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On uniqueness and ill-posedness for the deautoconvolution problem in the multi-dimensional case

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Abstract

This paper analyzes the inverse problem of deautoconvolution in the multi-dimensional case with respect to solution uniqueness and ill-posedness. Deautoconvolution means here the reconstruction of a real-valued L^2-function with support in the n-dimensional unit cube $[0,1]^n$ from observations of its autoconvolution either in the full data case (i.e. on $[0,2]^n$) or in the limited data case (i.e. on $[0,1]^n$). Based on multi-dimensional variants of the Titchmarsh convolution theorem due to Lions and Mikusiński, we prove in the full data case a twofoldness assertion, and in the limited data case uniqueness of non-negative solutions for which the origin belongs to the support. The latter assumption is also shown to be necessary for any uniqueness statement in the limited data case. A glimpse of rate results for regularized solutions completes the paper.

Keywords: deautoconvolution, multi-dimensional inverse problem, uniqueness and ambiguity, nonlinear integral equation, local ill-posedness, Titchmarsh convolution theorem.

(Some figures may appear in colour only in the online journal)

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1. Introduction

Motivated by applications to spectroscopy, to the structure of solid surfaces and to nanostructures (see, e.g. [2, 8, 16, 31]) the inverse problem of *deautoconvolution*, which means that a function x *with compact support* is to be reconstructed from its autoconvolution $y = x * x$, has been considered for the *one-dimensional case* extensively in the literature of the past decades. Ill-posedness, uniqueness and ambiguity as well as regularization of the deautoconvolution problem for a *real-valued function* with compact support had been first analyzed in [18]. Subsequent studies in this direction can be found in [3, 5–7, 9, 13, 14, 25]. After the turn of the millennium, the one-dimensional deautoconvolution problem for a *complex-valued function* with compact real support became of interest for modern methods of ultrashort laser pulse characterization, and we refer in this context to the article [17] as well as to the further mathematical studies in [1, 4, 15].

In a multi-dimensional setting, the problem of deautoconvolution occurs when one wants to recover a square integrable *density function* x of an n-dimensional random variable $\mathfrak{X}$ with support in the unit n-cube $[0,1]^n$ from observations of the density function $y = x * x$ of the n-dimensional random variable $\mathfrak{Y} := \mathfrak{X}_1 + \mathfrak{X}_2$, where $\mathfrak{X}, \mathfrak{X}_1$ and $\mathfrak{X}_2$ are assumed to be of i.i.d. type. Then by definition the function x obeys the conditions $x(t) \geqslant 0$ a.e. on $[0,1]^n$ and $\int_{\mathbb{R}^n} x(t)\,\mathrm{d}t = \int_{[0,1]^n} x(t)\,\mathrm{d}t = 1$. The recovery of x from observations of $\mathfrak{Y}$ is one specific instance of a deconvolution problem, and the class of such problems has received lots of attention in the literature, see e.g. [28].

The object of research in this article is to present an ensemble of results for the deautoconvolution problem in the multi-dimensional case in an L^2-setting. We are going to extend, with respect to the reconstruction of real functions with n real variables, assertions on *uniqueness, ambiguity* and *ill-posedness* that previously had been proven in the one-dimensional case. We also complement and generalize findings of our recent paper [10], where such results have been stated for the two-dimensional case. Our focus is on the reconstruction of a square integrable real function $x = x(t)$ with $t = (t_1, t_2, \ldots, t_n)^{\mathrm{T}} \in \mathbb{R}^n$ of $n \geqslant 2$ variables with support in the unit n-cube $[0,1]^n$ from its autoconvolution $[x * x](s) = y(s)$ with $s = (s_1, s_2, \ldots, s_n)^{\mathrm{T}} \in \mathbb{R}^n$. In this context, the elements x and y both can be considered as tempered distributions with compact support, where $\mathrm{supp}(\cdot)$ is regarded as the essential support with respect to the Lebesgue measure λ in $\mathbb{R}^n$. Precisely, we consider x as an element of the real Hilbert space $L^2(\mathbb{R}^n)$ with $\mathrm{supp(x)} \subseteq [0,1]^{\mathrm{n}}$. For short, in such a case we write $x \in L^2([0,1]^n)$ by taking into account that $x(t)$ is assumed to be zero for $t \in \mathbb{R}^n \setminus [0,1]^n$. It is well-know that, for the convolution of two functions f and g with $f, g \in L^2(\mathbb{R}^n)$ and compact supports, it holds that $f * g \in L^2(\mathbb{R}^n)$ as well as

$$\mathrm{supp(f * g)} \subseteq \mathrm{supp(f)} + \mathrm{supp(g)}. \tag{1.1}$$

Here, we use the arithmetic sum $A + B$ of two subsets A and B of $\mathbb{R}^n$ defined as

$$A + B = \begin{cases} \{a + b \in \mathbb{R}^n : \, a \in A, \, b \in B\} & \text{if } A, B \neq \emptyset, \\ \emptyset & \text{else.} \end{cases}$$

As a consequence of (1.1) we have for $x \in L^2(\mathbb{R}^n)$ with $\mathrm{supp(x)} \subseteq [0,1]^{\mathrm{n}}$ that $y = x * x \in L^2(\mathbb{R}^n)$ with $\mathrm{supp(x * x)} \subseteq [0,2]^{\mathrm{n}}$, or in other words that $y \in L^2([0,2]^n)$.

The inverse problem of deautoconvolution is equivalent to the solution of an operator equation

$$F(x) = y \tag{1.2}$$

with the nonlinear forward operator $F:\mathcal{D}(F)\subseteq X\to Y$ mapping between the real Hilbert spaces $X:=L^2([0,1]^n)$ and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and having the domain $\mathcal{D}(F)$, where the nonlinear operator F possesses the convolution integral form

$$[F(x)](s) := [x*x](s) = \int_{\mathbb{R}^n} x(s-t)\,x(t)\mathrm{d}t \qquad (s,t\in\mathbb{R}^n)\,. \tag{1.3}$$

Note that (1.2) is a quadratic-type nonstandard Volterra integral equation in the sense that the integration boundary depends on the evaluation variable of the image function, see (1.4) and (1.6) below and see also [34].

To ease the notation in the sequel, we will make use of the following abbreviations of n-dimensional cuboids and cubes. If $s,t\in\mathbb{R}^n$ are given, we denote by

$$[s,t]^n := [s_1,t_1]\times\ldots\times[s_n,t_n]$$

the corresponding n-cuboids spanned by s and t. Clearly, if $s_j>t_j$ for some $j\in\{1,\ldots,n\}$, then $[s,t]=\emptyset$. Note that—with a slight abuse of this notation—for $s,t\in\mathbb{R}$ we also write $[s,t]^n$ for the n-cube of the form $[s,t]\times\ldots\times[s,t]$.

In this paper, we will distinguish two data situations. First we consider the *full data case* with $X:=L^2([0,1]^n)$, $Y:=L^2([0,2]^n)$ and forward operator F as

$$[F(x)](s) := \int_{[\max(s-1,0),\min(s,1)]^n} x(s-t)\,x(t)\,\mathrm{d}t\,, \tag{1.4}$$

where $y(s)=[F(x)](s)$ is observable for all $s\in[0,2]^n$, which implies due to (1.1) that all relevant information about $x*x$ is available, but in practice based on noisy data $y^\delta\in Y$ with noise level $\delta>0$ and deterministic noise model

$$\|y-y^\delta\|_Y\leqslant\delta. \tag{1.5}$$

Secondly, we are treating again with noise model (1.5) the *limited data case* with $X=Y:=L^2([0,1]^n)$ and forward operator F as

$$[F(x)](s) := \int_{[0,s]^n} x(s-t)\,x(t)\,\mathrm{d}t\,. \tag{1.6}$$

Here, $y(s)=[F(x)](s)$ is only available for s on the unit n-cube $[0,1]^n$. Since here the scope of the data is only $1/2^n$ compared to the full data case, the chances of accurately recovering x from noisy observations of y are decreasing more and more in the limited data case as n gets larger. In contrast to the full data case, where we assume in the sequel that $\mathcal{D}(F)=X=L^2([0,1]^n)$, we focus in the limited data case also on the domain $\mathcal{D}(F)=\mathcal{D}^+$ defined as

$$\mathcal{D}^+ := \{x\in X=L^2([0,1]^n):\, x\geqslant 0 \ \text{ a.e. on } [0,1]^n\}\,. \tag{1.7}$$

This set $\mathcal{D}^+$ collects the non-negative functions from $L^2([0,1]^n)$ and contains as a subset the square integrable density functions with support in the unit n-cube.

We emphasize that a further restriction of the data $y(s)$ to a smaller n-cube $s\in[0,a]^n$ with $0<a<1$ is not meaningful, as the Volterra structure of the integral equation then implies that the data does not contain any information about $x(t)$ for $t\in[0,1]^n\setminus[0,a]^n$.

It is well known that, in an L^2-setting, the nonlinear autoconvolution operator F is weakly sequentially continuous and *non-compact*. Proofs that can be extended to the multi-dimensional case (see remark 1 below) are given in [18, theorem 2 with example 4 and proposition 4]. On the other hand, the Fréchet derivative of $F: X = L^2([0,1]^n) \to Y$ in both data cases $Y = L^2([0,2]^n)$ and $Y = L^2([0,1]^n)$ is given, for all $x \in X$, by

$$F'(x): X \to Y, \qquad F'(x)h = 2x * h \qquad (h \in X). \tag{1.8}$$

Thus, the autoconvolution operator F possesses everywhere a *compact* Fréchet derivative, which is a linear convolution integral operator of Hilbert–Schmidt-type.

In [23, definition 1] a (local) degree (c_1, c_2, c_3) of nonlinearity of a nonlinear operator F at the point $x_0 \in \mathcal{D}(F)$ was introduced by the condition that the inequality

$$\|F(x) - F(x_0) - F'(x_0)(x - x_0)\|_Y \leqslant K \|F'(x_0)(x - x_0)\|_Y^{c_1} \|F(x) - F(x_0)\|_Y^{c_2} \|x - x_0\|_X^{c_3} \tag{1.9}$$

holds with some constant $K > 0$ for all elements $x \in \mathcal{D}(F)$ in a neighborhood of x_0.

Now with the Fréchet derivative (1.8), the autoconvolution operator F satisfies the nonlinearity condition

$$\|F(x) - F(x_0) - F'(x_0)(x - x_0)\|_Y = \|F(x - x_0)\|_Y \leqslant \|x - x_0\|_X^2 \text{ for all } x, x_0 \in X, \tag{1.10}$$

which implies that a degree of nonlinearity $(0,0,2)$ of F is true on the whole space $X = L^2([0,1]^n)$. On the other hand, proposition 2.2 in [5] and its proof ensure that there is no $c_1 > 0$ such that F admits a degree of nonlinearity $(c_1, 0, 0)$ at any $x_0 \in X$. This is a consequence of the non-compactness of F in contrast to the compactness of $F'(x_0)$ for all $x_0 \in X$. As a corollary one finds that also for the triple $(0, c_2, 0)$ with some $c_2 > 0$ the inequality (1.9), which is then of *tangential cone condition* type, cannot hold for any constant $K < 1$ (see [5, corollary 2.3]). However, even in the one-dimensional case no result about a tangential cone condition in the sense of a degree of nonlinearity $(0, c_2, 0)$ with $c_2 > 0$ and $K \geqslant 1$ has yet been demonstrated for the autoconvolution operator F.

For any function $x \in L^2([0,1]^n)$ the autoconvolution products $F(x) = x * x$ and $F(-x) = (-x) * (-x)$ coincide for both data cases. However, it is of interest whether for $y = x * x$ the elements x and $-x$ are the only solutions of equation (1.2) with $F: L^2([0,1]^n) \to L^2([0,2]^n)$ from (1.4) in the full data case or not. In the limited data case, for $F: \mathcal{D}^+ \subset L^2([0,1]^n) \to L^2([0,1]^n)$, it is of interest whether the solution x is under non-negativity constraints even uniquely determined. Based on different versions of the Titchmarsh convolution theorem some answers to both questions are given in section 3 below. Before that, we recall in section 2 some basic assertions on convolution in form of three lemmas and the definitional concept of local ill-posedness for nonlinear operator equations. In section 4, it will be shown that the n-dimensional deautoconvolution problem leads in both data cases to operator equations (1.2), which are locally ill-posed everywhere. This requires the use of some kind of regularization in order to construct stable approximate solutions. Even though a detailed study on regularization of the problem is beyond the scope of this manuscript, we briefly report on some error norms and rate results for regularized solutions occurring in a numerical case study in section 5. There, we restrict ourselves for simplicity to the classical variant of quadratic Tikhonov regularization for nonlinear operator equations along the lines of the seminal paper [12] and best possible regularization parameters. For a more detailed numerical study in the two-dimensional case we refer to [10].

2. Preliminaries

Unfortunately, the formula (1.1) concerning the support of the convolution function $f * g$ is an inclusion and not an equation. However, for $n = 1$ and functions $f, g \in L^2(\mathbb{R})$ with compact supports, which are not identically zero a.e. one can formulate an equation for the minima (smallest values) of the supports as

$$\min \mathrm{supp}(\mathrm{f} * \mathrm{g}) = \min \mathrm{supp}(\mathrm{f}) + \min \mathrm{supp}(\mathrm{g}), \tag{2.1}$$

which is a consequence of the Titchmarsh convolution theorem from [32]. Based on (2.1) it could be shown in [18, theorem 1] that the one-dimensional deautoconvolution problem has a uniquely determined solution in the limited data case under non-negativity constraints. By the same argument it could be shown in [17, theorem 4.2] that x and $-x$ are the only solutions in the full data case of the one-dimensional deautoconvolution problem. An extension of those uniqueness and twofoldness results to the n-dimensional deautoconvolution problem require extensions of Titchmarsh's theorem to the multi-dimensional case, and we recall two versions of such extension by the following two lemmas.

The first lemma goes back to Lions (see [26, 27]) and replaces $\min \mathrm{supp}(\mathrm{f})$, the *support minimum* occurring in (2.1) for $n = 1$, with the *convex support* occurring in lemma 1 for general $n \in \mathbb{N}$. Here, $\mathrm{conv\,supp}(\mathrm{f})$ denotes the convex hull of $\mathrm{supp}(\mathrm{f})$, i.e. the smallest closed convex set outside which the function f vanishes a.e. on $\mathbb{R}^n$.

Lemma 1. *Let the functions $f, g \in L^2(\mathbb{R}^n)$ with $n \in \mathbb{N}$ have compact supports* $\mathrm{supp}(\mathrm{f})$ *and* $\mathrm{supp}(\mathrm{g})$. *Then we have for the convolution that $f * g \in L^2(\mathbb{R}^n)$ and that the equation*

$$\mathrm{conv\,supp}(\mathrm{f} * \mathrm{g}) = \mathrm{conv\,supp}(\mathrm{f}) + \mathrm{conv\,supp}(\mathrm{g}) \tag{2.2}$$

holds true. In the special case that $\mathrm{supp}(\mathrm{f} * \mathrm{g}) = \emptyset$, *i.e. the function $f * g$ vanishes a.e. on $\mathbb{R}^n$, then we have that at least one of the sets* $\mathrm{supp}(\mathrm{f})$ *or* $\mathrm{supp}(\mathrm{g})$ *is the empty set, which means that the underlying function f or g vanishes a.e. on $\mathbb{R}^n$.*

Lemma 1 will allow us to prove the twofoldness assertion for the full data case of the multi-dimensional deautoconvolution problem in theorem 1 below.

We also present an extension of the Titchmarsh convolution theorem to the multi-dimensional case by using Mikusiński's *n-simplex* technique adapted to our situation as lemma 2, and we refer in this context to [29, theorem VIIIb].

Lemma 2. *Let us introduce for $\gamma \geqslant 0$ the n-simplices*

$$\Delta(\gamma) := \{(t_1, t_2, \ldots, t_n)^{\mathrm{T}} \in \mathbb{R}^n : \; 0 \leqslant t_1, \, 0 \leqslant t_2, \, \ldots, \, 0 \leqslant t_n, \, t_1 + t_2 + \ldots + t_n \leqslant \gamma\}$$

in $\mathbb{R}^n$. For functions $f, g \in L^2(\mathbb{R}^n)$ with compact supports $\mathrm{supp}(\mathrm{f})$ *and* $\mathrm{supp}(\mathrm{g})$ *covered by $[0, \infty)^n$, we conclude from*

$$[f * g](s) = \int_{\mathbb{R}^n} f(s-t)\, g(t)\, \mathrm{d}t = 0 \quad \textit{a.e. for} \quad s \in \Delta(\gamma) \quad (\gamma \geqslant 0)$$

that there are numbers $\gamma_1, \gamma_2 \geqslant 0$ with $\gamma_1 + \gamma_2 \geqslant \gamma$ such that

$$f(t) = 0 \quad \textit{a.e. for } t \in \Delta(\gamma_1) \quad \textit{and} \quad g(t) = 0 \quad \textit{a.e. for } t \in \Delta(\gamma_2).$$

Lemma 2 will be used in theorem 3 below to prove that for the limited data case of the multi-dimensional deautoconvolution problem under non-negativity constraints the solution is uniquely determined.

As an inverse problem the operator equation (1.2) with forward operator (1.4) mapping from the real Hilbert space $X = L^2([0,1]^n)$ to the Hilbert space $Y = L^2([0,2]^n)$ in the full data case of multi-dimensional deautoconvolution tends to be ill-posed. A probably stronger ill-posedness phenomenon is to be expected for the limited data case under non-negativity constraints where $F : \mathcal{D}(F) \subset X = L^2([0,1]^n) \to Y = L^2([0,1]^n)$ and $\mathcal{D}(F) = \mathcal{D}^+$ characterize the forward operator. For a precise theoretical verification of the ill-posedness phenomenon we adopt the *concept of local ill-posedness* for nonlinear operator equations, and we recall this concept by the following definition (see e.g. [24, definition 1.1]).

Definition 1. An operator equation $F(x) = y$ with nonlinear forward operator $F : \mathcal{D}(F) \subseteq X \to Y$ mapping between the Hilbert spaces X and Y with domain $\mathcal{D}(F)$ is called *locally ill-posed* at a solution point $x^\dagger \in \mathcal{D}(F)$ if there exist, for all closed balls $\overline{\mathcal{B}_r(x^\dagger)}$ with radius $r > 0$ and center $x^\dagger$, sequences $\{x_k\} \subset \overline{\mathcal{B}_r(x^\dagger)} \cap \mathcal{D}(F)$ that satisfy the condition

$$\|F(x_k) - F(x^\dagger)\|_Y \to 0, \quad \text{but} \quad \|x_k - x^\dagger\|_X \not\to 0, \quad \text{as} \quad k \to \infty .$$

Otherwise, the operator equation is called *locally well-posed* at $x^\dagger$.

For $n = 1$, *local ill-posedness everywhere* on the non-negativity domain

$$\mathcal{D}(F) = \{x \in X = L^2([0,1]) : x \geqslant 0 \text{ a.e. on } [0,1]\}$$

was proven for the deautoconvolution problem in the limited data case in [18, lemma 6]. We will extend this result to the multi-dimensional situation below in theorem 4.

Local ill-posedness everywhere on $L^2([0,1])$ could also be shown for the full data case of deautoconvolution and $n = 1$ in [14, proposition 2.3] by perturbing the solution with an appropriate sequence of square integrable real functions, which is weakly convergent in $L^2([0,1])$. By considering such sequences as 'rank one perturbations' we can also show local ill-posedness everywhere on $L^2([0,1]^n)$ in the multi-dimensional situation of deautoconvolution with full data. For preparation we present here the following lemma, the proof of which is an immediate consequence of Lebesgue's dominated convergence theorem.

Lemma 3. *Let $\{h_k\}_{k=1}^\infty \subset L^2([0,1])$ be a sequence of real functions of one real variable, which is weakly convergent to zero, i.e. $h_k \rightharpoonup 0$ in $L^2([0,1])$ as $k \to \infty$. Then we have for arbitrary real functions f of n real variables with $f \in L^2([0,1]^n)$ that the sequence $\{f_k = f + h_k\}_{k=1}^\infty \subset L^2([0,1]^n)$ defined as*

$$f_k(t_1, t_2, \ldots, t_n) := f(t_1, t_2, \ldots, t_n) + h_k(t_1) \qquad ((t_1, t_2, \ldots, t_n)^{\mathrm{T}} \in \mathbb{R}^n,\ k \in \mathbb{N}) \tag{2.3}$$

is weakly convergent to f, i.e. $f_k \rightharpoonup f$ in $L^2([0,1]^n)$ as $k \to \infty$. In this context, we have that $\|f_k - f\|_{L^2([0,1]^n)} = \|h_k\|_{L^2(0,1)}$ for all $k \in \mathbb{N}$.

Remark 1. The idea of 'rank one perturbations' in the sense of formula (2.3) can be exploited for the autoconvolution operator F at different places in order to extend properties from the one-dimensional to the multi-dimensional situation. Besides the issue of weak convergence addressed in lemma 3 with applications to ill-posedness assertions in section 4 below, this technique is, for example, also helpful for proving the non-compactness of F in the multi-dimensional situation based on the one-dimensional example 4 from [18] in view of the separable structure of F.

3. New assertions on twofoldness and uniqueness for the multi-dimensional deautoconvolution problem

To formulate our results on twofoldness and uniqueness of the deautoconvolution problem, we need to specify what is understood as solution.

Definition 2. For given $y \in L^2([0,2]^n)$, we call $x^\dagger \in L^2([0,1]^n)$ a *solution* to the operator equation (1.2) with $F : L^2([0,1]^n) \to L^2([0,2]^n)$ according to (1.4)

in the full data case if it satisfies the condition

$$[x^\dagger * x^\dagger](s) = y(s) \quad \text{a.e. for} \quad s \in [0,2]^n\,, \tag{3.1}$$

in the limited data case if it satisfies the condition

$$[x^\dagger * x^\dagger](s) = y(s) \quad \text{a.e. for} \quad s \in [0,1]^n\,. \tag{3.2}$$

If moreover $x^\dagger \in \mathcal{D}^+$ with $\mathcal{D}^+$ as in (1.7), then we call it a *non-negative solution*.

3.1. Results for the full data case

Lemma 1 allows us to prove the following theorem on solution twofoldness in the full data case of multi-dimensional deautoconvolution.

Theorem 1. *If, for given $y \in L^2([0,2]^n)$, the function $x^\dagger \in L^2([0,1]^n)$ is a solution in the full data case in the sense of definition 2, then $x^\dagger$ and $-x^\dagger$ are the only solutions in this sense.*

Proof. Let $x^\dagger \in L^2([0,1]^n)$ be a solution in the full data case in the sense of definition 2 and consider an arbitrary function $h \in L^2([0,1]^n)$ such that $x^\dagger + h$ is also a solution in the full data case in the sense of definition 2. This means that $[x^\dagger * x^\dagger](s) = [(x^\dagger + h) * (x^\dagger + h)](s)$ a.e. for $s \in [0,2]^n$, which implies that

$$[h * (2x^\dagger + h)](s) = 0 \tag{3.3}$$

a.e. for $s \in [0,2]^n$. By setting $f := h$ and $g := 2x^\dagger + h$ we can apply lemma 1. Taking into account that $\mathrm{supp}(h * (2x^\dagger + h)) \subseteq [0,2]^n$, we then have (3.3) a.e. for $s \in \mathbb{R}^n$, or in other words $\mathrm{supp}(h * (2x^\dagger + h)) = \emptyset$ and consequently $\mathrm{conv\,supp}(h * (2x^\dagger + h)) = \emptyset$. This implies, due to equation (2.2), that either $\mathrm{supp}(h) = \emptyset$ or $\mathrm{supp}(2x^\dagger + h) = \emptyset$ is true. On the one hand, $\mathrm{supp}(h) = \emptyset$ leads to the solution $x^\dagger$ itself, whereas on the other hand $\mathrm{supp}(2x^\dagger + h) = \emptyset$ leads to $[2x^\dagger + h](t) = 0$ a.e. for $t \in [0,1]^n$ and thus with $h = -2x^\dagger$ to the second solution $-x^\dagger$. Alternative solutions are therefore excluded. This proves the theorem. □

3.2. Results for the limited data case

For solutions $x^\dagger$ in the limited data case in the sense of definition 2 it is important whether the condition $0 \in \mathrm{supp}(x^\dagger)$ or its counterpart $0 \notin \mathrm{supp}(x^\dagger)$ is fulfilled. In this context, $0 \in \mathrm{supp}(x^\dagger)$ means that for any ball $B_r(0)$ around the origin with arbitrary small radius $r > 0$ there exists a set $M_r \subset B_r(0) \cap [0,1]^n$ with Lebesgue measure $\lambda(M_r) > 0$ such that $x^\dagger(t) \neq 0$ a.e. for $t \in M_r$. Vice versa, for $0 \notin \mathrm{supp}(x^\dagger)$ we have some sufficiently small radius $r > 0$ such that $x^\dagger(t) = 0$ a.e. for $t \in B_r(0) \cap [0,1]^n$.

In a first step we generalize by theorem 2 those aspects that had been fixed for $n = 1$ in [18, theorem 1] concerning the strong non-injectivity of the autoconvolution operator in the limited data case to the multi-dimensional situation with arbitrary $n \in \mathbb{N}$.

Theorem 2. *If, for given $y \in L^2([0,1]^n)$, the function $x^\dagger \in L^2([0,1]^n)$ is a solution in the limited data case in the sense of definition 2 that fulfills the condition*

$$0 \notin \mathrm{supp}(x^\dagger)\,, \tag{3.4}$$

then there exist infinitely many other solutions $\hat{x}^\dagger \in L^2([0,1]^n)$ in this sense.

Proof. Under the condition (3.4) there is some $0 < \varepsilon < 1/2$ such that $x^\dagger(t) = 0$ a.e. for $t \in [0,\varepsilon]^n$. Now there exist infinitely many $h \in L^2([0,1]^n)$ such that

$$h(t) = 0 \quad \text{a.e. for } t \in [0,1]^n \setminus [1-\varepsilon,1]^n. \tag{3.5}$$

For any such h we have

$$[h*(2x^\dagger + h)](s) = \int_{[0,s]^n} (2x^\dagger + h)(s-t)\,h(t)\,\mathrm{d}t = \int_{[1-\varepsilon,s]^n} (2x^\dagger + h)(s-t)\,h(t)\,\mathrm{d}t.$$

But for $s \in [0,1]^n$ and $t \in [1-\varepsilon,s]^n$, we have component-wise that

$$0 \leqslant s_i - t_i \leqslant 1 - t_i \leqslant \varepsilon < 1 - \varepsilon$$

due to $\varepsilon < \frac{1}{2}$, so that $(2x^\dagger + h)(s-\cdot) = 0$ a.e. for $[1-\varepsilon,s]^n$. Therefore, (3.3) holds a.e. for $s \in [0,1]^n$, which implies

$$[(x^\dagger + h)*(x^\dagger + h)](s) = y(s) \quad \text{a.e. for} \quad s \in [0,1]^n.$$

This yields the claim. □

Now we are ready to formulate and to prove with the following theorem a main new result of this paper, which extends the solution uniqueness assertion for the limited data case under non-negativity constraints published for $n = 1$ in [18, theorem 1] to the multi-dimensional situation with arbitrary $n \in \mathbb{N}$. The proof of this theorem is based on Mikusiński's n-simplex technique introduced above by lemma 2.

Theorem 3. *If, for given $y \in L^2([0,1]^n)$, the function $x^\dagger \in L^2([0,1]^n)$ is a non-negative solution in the limited data case in the sense of definition 2 that fulfills the condition*

$$0 \in \mathrm{supp}(\mathrm{x}^\dagger), \tag{3.6}$$

then $x^\dagger$ is the uniquely determined non-negative solution in this case.

Proof. First we will show that under the condition (3.6) the non-negative solution $x^\dagger(t)$ is uniquely determined a.e. for $t \in \Delta(1)$. Namely, supposed that there exists a function $h \in L^2([0,1]^n)$ with $x^\dagger + h \geqslant 0$ satisfying the equation

$$[(x^\dagger + h)*(x^\dagger + h)](s) = y(s) \quad \text{a.e. for} \quad s \in [0,1]^n, \tag{3.7}$$

we would have that (3.3) holds a.e. for $s \in [0,1]^n$. Because of $[0,1]^n \supset \Delta(1)$, lemma 2 applies with $f := h$, $g := 2x^\dagger + h$ and $\gamma = 1$. Obviously, we have $\gamma_2 = 0$ due to the fact that $[2x^\dagger + h](t) \geqslant x^\dagger(t)$ a.e. for $t \in [0,1]^n$, which implies together with condition (3.6) that $0 \in \mathrm{supp}(2\mathrm{x}^\dagger + \mathrm{h})$. Then we find as a consequence of $\gamma_1 + \gamma_2 \geqslant \gamma$ that $\gamma_1 \geqslant 1$ must hold, which yields $h(t) = 0$ a.e. for $t \in \Delta(1)$.

In a second step of the proof we show that also perturbations $h \in L^2([0,1]^n)$ with $x^\dagger + h \geqslant 0$ and $\mathrm{supp}(\mathrm{h}) \subseteq \overline{[0,1]^n \setminus \Delta(1)}$ are only possible if h is the zero function almost everywhere on $[0,1]^n \cap \Delta(2)$. Now assume, for such function h, that it obeys the condition (3.7) and consequently (3.3) holds a.e. for $s \in [0,1]^n$, from which we derive that

$$[h*(-2x^\dagger)](s) = [h*h](s) \quad \text{a.e. for} \quad s \in [0,1]^n. \tag{3.8}$$

This allows us to apply lemma 2 with $f := -2x^\dagger$, $g := h, f*g = h*h$ and the associated values γ_1, γ_2 and γ, respectively. Evidently, we have

$$\mathrm{supp}(\mathrm{h}*\mathrm{h}) \subseteq 2\mathrm{supp}(\mathrm{h}) \subseteq \overline{[0,2]^n \setminus \Delta(2)}$$

and thus $\gamma = 2$. This yields $\gamma_2 = 2$ and hence $h = 0$ a.e. on $[0,1]^n \cap \Delta(2)$, because $\gamma_1 = 0$ as a consequence of condition (3.6). Now, for $n = 2$ the proof is complete, because of $[0,1]^2 \subset [0,1]^2 \cap \Delta(2)$. For $n > 2$, however we must repeat the second step in an analog manner m times until $2^m \geqslant n$ such that $h = 0$ a.e. on $[0,1]^n \cap \Delta(2^m) \supseteq [0,1]^n \cap \Delta(n) = [0,1]^n$. Then the proof is complete. $\square$

4. Ill-posedness phenomena

For nonlinear inverse problems modelled by operator equations (1.2) in Hilbert spaces, the character and strength of ill-posedness may be a local property and may depend on nonlinearity conditions of the forward operator F, see for discussions and examples of the articles [20, 22, 23]. Therefore, the concept of local ill-posedness at a solution point $x^\dagger$ (see definition 1 above) applies for (1.2) with the autoconvolution operator F from (1.3). It could be proven for the one-dimensional situation that the deautoconvolution problem is *locally ill-posed everywhere* on $\mathcal{D}(F) = L^2([0,1])$ for the full data case (see [14, proposition 2.3]) and on $\mathcal{D}(F) = \mathcal{D}^+ \subset L^2([0,1])$ with $\mathcal{D}^+$ from (1.7) with $n = 1$ for the limited data case (see [18, lemma 6]). The following two theorems extend the results to the multi-dimensional situation for arbitrary $n \in \mathbb{N}$.

Theorem 4. *For the limited data case of deautoconvolution, the operator equation (1.2) with $X = Y = L^2([0,1]^n)$ and forward operator $F : \mathcal{D}^+ \subset X \to Y$ from (1.6) with non-negativity domain $\mathcal{D}^+$ from (1.7) is locally ill-posed everywhere on $\mathcal{D}(F) = \mathcal{D}^+$.*

Proof. Let $x^\dagger \in \mathcal{D}^+$ be a non-negative solution in the limited data case in the sense of definition 2. To show local ill-posedness at $x^\dagger$ we introduce for fixed $r > 0$ the sequence $\{h_k\}_{k=3}^\infty \subset L^2([0,1]^n)$ of perturbations of the form

$$h_k(t) := \begin{cases} k^{n/2}\, r & \text{for} \quad t \in [1 - \frac{1}{k}, 1]^n \\ 0 & \text{for} \quad t \in [0,1]^n \setminus [1 - \frac{1}{k}, 1]^n \end{cases}$$

with $x_k := x^\dagger + h_k \in \mathcal{D}^+$, $\|h_k\|_{L^2([0,1]^n)} = r$ and consequently $x_k \in \overline{\mathcal{B}_r(x^\dagger)} \cap \mathcal{D}^+$ for all $k \geqslant 3$. To complete the proof of the theorem we still need to show that the norm $\|F(x_k) - F(x^\dagger)\|_{L^2([0,1]^n)}$ tends for all $r > 0$ to zero as k tends to infinity. Owing to $F(x_k) - F(x^\dagger) = 2x^\dagger * h_k + h_k * h_k$ and $\|h_k * h_k\|_{L^2([0,1]^n)} = 0$, this rewrites as

$$\|x^\dagger * h_k\|_{L^2([0,1]^n)} \to 0 \quad \text{as} \quad k \to \infty.$$

Evidently, for $s = (s_1, s_2, \ldots, s_n)^{\mathrm{T}}, t = (t_1, t_2, \ldots, t_n)^{\mathrm{T}} \in \mathbb{R}^n$, the non-negative values

$$[x^\dagger * h_k](s) = \int\limits_{[0,s]^n} h_k(s - t)\, x^\dagger(t)\, \mathrm{d}t$$

can be different from zero only for $s \in [1 - \frac{1}{k}, 1]^n$. Using the Cauchy–Schwarz inequality and taking into account that $x^\dagger \in \mathcal{D}^+$ we have for those $s \in [1 - \frac{1}{k}, 1]^n$ the estimate

$$\begin{aligned} [x^\dagger * h_k](s) &= k^{n/2}\, r \int\limits_{\left[0, s - \left(1 - \frac{1}{k}\right)\right]^n} x^\dagger(t)\, \mathrm{d}t \\ &\leqslant r\, \|x^\dagger\|_{L^2([0,1]^n)}. \end{aligned}$$

This, however, yields

$$\|x^\dagger * h_k\|_{L^2([0,1]^n)} \leqslant r\,\|x^\dagger\|_{L^2([0,1]^n)} \left(\int\limits_{[1-\frac{1}{k},1]^n} 1\,\mathrm{d}s \right)^{1/2} = \frac{r\,\|x^\dagger\|_{L^2([0,1]^n)}}{k^{n/2}}$$

tending for all $r > 0$ to zero as k tends to infinity. This completes the proof of the theorem. □

Theorem 5. *For the full data case of deautoconvolution, the operator equation (1.2) with $X = L^2([0,1]^n)$, $Y = L^2([0,2]^n)$ and forward operator $F : X \to Y$ from (1.4) is locally ill-posed everywhere on $\mathcal{D}(F) = X$.*

Proof. Let $x^\dagger \in L^2([0,1]^n)$ be a solution in the full data case in the sense of definition 2. For showing local ill-posedness at $x^\dagger$ we fix $r > 0$ arbitrary and introduce the sequence $\{h_k\}_{k=1}^\infty \subset L^2([0,1])$ of functions of one real variable of the form

$$h_k(t) := \sqrt{2}\,r\sin(k^2t^2) \qquad (t \in [0,1],\ k \in \mathbb{N}). \tag{4.1}$$

For finding properties of h_k and $F(h_k) = h_k * h_k$ one needs to use the Fresnel integrals

$$S(s) := \int_0^s \sin(t^2)\mathrm{d}t \quad \text{and} \quad C(s) := \int_0^s \cos(t^2)\mathrm{d}t.$$

For $s \in [0,\infty)$ the range of both continuous functions is covered by the interval $[0,1]$. One easily finds that $0.5\,r < \|h_k\|_{L^2([0,1])} < r = \lim_{k\to\infty} \|h_k\|_{L^2([0,1])}$ and that the weak convergence $h_k \rightharpoonup 0$ in $L^2([0,1])$ as $k \to \infty$ takes place. The latter is a consequence of the fact that, for all $0 \leqslant s \leqslant 1$,

$$0 \leqslant \int_0^s h_k(t)\mathrm{d}t = \frac{\sqrt{\pi}\,r\,S(k\sqrt{2/\pi})s)}{k} \leqslant \frac{\sqrt{\pi}\,r}{k} \to 0 \quad \text{as} \quad k \to \infty.$$

Now we consider the perturbed functions $x_k := x^\dagger + h_k \in L^2([0,1]^n)$ defined as

$$x_k(t_1,t_2,\ldots,t_n) := x^\dagger(t_1,t_2,\ldots,t_n) + h_k(t_1) \qquad ((t_1,t_2,\ldots,t_n)^{\mathrm{T}} \in \mathbb{R}^n,\ k \in \mathbb{N}),$$

with $x_k \in \overline{\mathcal{B}_r(x^\dagger)}$ and $\|x_k - x^\dagger\|_{L^2([0,1]^n)} = \|h_k\|_{L^2([0,1])} \not\to 0$ as $k \to \infty$. To complete the proof, we still have to show that

$$\|F(x_k) - F(x^\dagger)\|_{L^2([0,2]^n)} \to 0 \quad \text{as} \quad k \to \infty\,.$$

Since $F(x_k) - F(x^\dagger) = F'(x^\dagger)(x_k - x^\dagger) + F(x_k - x^\dagger)$ and $x_k - x^\dagger \rightharpoonup 0$ as $k \to \infty$, we have $\lim_{k\to\infty} \|F(x_k) - F(x^\dagger)\|_{L^2([0,2]^n)} \leqslant \lim_{k\to\infty} \|F(x_k - x^\dagger)\|_{L^2([0,2]^n)} \leqslant \lim_{k\to\infty} \|h_k * h_k\|_{L^2([0,2])}$ by taking into account lemma 3 and that $F'(x^\dagger)$ is a compact operator. To complete the proof we finally show that $\lim_{k\to\infty} \|h_k * h_k\|_{L^2([0,2])} = 0$. Owing to the properties of Fresnel integrals mentioned above, this is a consequence of $|[h_k * h_k](\xi)| \leqslant \frac{\bar{C}}{k}$ for $\xi \in [0,2]$ with a uniform constant $\bar{C} > 0$, which follows from the two formulas

$$\int\limits_0^s \sin(k^2(s-t)^2)\sin(k^2t^2)\mathrm{d}t = \frac{\sqrt{\pi}ks\big(S(ks/\sqrt{\pi})\sin(k^2s^2/2) - C(ks/\sqrt{\pi})\cos(k^2s^2/2)\big) + \sin(k^2s^2)}{2k^2s}$$

valid for $0 \leqslant s \leqslant 1$, and

$$\int\limits_{s-1}^1 \sin(k^2(s-t)^2)\sin(k^2t^2)\mathrm{d}t = \frac{\sqrt{\pi}ks\big(S(k(2-s)/\sqrt{\pi})\sin(k^2s^2/2) - C(k(2-s)/\sqrt{\pi})\cos(k^2s^2/2)\big) + \sin(k^2s(2-s))}{2k^2s}$$

valid for $1 < s \leqslant 2$. □

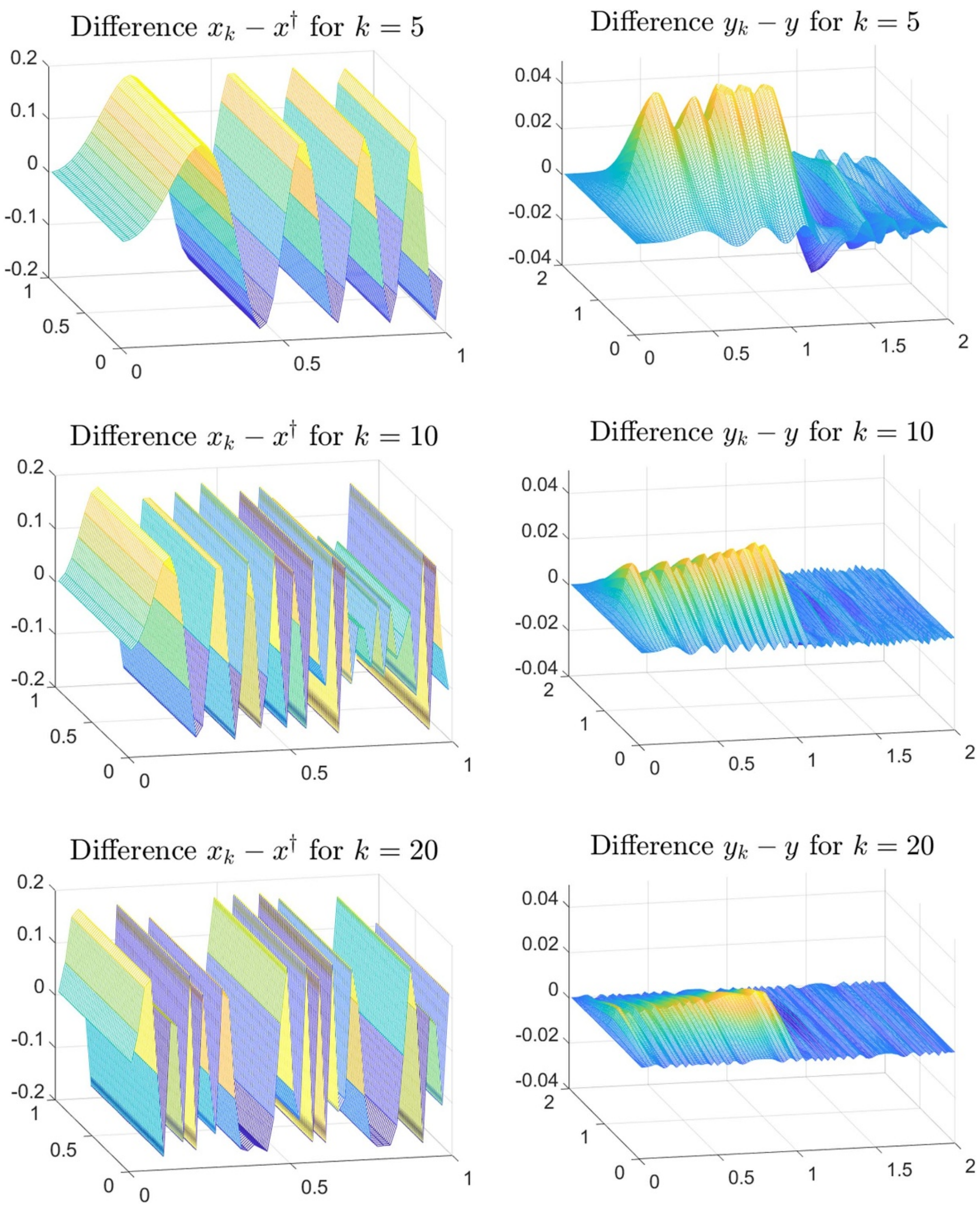

Figure 1. Development of differences $x_k - x^\dagger$ and $y_k - y$ for increasing k.

We are going to illustrate with figure 1 the ill-posedness phenomenon for the full data case of deautoconvolution along the lines of the ideas of the proof of theorem 5. For this purpose we exploit as an example solution the function

$$x^\dagger(t_1,t_2) = \left[\frac{2}{3}(t_1+1)\right] \cdot \left[\frac{\pi}{2+\pi}\left(\cos\left(\left(t_2-\frac{1}{2}\right)\pi\right)+1\right)\right],$$

which characterizes a factorable probability density function of a two-dimensional random vector with two uncorrelated one-dimensional components. For the sequence introduced in (4.1) we use the function $h_k(t) = \frac{\sqrt{2}}{8}\sin(k^2t^2)$, which leads to the perturbed solution $x_k(t_1,t_2) = x^\dagger(t_1,t_2) + h_k(t_1)$ that converges weakly in $L^2([0,1]^2)$ to $x^\dagger$ as $k\to\infty$, but not in norm as the pictures of $x_k - x^\dagger$ on the left in figure 1 for $k = 5, 10$ and 20 clearly show. However, the pictures on the right indicate convincingly the norm convergence of $y_k = x_k * x_k$ to $y = x^\dagger * x^\dagger$ in the space $L^2([0,2]^2)$.

5. A glimpse of rate results for regularized solutions

The goal of this concluding section is to mention some behavior of regularized solutions occurring in a brief case study on deautoconvolution. This behavior is unexpected in the sense that the numerical experiments show Hölder convergence rates, even though the usually required smoothness conditions are probably not satisfied. Here, in a setting analogous to [12] and [11, section 10.2] the regularized solutions

$$x_\alpha^\delta \in \underset{x\in\mathcal{D}(F)}{\arg\min}\left[\|F(x)-y^\delta\|_Y^2 + \alpha\|x-\bar{x}\|_X^2\right] \tag{5.1}$$

are minimizers of the Tikhonov functional. For both operators (1.4) and (1.6) under consideration, the element $y^\delta \in Y$ denotes the available data satisfying (1.5), $\bar{x}\in X$ is a reference element (initial guess), and $\alpha > 0$ is a regularization parameter. Our study is reduced to the case that best possible regularization parameters $\alpha = \alpha_{\mathrm{opt}}$ in the sense of

$$\alpha_{\mathrm{opt}}(\delta) = \min_{\alpha>0}\|x_\alpha^\delta - x^\dagger\|_X \tag{5.2}$$

are evaluated. From the three density functions of one real variable with supports in $[0,1]$,

$$x_1(t_1) = \frac{2(t_1+1)}{3},\quad x_2(t_2) = \frac{\pi}{2+\pi}\left(\cos\left(\left(t_2-\frac{1}{2}\right)\pi\right)+1\right),\quad x_3(t_3) = \begin{cases} \frac{5}{4} & 0\leqslant t_1 < \frac{1}{2} \\ t_1 & \frac{1}{2}\leqslant t_1\leqslant 1\end{cases},$$

we assemble two solutions $x^\dagger$ for the two- and three-dimensional situation of deautoconvolution as

$$x^\dagger(t_1,t_2) = x_1(t_1)x_2(t_2) \qquad \text{for } n=2$$

and

$$x^\dagger(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_3(t_3) \qquad \text{for } n=3,$$

which are density functions with supports $[0,1]^n$. To the discretization level with a uniform mesh width of $\frac{1}{50}$ in each direction, the regularized solutions $x^\delta_{\alpha_{\mathrm{opt}}}$ according to (5.1) have been calculated with a constant initial guess $\bar{x}\equiv 0.5$ in the discretized form for $n = 2, 3$ and randomly generated noisy data $y^\delta \in L^2([0,2]^n)$ (full data case) as well as for $y^\delta\in L^2([0,1]^n)$ (limited data case).

The discretization is achieved via the composite midpoint rule, and the corresponding discretized nonlinear optimization problem (5.1) is solved by using a damped Newton method. More details and a conceptional algorithm can be found in [10].

The relative empirical errors in % measured in the discrete L^2-norm for different δ, each simulated from ten independent runs, are listed in table 1. The bottom line of the table contains the Hölder exponent $0 < \kappa < 1$ of the convergence rate $\|x^\delta_{\alpha_{\mathrm{opt}}} - x^\dagger\| = \mathcal{O}(\delta^\kappa)$ as $\delta\to 0$ for the different situations, which had been estimated by regression from the selection of δ-values under consideration in the table.

Table 1. Relative error norms of regularized solutions.

Relative input errors $\frac{\|y^\delta - y^\dagger\|_Y}{\|y^\dagger\|_Y}$	Relative output errors of $x^\delta_{\alpha_{\mathrm{opt}}}$ $\frac{\|x^\delta_{\alpha_{\mathrm{opt}}} - x^\dagger\|_X}{\|x^\dagger\|_X}$			
	Full data case		Limited data case	
	$n=2$	$n=3$	$n=2$	$n=3$
10%	9.85%	13.48%	17.54%	23.59%
8%	8.70%	12.12%	17.21%	22.59%
5%	6.38%	9.82%	15.17%	19.99%
2%	3.61%	6.26%	9.74%	14.54%
1%	2.31%	4.12%	7.95%	11.58%
0.8%	1.98%	3.57%	7.39%	10.50%
0.5%	1.44%	2.61%	5.94%	9.24%
0.2%	0.78%	1.42%	4.10%	6.85%
0.1%	0.48%	0.87%	2.70%	5.47%
0.05%	0.30%	0.53%	1.76%	4.31%
Estim. Hölder exponent κ	0.66	0.61	0.43	0.32

An inspection of table 1 shows for both dimensions $n = 2$ and $n = 3$ a substantial reduction of the regularization error norms in the full data case compared to the limited data case. This is intuitively explained by the lack of data in $[0,2]^n \setminus [0,1]^n$, but even though this lack is considerably larger in dimension $n = 3$ (factor 8) compared to $n = 2$ (factor 4), the error norms do not fully reflect this behavior.

Based on ten different noise levels δ, a rough estimation of convergence rates of the corresponding error norms as δ tends zero indicates Hölder exponents $\kappa > 0.5$ in the full data case and $\kappa < 0.5$ in the limited data case. However, both results cannot fully be explained by available theory. It is known from [12] and [11, theorem 10.4] that a $\kappa = 0.5$ rate (i.e. $\mathcal{O}(\sqrt{\delta})$) is obtained under a range-type source condition $x^\dagger - \bar{x} = (F'(x^\dagger))^* w$ in combination with a smallness condition on $\|w\|_Y$. On the other hand, it has been shown in [5, proposition 2.6] that such theory is hard to apply for the autoconvolution operator F even in the one-dimensional case. To obtain rates with $\kappa > 0.5$, it is i.e. known from [30] and [11, theorem 10.7], that a rate $\mathcal{O}(\delta^{\frac{2}{3}})$ can be obtained under the higher-order range condition $x^\dagger - \bar{x} = (F'(x^\dagger))^* F'(x^\dagger) v$ in combination with a smallness condition on $\|v\|_X$. But in view of [5, proposition 2.6] it is also questionable whether such a result can be applied for the autoconvolution operator F at hand. In both situations, one reason seems to be fact that the compact Fréchet derivatives $F'(x)$ carry too little information about the non-compact operator F. Also *variational source conditions* introduced in [21] and, for example, further analyzed in [19, 33] could not be successfully exploited for obtaining convergence rates in deautoconvolution. Solely in [4, proposition 5.1 and corollary 5.2] a convergence rate could be derived by means of variational source conditions, but only under strong sparsity assumptions on the solution $x^\dagger$. Nevertheless, the numerical experiment in the context of table 1 indicates the practical occurrence of Hölder convergence rates for regularized solutions to the multi-dimensional deautoconvolution problem.

Data availability statement

The data cannot be made publicly available upon publication because they are not available in a format that is sufficiently accessible or reusable by other researchers. The data that support the findings of this study are available upon reasonable request from the authors.

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