) begins the splitting of time for compliance with a step-wise integration scheme. In this scheme operators with an i index are terms that have been evaluated for the previous time step, a known value, and operators with an $(i+1)$ index represent terms for the time step currently being evaluated, and may be known or unknown. This equation has introduced the term $\dot{\boldsymbol{\Delta}}_i$, which is the velocity of the previous time step, $\dot{\boldsymbol{\Delta}}_{i+1}$, which is the velocity for the current time step, Δt, the change in time per step, and t_i, the time at the previous step. Time, t, can be anytime between the time of the previous step and the time of the current step. It will become increasingly important to understand which operators represent current or future time and which are still variable.

Equations (37) and (38) contain displacement rather than velocity, so it also necessary to integrate equation (39) to get a displacement that can be used. This displacement is

$$\boldsymbol{\Delta}(t) = \dot{\boldsymbol{\Delta}}_i (t - t_i) + \left(\frac{1}{2}\right)\left(\frac{(t-t_i)^2}{\Delta t}\right)\left(\dot{\boldsymbol{\Delta}}_{i+1} - \dot{\boldsymbol{\Delta}}_i\right) + \boldsymbol{\Delta}_i \text{ for } t_i \leqslant t \leqslant t_{i+1} \tag{40}$$

which similarly is accurate only for anytime, t, between steps.

Substituting equation (40) into equations (36)–(38) and evaluating at time $t = t_{i+1}$, such that the solution is the solution for the current time step yields

$$\mathbf{A}_0 = \mathbf{K}\boldsymbol{\Delta}_{i+1} \tag{41}$$

$$\mathbf{A}_G = \left(-\mathbf{g}_i + \mathbf{G}_i\dot{\boldsymbol{\Delta}}_{i+1}\right) \tag{42}$$

$$\mathbf{A}_B = \left(-\mathbf{b}_i + \mathbf{B}_i\dot{\boldsymbol{\Delta}}_{i+1}\right) \tag{43}$$

where a series of mathematical definitions have taken place such that the resulting equations can be displayed in this format. These definitions are

$$\mathbf{K} = h\left[G(0)\,\mathbf{J}_G + B(0)\,\mathbf{J}_B\right] \tag{44}$$

$$\mathbf{G}_i = h\mathbf{J}_G \int_{t_i}^{t_{i+1}} G'(t_{i+1} - \tau)\left(\frac{1}{2}\right)\left(\frac{(\tau - t_i)^2}{\Delta t}\right) \mathrm{d}\tau \tag{45}$$

$$\mathbf{B}_i = h\mathbf{J}_B \int_{t_i}^{t_{i+1}} B'(t_{i+1} - \tau)\left(\frac{1}{2}\right)\left(\frac{(\tau - t_i)^2}{\Delta t}\right) \mathrm{d}\tau \tag{46}$$

$$\begin{aligned}\mathbf{g}_i = &-h\mathbf{J}_G \int_{0+}^{t_i} G'(t_{i+1} - \tau)\,\boldsymbol{\Delta}(\tau)\,\mathrm{d}\tau \\ &- h\mathbf{J}_G \int_{t_i}^{t_{i+1}} G'(t_{i+1} - \tau)\left[\dot{\boldsymbol{\Delta}}_i(\tau - t_i) + \left(\frac{1}{2}\right)\left(\frac{(\tau - t_i)^2}{\Delta t}\right)\left(-\dot{\boldsymbol{\Delta}}_i\right) + \boldsymbol{\Delta}_i\right]\mathrm{d}\tau\end{aligned} \tag{47}$$

$$\begin{aligned}\mathbf{b}_i = &-h\mathbf{J}_B \int_{0+}^{t_i} B'(t_{i+1} - \tau)\,\boldsymbol{\Delta}(\tau)\,\mathrm{d}\tau \\ &- h\mathbf{J}_B \int_{t_i}^{t_{i+1}} B'(t_{i+1} - \tau)\left[\dot{\boldsymbol{\Delta}}_i(\tau - t_i) + \left(\frac{(\tau - t_i)^2}{2\Delta t}\right)\left(-\dot{\boldsymbol{\Delta}}_i\right) + \boldsymbol{\Delta}_i\right]\mathrm{d}\tau.\end{aligned} \tag{48}$$

In equation (44), $\mathbf{K}$ is a stiffness matrix that is consistent with a traditional use in finite elements. However, in equations (45)–(48) some terms unique to viscoelasticity have been created. The terms $\mathbf{G}_i$ and $\mathbf{B}_i$ have integrals only from t_i to t_{i+1}. When these bounds are applied over a convolution of $t_{i+1} - \tau$, the result is an expression with a fixed value, so it will remain on the left hand side of the resulting system of equations along with $\mathbf{K}$. The terms $\mathbf{g}_i$ and $\mathbf{b}_i$ however have time dependence only up to the previous time step, but a value that changes with each additional time step. These terms are then understood to be calculable contributions from previous time steps and are moved to the right side of the system of equations as forcing terms.

Formulating these equations as such and substituting back into equation (28), an $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ system consistent with time step integration can be written as

$$\mathbf{M}\ddot{\Delta}_{i+1} + \mathbf{C}\dot{\boldsymbol{\Delta}}_{i+1} + \mathbf{K}\boldsymbol{\Delta}_{i+1} = \mathbf{f}_{i+1} + \mathbf{Q}_{i+1} + \mathbf{g}_i + \mathbf{b}_i \tag{49}$$

where $\mathbf{C}$ incorporates the elements $\mathbf{G}_i$ and $\mathbf{B}_i$ and is

$$\mathbf{C} = \mathbf{G}_i + \mathbf{B}_i. \tag{50}$$

4. Constitutive relations

Constitutive relations are traditionally used to provide an analytical function for the relaxation moduli G and B. Section 4 lays out a range of constitutive relations used widely today and provides a form that could be substituted into this generalized framework. However, not all the relations provided here can be utilized as easily as others. Some of these constitutive relations require other changes in the model, such as the introduction of additional internal degrees of freedom. It remains a question of implementation to identify how best to allow for the full range of possible interpretations of the constitutive material data that the authors intend to enable. Furthermore, in this generalized framework it is not clear that a constitutive relation is needed at all. Instead, a simple interpolation of measured data can be used and some approaches to interpolating data in place of a constitutive relation are outlined in section 5. When considering these constitutive relations, as well as interpolation approaches, it is worth remembering some of the well established physical rules that govern relaxation moduli. One requirement is that the relaxation function be bounded, with definite values at infinity, as well as derivatives that approach zero at infinity [29]. Another requirement is that the relaxation modulus in time decrease monotonically as has been laid out by previous authors [29].

4.1. Generalized Maxwell model

The Generalized Maxwell model is likely the most common form of the constitutive relationship for a viscoelastic material. In this model, the material is viewed as a primary spring element with a fixed stiffness in parallel with a series combination of spring and dashpot. In the time-domain, viscoelastic modulus decreases monotonically under strain. These spring-dashpot components are represented in equation form by the constants S_{G_1} and C_{G_1} for shear, which are fit to experimental data. The relaxation functions are

$$G(t-\tau) = \sum_{n=1}^{N} C_{G_i} e^{\frac{\tau - t}{S_{G_i}}}. \tag{51}$$

Equation (51) is taken from [30] and provides a common statement of the modulus $G(t)$ in the relevant form for integration.

4.2. Kelvin–Voigt model

The Kelvin–Voigt model is an approach to constitutive behavior in viscoelastic materials that is similar to the generalized Maxwell model. However, rather than viewing the material as a series of spring-dashpot series components, the Kelvin–Voigt model views it as a parallel system of spring and dashpot. However, traditionally the model does not incorporate a series of components to model a material and rather allows the dashpot constant to vary over frequency [31].

Identifying a form of relaxation modulus from the Kelvin–Voigt model that is directly applicable to the generalized formulation here is not trivial. While the analogous Kelvin–Voigt model is well documented, a form equivalent to $G(t)$ is rarely stated. Close examination of [31] in reference to strain reveals a form of relaxation modulus that should be implementable in place of $G(\tau - t)$ and $K(\tau - t)$,

$$G(t-\tau) = \frac{1}{G_\infty}\left[1 - \exp\left(\frac{-G_\infty}{\eta}(\tau - t)\right)\right]. \tag{52}$$

In equation (52), G_∞ is the stiffness of the spring element, while η is the viscous term corresponding to the dashpot. An alternative form is often used which groups G_∞ and η into a term τ, which has not been used in this case, to avoid confusion with the variable τ used in this formulation to represent reduced time.

4.3. Golla–Hughes–McTavish model

The Golla–Hughes–McTavish model allows for a simple expression of the familiar $G(t)$ [32]. However, the model requires some additional steps for integration with the proposed generalized approach. While use of this form of $G(t)$ could give expressions for $\mathbf{g}_i$, $\mathbf{b}_i$, $\mathbf{G}_i$ and $\mathbf{B}_i$, the model works by incorporating internal degrees of freedom, which may also require further manipulation of stiffness and damping matrices [13],

$$G(t) = G_\infty\left[1 + \sum_{k=1}^{N}\alpha_k \frac{b_{2k}e^{-b_{1k}t} - b_{1k}e^{-b_{2k}t}}{b_{2k} - b_{1k}}\right]. \tag{53}$$

In equation (53), G_∞ and α_k are found by curve-fitting to experimental data. The operators b_{1k} and b_{2k} are also based on numerical fitting, but in the frequency domain.

4.4. Fractional derivative model

Another alternative constitutive relation is the Fractional derivative model. In form, the model makes for a reasonable mathematical choice since the constitutive relationship uses the same form of stress as presented in equation (1), with an equation for $G(t)$ that looks reasonably like similar models [14]. However, the FDM approach has additional complexities and will likely lead to stiffness and damping matrix contributions inconsistent with the generalized approach [13]. However, it may be possible to unify the approach in future works. The relaxation modulus $G(t)$ for the fractional derivative model can be expressed as

$$G(t) = \frac{G_1}{\Gamma(1-\alpha)\,t^\alpha}. \tag{54}$$

As with previous approaches, the operators α and G_1 are constants related to the experimental behavior of the material and Γ is the gamma function.

5. Analytic expressions for the relaxation integral

In this section, various interpolation assumptions are considered. This is done to provide examples of how $\mathbf{g}_i, \mathbf{b}_i, \mathbf{B}_i$, and $\mathbf{G}_i$ would be developed for interpolation approaches to the relaxation function. However, the schemes selected may be reasonable for various problems and so should be considered usable statements in this generalized approach.

5.1. Linear assumption

The vector $\mathbf{g}_i$ and matrix $\mathbf{G}_i$ are derived as an example of the linear interpretation of the relaxation $G(t)$ and $B(t)$. The distinction between final bulk and shear terms is trivial with the only change being the use of the bulk operator J_B, instead of J_G. One convenient feature of the linear interpolation is that so long as the measured data being interpolated does not violate the rule that relaxation modulus be monotonically decreasing, this interpolation will also not violate it. First, a linear statement of shear relaxation modulus would be

$$G(t_{i+1}-\tau) = a(t_{i+1}-\tau)+b \tag{55}$$

$$G'(t_{i+1}-\tau) = a \tag{56}$$

where a and b are constants which may not be explicitly expressed in the case of interpreting data directly. Substituting into equation (47) yields

$$\mathbf{g}_i = -hJ_G\int_0^{t_i} a\Delta(\tau)\,\mathrm{d}\tau - hJ_G\int_{t_i}^{t_{i+1}} a\left(\dot{\mathbf{\Delta}}_i(\tau-t_i)+\frac{1}{2}\frac{(\tau-t_i)^2}{\Delta t}\left(-\dot{\mathbf{\Delta}}_i\right)+\mathbf{\Delta}_i\right)\mathrm{d}\tau \tag{57}$$

$$\mathbf{G}_i = hJ_G\int_{t_i}^{t_{i+1}} a\frac{1}{2}\left(\frac{(\tau-t_i)^2}{\Delta t}\right)\mathrm{d}\tau. \tag{58}$$

Multiplying out terms and integrating from 0 to t_i with respect to τ further yields

$$\mathbf{g}_i = hJ_G\int_0^{t_i} a\Delta(\tau)\,\mathrm{d}\tau - \frac{ahJ_G(t_i-t_{i+1})}{2}\left(-t_i\dot{\mathbf{\Delta}}_i+t_{i+1}\dot{\mathbf{\Delta}}_i+2\mathbf{\Delta}_i\right) - \frac{ahJ_G\dot{\mathbf{\Delta}}_i(t_i-t_{i+1})^3}{6\Delta t}. \tag{59}$$

$$\mathbf{G}_i = -ahJ_G\frac{(t_{i+1}-t_i)^3}{6\Delta t} \tag{60}$$

Similar integration would yield expressions for $\mathbf{b}_i$, and $\mathbf{B}_i$. One integral remains in $\mathbf{g}_i$ and would be best evaluated numerically via a trapezoidal rule over each timestep.

5.2. Quadratic assumption

A quadratic fit is also a possible approach, that may prove more accurate in some cases. However, for quadratic interpolation, even if measured data is monotonically decreasing, the interpolation could violate the rule and produce instances in which the modulus increases instead of decreasing. Quadratic approximations of $G(t)$ and its derivative are

$$G(t_{i+1}-\tau) = a(t_{i+1}-\tau)^2+b(t_{i+1}-\tau)+c \tag{61}$$

$$G'\left(t_{i+1}-\tau\right)=2a\left(t_{i+1}-\tau\right)+b. \tag{62}$$

As with the linear case, these forms can be substituted into the expressions for $\mathbf{g}_i$ and $\mathbf{G}_i$, which yields

$$\begin{aligned}\mathbf{g}_i=&-hJ_G\int_0^{t_i}\left(2a\left(t_{i+1}-\tau\right)+b\right)\Delta\left(\tau\right)\mathrm{d}\tau\\&-hJ_G\int_{t_i}^{t_{i+1}}\left(2a\left(t_{i+1}-\tau\right)+b\right)\left(\dot{\mathbf{\Delta}}_i\left(\tau-t_i\right)+\frac{1}{2}\frac{\left(\tau-t_i\right)^2}{\Delta t}\left(-\dot{\mathbf{\Delta}}_i\right)+\mathbf{\Delta}_i\right)\mathrm{d}\tau\end{aligned} \tag{63}$$

$$\mathbf{G}_i=hJ_G\int_{t_i}^{t_{i+1}}\left(2a\left(t_{i+1}-\tau\right)+b\right)\frac{1}{2}\left(\frac{\left(\tau-t_i\right)^2}{\Delta t}\right)\mathrm{d}\tau \tag{64}$$

Evaluating these expressions symbolically yields

$$\begin{aligned}\mathbf{g}_i=&-hJ_G\int_0^{t_i}(2a(t_{i+1}-\tau)+b)\Delta(\tau)\mathrm{d}\tau+\left(\frac{hJ_G}{12}(t_i-t_{i+1})(12b\mathbf{\Delta}_i-6\dot{\mathbf{\Delta}}_i bt_{i+1}\right.\\&+6\dot{\mathbf{\Delta}}_i bt_i-12a\mathbf{\Delta}_i t_i+12a\mathbf{\Delta}_i t_{i+1}+4\dot{\mathbf{\Delta}}_i at_i^2+4\dot{\mathbf{\Delta}}_i at_{i+1}^2\\&\left.-8\dot{\mathbf{\Delta}}_i at_i t_{i+1})\right)-\frac{\dot{\mathbf{\Delta}}_i J_G h(t_i-t_{i+1})^3(2b-at_i+at_{i+1})}{12\Delta t}\end{aligned} \tag{65}$$

$$\mathbf{G}_i=-hJ_G\frac{\left(-t_{i+1}+t_i\right)^3\left(-a\left(t_i-t_{i+1}\right)+2b\right)}{12\Delta t} \tag{66}$$

where it is worth noting that interpolation would incorporate data from more than one pair of points in the measured data.

5.3. Exponential assumption

For the case of an exponential fit, a single Prony series term will be used. If a series of terms were to be used, this assumption would then be equivalent to the application of the generalized Maxwell model to this approach. Exponential approximation and it is derivative are

$$G\left(t_{i+1}-\tau\right)=C_{G_1}e^{\frac{\tau-t_{i+1}}{S_{G_1}}} \tag{67}$$

$$G'\left(t_{i+1}-\tau\right)=-\frac{C_{G_1}}{S_{G_1}}e^{\frac{\tau-t_{i+1}}{S_{G_1}}} \tag{68}$$

where the coefficients in G are the same as those understood in section 4.1 for the generalized Maxwell model. Substituting these expressions into equations (45) and (47) yields

$$\begin{aligned}\mathbf{g}_i=hJ_G\int_0^{t_i}\frac{C_{G_1}}{S_{G_1}}e^{\frac{\tau-t_{i+1}}{S_{G_1}}}\Delta\left(\tau\right)\mathrm{d}\tau+hJ_G\int_{t_i}^{t_{i+1}}\frac{C_{G_1}}{S_{G_1}}e^{\frac{\tau-t_{i+1}}{S_{G_1}}}&\left(\dot{\mathbf{\Delta}}_i\left(\tau-t_i\right)\right.\\+\frac{1}{2}\frac{\left(\tau-t_i\right)^2}{\Delta t}\left(-\dot{\mathbf{\Delta}}_i\right)+\mathbf{\Delta}_i&\left.\right)\mathrm{d}\tau\end{aligned} \tag{69}$$

$$\mathbf{G}_i = -hJ_G \int_{t_i}^{t_{i+1}} \frac{C_{G_1}}{S_{G_1}} e^{\frac{\tau - t_{i+1}}{S_{G_1}}} \frac{1}{2} \left(\frac{(\tau - t_i)^2}{\Delta t} \right) \mathrm{d}\tau \tag{70}$$

Finally, evaluation of these integrals once again leads to expressions for $\mathbf{g}_i$ and $\mathbf{G}_i$,

$$\begin{aligned}
\mathbf{g}_i = hJ_G \int_0^{t_i} \frac{C_{G_1}}{S_{G_1}} e^{\frac{\tau - t_{i+1}}{S_{G_1}}} \Delta(\tau)\mathrm{d}\tau + \frac{C_{G_1} J_G h}{2} \Big(& 2\mathbf{\Delta}_i - 2\dot{\mathbf{\Delta}}_i t_i + 2\dot{\mathbf{\Delta}}_i t_{i+1} - \mathbf{\Delta}_i e^{\frac{t_i - t_{i+1}}{S_{G_1}}} 2 \\
& - 2\dot{\mathbf{\Delta}}_i S_{G_1} + 2\dot{\mathbf{\Delta}}_i S_{G_1} e^{\frac{t_i - t_{i+1}}{S_{G_1}}} \Big) - \frac{C_{G_1} \dot{\mathbf{\Delta}}_i J_G h}{2\Delta t} \\
& \times \left(2S_{G_1} t_i - 2S_{G_1} t_{i+1} - 2S_{G_1}^2 e^{\frac{t_i - t_{i+1}}{S_{G_1}}} - 2t_i t_{i+1} + 2S_{G_1}^2 + t_i^2 + t_{i+1}^2 \right)
\end{aligned} \tag{71}$$

$$\mathbf{G}_i = \frac{C_{G_1} hJ_G S_{G_1}^2}{\Delta t} e^{\frac{t_i - t_{i+1}}{S_{G_1}}} - \frac{C_{G_1} J_G h}{2\Delta t} \left(2S_{G_1}^2 + 2S_{G_i} t_i - 2S_{G_i} t_{i+1} + t_i^2 - 2t_i t_{i+1} + t_{i+1}^2 \right). \tag{72}$$

6. Conclusion

To establish notation, we summarized the equations of motion for 2D elasticity and development of three-dimensional viscoelastic stress equations. Through time splitting in the relaxation integral and incorporation of a velocity assumption, a traditional mass, stiffness and damping matrix were formed with the benefit of having no time dependence. The final result is a finite element formulation for viscoelastic materials that is consistent with step-wise integration and independent of constitutive relation. This work presents analytical expressions for common constitutive laws which may be incorporated equivalently. Additionally, expressions are given that allow the linear, quadratic, and exponential interpolation of data directly for the first time.

This analysis offers two critical and perhaps unexpected takeaways beyond the prospect of direct data implementation. First, many efforts in fitting Prony terms of the General Maxwell Model focus accuracy in expressing the relaxation function itself. However, in the Newmark–Beta scheme only the derivative of the relaxation function actually appears in the equation of motion. Second, this analysis highlights that there is nothing unique about the mathematical form of the general Maxwell model or any other constitutive relation that enables a step-wise integration scheme. In fact, any number of expressions of relaxation function are possible.

By generalizing the mathematical development, without early incorporation of a viscoelastic constitutive relation, the final formulation can be used with any constitutive relation. Furthermore, this effort paves the way for calculations that do not require a constitutive relation at all. Direct data implementation should be possible with this formulation, but future works will be needed to test and implement this new approach. A direct data implementation scheme may ultimately benefit the efficiency of the solver, the ease of use for the analyst, and the accuracy in accounting for viscoelastic behavior in material.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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