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Applications of kinetic tools to inverse transport problems

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Abstract
We show that the inverse problems for a class of kinetic equations can be solved by classical tools in PDE analysis including energy estimates and the celebrated averaging lemma. Using these tools, we give a unified framework for the reconstruction of the absorption coefficient for transport equations in the subcritical and critical regimes. Moreover, we apply this framework to obtain, to the best of our knowledge, the first result in a nonlinear setting. We also extend the result of recovering the scattering coefficient in Choulli and Stefanov (1998 *Osaka J. Math.* **36** 87–104) from 3D to 2D strictly convex domains.

Keywords: kinetic theory, optical tomography, inverse transport problems, averaging lemma

(Some figures may appear in colour only in the online journal)

1. Introduction

Kinetic theory describes the behaviour of a large number of particles that follow the same physical laws in a statistical manner. Depending on the particular type of particles, various equations are derived. These include, among many others, the Boltzmann equation for the rarified gas, Vlasov–Poisson equation for charged plasma particles, the radiative transfer equation for photons, and the neutron transport equation for neutrons. In the kinetic theory, one uses $f(t, x, v)$ to denote the density distribution function of the particles in the phase space (x, v) at time t. The kinetic equation that f satisfies is of the form

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = Q[f],$$

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where the terms on the left characterizes the trajectory of particles moving with velocity v and accelerated/decelerated by the external field E, namely,

$$\dot{x} = v, \quad \dot{v} = E.$$

In practice, for example for the plasma system, E is the electric field generated by the electric potential. The term on the right collects information about particles colliding with each other and/or with the media. The specific form of Q depends on the particular type of particles studied.

During the past three decades, analysis of kinetic equations has seen drastic progresses. In particular, with the introduction of averaging lemma and application of the concept of entropy combined with traditional energy estimates, the well-posedness and the convergence to equilibria can now be shown for many kinetic equations.

Despite their wide applications for forward problems, such techniques are barely used in the inverse setting, where the goal is to recover certain unknown parameters (in E or Q for example). These parameters are usually set constitutively or 'extracted' from lab experiments. Mathematically, such 'extraction' is a process termed inverse problem, which is generally hard to solve rigorously. Aside from very limited examples [8, 9, 7, 14–16, 28, 30, 37–40] along with some analysis on stability [5, 6, 12, 22, 25, 27, 28, 32, 41, 42], it is unknown in general, what kind of data would be enough to guarantee a unique reconstruction or when the reconstruction is stable. Moreover, in the few solved examples, the techniques used rely on careful and rather explicit calculation of the solutions to the PDEs, or on the experimental advances that involve hybrid imaging to reveal internal data [10, 17]. As a consequence, it is challenging to extend these results to general models (see reviews in [4, 36]). There are, however, a large amount of studies addressing the related computational issues [1, 13, 29, 31, 34, 35] (also see reviews in [2, 3, 33]).

In this paper we propose to use energy methods and the averaging lemma to investigate the unique reconstruction of parameters in transport equations in a rather general setup. Since our methods do not rely on fine details of the equation as much as in the previous works, we can apply our results to a class of models including a nonlinear transport equation. We are also able to extend the study of the radiative transport equation in the subcritical case in [16, 39] to a unified analysis in both subcritical and critical regimes. Further comments regarding the dimensionality can be found in section 1.2 where precise statements of the main results are shown.

1.1. Singular decomposition

Throughout the paper we study the time-independent problem

$$v \cdot \nabla_x f(x, v) = -\sigma_a f(x, v) + F_f(x), \qquad x \in \Omega \subseteq \mathbb{R}^2, v \in \mathbb{S}^1, \tag{1.1}$$

where Ω is a bounded convex domain, $\mathbb{S}^1$ is the unit circle with a normalized measure, and $F_f(x)$ maps f into a function that only depends on x. For example, in the first case solved in theorem 1.1, we consider

$$F : L^p(\Omega \times \mathbb{S}^1) \to L^p(\Omega), \qquad p \geqslant 1.$$

The function space for f may vary in different problems.

We assume that σ_a is isotropic in the sense that $\sigma_a = \sigma_a(x)$. One example is the radiative transfer equation (RTE) where F_f is simply defined by taking the zeroth moment of f:

$$F_f(x) = \sigma_s(x) \int_{\mathbb{S}^1} f(x, v)\mathrm{d}v.$$

The data we will be using is of the Albedo type, namely, we can impose an incoming boundary condition and measure the associated outgoing boundary data and define the Albedo operator as

$$\mathcal{A}: \quad f|_{\Gamma_-} \to f|_{\Gamma_+}.$$

Here $\Gamma_\pm$ are the collections of all coordinates on the physical boundary with the velocity pointing either in or out of the domain defined by

$$\Gamma_\pm = \{(x, v): \quad x \in \partial\Omega, \ \pm n(x) \cdot v \leqslant 0\},$$

where $n(x)$ is the outward normal at $x \in \partial\Omega$. The goal is to reconstruct parameters in (1.1) such as σ_a or unknown parameters in F_f by taking multiple sets of incoming-to-outgoing data.

The basic approach we adopt here is the method of singular decomposition. It is introduced in [16] to recover the absorption and scattering kernel in the radiative transfer equation. The main idea of this method is built upon the observation that the solution $f(x, v)$ to (1.1) can be decomposed into parts with different regularity. Each part contains information of different terms in equation (1.1). Hence if one is able to separate these parts with different regularity by imposing proper test functions on Γ_-, then there is hope to recover various terms in equation (1.1).

As an illustration, we explain the basic procedures to reconstruct σ_a in (1.1). We start with splitting the solution as $f = f_1 + f_2$ where f_1,f_2 satisfy

$$\begin{cases} v \cdot \nabla_x f_1 = -\sigma_a f_1, \\ f_1|_{\Gamma_-} = f|_{\Gamma_-} \end{cases} \quad \text{and} \quad \begin{cases} v \cdot \nabla_x f_2 = -\sigma_a f_2 + F_f, \\ f_2|_{\Gamma_-} = 0. \end{cases}$$

With a relatively singular and concentrated input, e.g. $f|_{\Gamma_-} = \phi(\frac{x-x_0}{\epsilon})\phi(\frac{v-v_0}{\epsilon})$, f_1 will be more singular compared with f_2: the information of f_1 propagates only in a narrow neighborhood of a ray while f_2 is more spread out. Hence one is able to isolate f_1 from f_2 by measuring the outgoing data only in a small neighborhood of the exit point for f_1. It is then clear from the equation for f_1 that the absorption coefficient σ_a can be fully recovered once f_1 known. The details of such analysis is shown in section 2.

The method of singular decomposition has been extensively used in many variations of RTE, including the time-dependent model, when data is angular-averaging type, models with internal source, and models with adjustable frequencies, among some others [8, 9, 7, 28, 30, 38–40]. See also reviews [2, 4, 33]. Stability was discussed in [5, 6, 12, 25, 27, 41, 42]. To our knowledge, all these discussions are centered around linear RTEs. Since linearity plays the central role, so far there has been no result in a nonlinear setup. One of our goals in this paper is to extend singular decomposition to a nonlinear system.

1.2. Main results

We show two main results in this paper. The first result gives a general framework for recovering the absorption coefficient. To present our idea in the simplest form, we set our proof in two dimension. General dimensions can be treated similarly.

The domain Ω considered in this paper is strictly convex with a C^2 boundary. For such a domain, there exists a C^2-function $\xi : \mathbb{R}^2 \to \mathbb{R}$ (which is called the defining function of Ω) such that $\overline{\Omega}$ and its boundary are described by

$$\overline{\Omega} = \{x \big| \, \xi(x) \leqslant 0\} \qquad \text{and} \qquad \partial\Omega = \{x \big| \, \xi(x) = 0\}. \tag{1.2}$$

Moreover, $\nabla_x \xi(x) \neq 0$ for any $x \in \partial\Omega$ and there exists a constant $D_0 > 0$ such that

$$\sum_{i,j=1}^{2} \partial_{ij}\xi(x) a_i a_j \geqslant D_0 |a|^2, \qquad \forall x \in \Omega. \tag{1.3}$$

The construction of ξ uses the distance function $\mathrm{dist}(x, \partial\Omega)$ whose regularity is the same as the regularity of $\partial\Omega$. We refer the reader to section 14.6 in [21] for more details. The outward normal $n(x)$ at $x \in \Omega$ is then given by

$$n(x) = \frac{\nabla_x \xi(x)}{|\nabla_x \xi(x)|}, \qquad \forall x \in \partial\Omega.$$

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a strictly convex and bounded domain with a C^2 boundary. Suppose $\sigma_a \geqslant 0$ is isotropic and $\sigma_a \in C(\overline{\Omega})$. Suppose there exists $p \geqslant 1$ such that for any given incoming data ϕ satisfying*

$$\phi \in L^p(\Gamma_-), \qquad \phi \geqslant 0,$$

equation (1.1) with a given mapping F_f has a unique solution with the bound

$$||F_f||_{L^p(\Omega)} \leqslant C_{0,F} ||\phi||_{L^p(\Gamma_-)}, \tag{1.4}$$

where $C_{0,F}$ is independent of ϕ. Then with proper choices of the incoming data and outgoing measurements, the absorption coefficient σ_a can be uniquely reconstructed. Moreover, such σ_a is independent of the particular form of F_f as long as F_f satisfies (1.4) with $C_{0,F}$ independent of the incoming data ϕ.

We remark that although the specific form of F_f is not needed in the proof of theorem 1.1, when applying theorem 1.1 to particular examples, one needs to make use of the specific definition of F_f to verify (1.4) and the well-posedness of (1.1) with such F_f. We also comment that this assumptions on F_f is not as restrictive as they may appear. In fact it is common for a vast class of kinetic equations that F_f only depends on the moments of f and satisfies the bound in (1.4). Upon proving theorem 1.1 in section 2, we will give two examples to demonstrate its effectiveness.

In the second result, we show the unique recovery of the scattering coefficient σ_s in the classical RTE (radiative transfer equation):

$$v \cdot \nabla_x f = -\sigma_a f + \sigma_s \langle f \rangle, \qquad x \in \Omega \subseteq \mathbb{R}^2, v \in \mathbb{S}^1, \tag{1.5}$$

where $\langle f \rangle = \int_{\mathbb{S}^1} f \mathrm{d}v$ with $\mathrm{d}v$ normalized in the way that $\int 1 \mathrm{d}v = 1$. This equation describes the dynamics of photon particles in a bounded domain Ω. The media is characterized by the total cross section σ_a and the scattering cross section σ_s, which are both functions of x. These cross sections are determined by the optical properties of the media.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^2$ be a strictly convex and bounded domain with a C^2 boundary (see the precise definition in section 2). Suppose $\sigma_a, \sigma_s \in C(\overline{\Omega})$ with σ_a given and $0 < \sigma_0 \leqslant \sigma_s \leqslant \sigma_a$. Then with proper choices of the incoming data and outgoing measurements, the scattering coefficient σ_s in (1.5) can be uniquely reconstructed from the measurement of the outgoing data.*

Two comments are in place for theorem 1.2: first, we only show the result in $\mathbb{R}^2$ since this is the case not covered in [16]. Similar strategy used to prove theorem 1.2 can also be applied to any higher dimension by using the same incoming data and measurement as in [16]. In this sense, our result would be an extension of [16]. Second, in $\mathbb{R}^2$ so far we can only treat the case where σ_s is isotropic, that is, $\sigma_s = \sigma_s(x)$. Similar as in [16], such constraint is not needed for higher dimensions. We also note that 2D case was studied in [39]. However, there smallness of the scattering kernel is assumed while we can deal with the critical and general subcritical cases.

This paper is laid out as follows. In section 2, we show the proof of theorem 1.1 together with its applications to the classical linear RTE and a nonlinear RTE coupled with a temperature equation. In section 3, we show the proof of theorem 1.2. Some technical parts in the proofs of these two theorems are left in the appendices.

2. Absorption coefficient for radiative transfer equations

For each $(x, v) \in \overline{\Omega} \times \mathbb{S}^1$, we use $\tau_-(x, v)$ and $\tau_+(x, v)$ to denote the nonnegative backward and forward exit times, which are the instances where

$$x - \tau_-(x, v)v \in \partial\Omega, \qquad x + \tau_+(x, v)v \in \partial\Omega, \quad \text{for any } (x, v) \in \overline{\Omega} \times \mathbb{S}^1. \tag{2.1}$$

Recall the basic properties of the backward exit time from lemma 2 in [24]:

Lemma 2.1 ([24]). *Suppose $\Omega \subseteq \mathbb{R}^2$ is strictly convex and has a C^2 boundary. Suppose ξ is the characterizing function of Ω and $\partial\Omega$ which satisfies (1.2) and (1.3). For any $(x, v) \in \overline{\Omega} \times \mathbb{S}^1$, let τ_- be the backward exit time defined in (2.1) and $x_- \in \partial\Omega$ be the exit point given by $x_- = x - \tau_-(x, v)v$. Then*

(a) (τ_-, x_-) are uniquely determined for each $(x, v) \in \overline{\Omega} \times \mathbb{S}^1$;
(b) Suppose $\xi \in C^1(\mathbb{R}^3)$ and $v \cdot n(x_-) \neq 0$. Then τ_-, x_- are differentiable at (x, v) with

$$\nabla_x \tau_-(x, v) = \frac{n(x_-)}{v \cdot n(x_-)}, \qquad \nabla_v \tau_-(x, v) = \frac{\tau_- n(x_-)}{v \cdot n(x_-)},$$
$$\nabla_x x_-(x, v) = \mathcal{I} - \nabla_x \tau_- \otimes v, \qquad \nabla_v x_-(x, v) = -\tau_- \mathcal{I} - \nabla_v \tau_- \otimes v.$$

The rest of this section is devoted to the proof of theorem 1.1. As introduced in the previous section, the idea of the proof is to separate the terms in the equations and compare the induced singularities. In particular, let f the solution to the equation (1.1) with boundary condition $f|_{\Gamma_-} = \phi(x, v)$. We separate it as $f = f_1 + f_2$ so that f_1 satisfies

$$v \cdot \nabla_x f_1 = -\sigma_a f_1, \qquad f_1\big|_{\Gamma_-} = \phi(x, v), \tag{2.2}$$

and f_2 satisfies

$$v \cdot \nabla_x f_2 = -\sigma_a f_2 + F_f(x), \qquad f_2\big|_{\Gamma_-} = 0. \tag{2.3}$$

If we choose $\phi(x, v)$ to be a delta-like function concentrating at a point $(x^{\text{in}}, v^{\text{in}}) \in \Gamma_-$, then it is clear through equation (2.2) that the leading singularity of f will be propagating along the ray

$$x = x^{\text{in}} + \tau v^{\text{in}}, \quad \tau \in [0, \tau_+].$$

Defining

$$v^{\mathrm{out}} = v^{\mathrm{in}}, \qquad x^{\mathrm{out}} = x^{\mathrm{in}} + \tau_+(x^{\mathrm{in}}, v^{\mathrm{in}})v^{\mathrm{in}}, \tag{2.4}$$

and letting the test function ψ concentrate on $(x^{\mathrm{out}}, v^{\mathrm{out}})$, we will split the measurement of the outgoing data into two components (with $\mathrm{d}\Gamma_+ = n(x) \cdot v\mathrm{d}S_x\mathrm{d}v$):

$$\begin{aligned} M_\psi(f) &= \int\int_{\Gamma_+} \psi(x,v)f(x,v)\,\mathrm{d}\Gamma_+ \\ &= \int\int_{\Gamma_+} \psi(x,v)f_1(x,v)\,\mathrm{d}\Gamma_+ + \int\int_{\Gamma_+} \psi(x,v)f_2(x,v)\,\mathrm{d}\Gamma_+ \\ &= M_\psi(f_1) + M_\psi(f_2). \end{aligned} \tag{2.5}$$

The estimates in the proof are designated to show that

$$M_\psi(f_1) \text{ is determined by the X-ray transform of } \sigma_a \tag{2.6}$$

and for concentrated incoming data ϕ and concentrated test function ψ,

$$M_\psi(f_2) \ll M_\psi(f_1). \tag{2.7}$$

The concentration of ϕ and ψ will be described by ϵ and δ in the proof.

From this separation one can reconstruct σ_a via the unique recovery of σ_a in the x-ray transform. Details of the proof are shown below. One convention that we follow in the rest of this paper is that we repeatedly use c_0 and C_0 to denote constants that may change from line to line.

Proof of theorem 1.1. Let $\epsilon, \delta > 0$ be arbitrary constants to be chosen later and let ϕ_0 be a smooth function on $\mathbb{R}$ such that

$$0 \leqslant \phi_0(r) \leqslant 1, \qquad \phi_0 \in C_c^\infty([0,\infty)), \qquad \phi_0(0) = 1, \qquad \int_0^\infty \phi_0(r)\mathrm{d}r = 1.$$

For any $(x^{\mathrm{in}}, v^{\mathrm{in}}) \in \Gamma_-$ such that

$$v^{\mathrm{in}} \cdot n(x^{\mathrm{in}}) = -c^{\mathrm{in}} < 0, \tag{2.8}$$

choose the incoming data for equation (1.1) as

$$\phi(x,v) = \frac{1}{\epsilon\delta}\phi_0\left(\frac{|x - x^{\mathrm{in}}|}{\epsilon}\right)\phi_0\left(\frac{|v - v^{\mathrm{in}}|}{\delta}\right), \qquad (x,v) \in \Gamma_-.$$

Let $(x^{\mathrm{out}}, v^{\mathrm{out}})$ be defined in (2.4), and we take the test function for measurement to be:

$$\psi(x,v) = \psi_0(x - x^{\mathrm{out}})\psi_0\left(\frac{v - v^{\mathrm{out}}}{\delta}\right) = \psi_0(x - x^{\mathrm{out}})\psi_0\left(\frac{v - v^{\mathrm{in}}}{\delta}\right), \qquad (x,v) \in \Gamma_+,$$

where $\psi_0(r)$ is a smooth function that satisfies

$$0 \leqslant \psi_0(r) \leqslant 1, \qquad \psi_0 \in C_c^\infty([0,\infty)), \qquad \psi_0(0) = 1, \qquad \int_0^\infty \psi_0(r)\mathrm{d}r = 1. \tag{2.9}$$

We can solve along characteristics in (2.2) and (2.3) to obtain explicit and semi-explicit formulas for f_1 and f_2 as

$$f_1(x,v) = \mathrm{e}^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv)\mathrm{d}s}\phi(x-\tau_-(x,v)v, v), \qquad \forall\,(x,v) \in \overline{\Omega}\times\mathbb{S}^1 \tag{2.10}$$

and

$$f_2(x,v) = \int_0^{\tau_-(x,v)} \mathrm{e}^{-\int_0^s \sigma_a(x-\tau v)\,\mathrm{d}\tau} F_f(x-sv)\mathrm{d}s, \qquad \forall\,(x,v) \in \overline{\Omega}\times\mathbb{S}^1. \tag{2.11}$$

For future use, define the sets $\mathbb{S}^1_{x,+}$ and $\partial\Omega_v^+$ by

$$\partial\Omega_v^+ = \left\{x \in \partial\Omega \middle| n(x)\cdot v > 0\right\}, \qquad \text{for all } v \in \mathbb{S}^1, \tag{2.12}$$

$$\mathbb{S}^1_{x,+} = \left\{v \in \mathbb{S}^1 \middle| v\cdot n(x) > 0\right\}, \qquad \text{for all } x \in \partial\Omega. \tag{2.13}$$

We show (2.6) in two steps.

Step 1: limit of $M_\psi(f_1)$ Using (2.10), we have

$$M_\psi(f_1) = \int\int_{\Gamma_+} \psi(x,v)\mathrm{e}^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv)\mathrm{d}s}\phi(x-\tau_-(x,v)v,v)\,\mathrm{d}\Gamma_+ \tag{2.14}$$

$$\begin{aligned}
&= \frac{1}{\epsilon\delta}\int_{\partial\Omega}\int_{\mathbb{S}^1_{x,+}} \mathrm{e}^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv)\mathrm{d}s}\psi_0(x-x^{\mathrm{out}})\psi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\\
&\qquad\times\phi_0\left(\frac{|x-\tau_-(x,v)v-x^{\mathrm{in}}|}{\epsilon}\right)\phi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)n(x)\cdot v\mathrm{d}v\mathrm{d}S_x\\
&= \frac{1}{\epsilon}\int_{\partial\Omega}\psi_0(x-x^{\mathrm{out}})\,G_{\epsilon,\delta}(x)\mathrm{d}S_x,
\end{aligned} \tag{2.15}$$

where $G_{\epsilon,\delta}(x)$ denotes the inner integral and it can be further simplified in notation as

$$\begin{aligned}
G_{\epsilon,\delta}(x) &= \frac{1}{\delta}\int_{\mathbb{S}^1_{x,+}} \mathrm{e}^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv)\mathrm{d}s}\psi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\\
&\qquad\times\phi_0\left(\frac{|x-\tau_-(x,v)v-x^{\mathrm{in}}|}{\epsilon}\right)\phi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)n(x)\cdot v\mathrm{d}v\\
&= \frac{1}{\delta}\int_{\mathbb{S}^1_{x,+}} H_\epsilon(x,v)\psi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\phi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\mathrm{d}v
\end{aligned}$$

with

$$H_\epsilon(x,v) = \mathrm{e}^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv)\mathrm{d}s}\phi_0\left(\frac{|x-\tau_-(x,v)v-x^{\mathrm{in}}|}{\epsilon}\right)n(x)\cdot v. \tag{2.16}$$

We will first pass $\delta\to 0$ and then $\epsilon\to 0$ in (2.15). Note that for each fixed ϵ and $x\in\partial\Omega$, the inner integral $G_{\epsilon,\delta}$ satisfies

$$0 \leqslant G_{\epsilon,\delta}(x) \leqslant \frac{1}{\delta}\int_{\mathbb{S}^1_{x,+}} \psi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\mathrm{d}v \leqslant \frac{1}{\delta}\int_0^{2\pi}\psi_0\left(\frac{\sin\theta/2}{\delta/2}\right)\mathrm{d}v$$
$$\leqslant \frac{1}{\delta}\int_0^{\alpha_0\delta}\psi_0\left(\frac{\sin\theta/2}{\delta/2}\right)\mathrm{d}v + \frac{1}{\delta}\int_{2\pi-\alpha_0\delta}^{2\pi}\psi_0\left(\frac{\sin\theta/2}{\delta/2}\right)\mathrm{d}v \leqslant 2\alpha_0,$$

where α_0 only depends on the size of the support of ψ_0. Such uniform bound ensures that the Lebesgue Dominated Convergence theorem can be applied when taking the δ-limit in (2.15). To compute the pointwise δ-limit of $G_{\epsilon,\delta}$, we denote D_{ϕ_0} as the set where

$$D_{\phi_0} = \left\{(x,v)\in\Gamma_+ \Big| \frac{|v-v^{\mathrm{in}}|}{\delta}, \frac{|x_- - x^{\mathrm{in}}|}{\epsilon} \in \mathrm{Supp}\phi_0\right\}, \qquad x_- = x - \tau_-(x,v)v.$$

Then the measurement $M_\psi(f_1)$ becomes

$$M_\psi(f_1) = \iint_{D_{\phi_0}} \psi(x,v)\mathrm{e}^{-\int_0^{\tau_-(x,v)}\sigma_a(x-sv)\mathrm{d}s}\phi(x-\tau_-(x,v)v,v)\,\mathrm{d}\Gamma_+.$$

By the non-degeneracy condition of $(x^{\mathrm{in}}, v^{\mathrm{in}})$ in (2.8) and the support of ϕ_0, the normal direction $n(\cdot)$ is continuous in a small neighbourhood of x^{in}. Hence, if we choose δ, ϵ to be small enough, then for any $(x,v)\in D_{\phi_0}$, we have

$$v\cdot n(x_-) < -\frac{1}{2}c^{\mathrm{in}} < 0, \qquad x_- = x-\tau_-(x,v)v. \tag{2.17}$$

Application of lemma 2.1 gives that

$$\tau_-(x,v)\in C^1(\overline{D}_{\phi_0}),$$

which implies that $\tau_-(x,v)$ is uniformly continuous on D_{ϕ_0}. Together with the continuity of σ_a, and ϕ_0, we deduce that for each ϵ, the function $H_\epsilon(\cdot,\cdot):\overline{D}_{\phi_0}\to\mathbb{R}$ is continuous. Hence H_ϵ is uniformly continuous on $\overline{D}_{\phi_0}$ and thus

$$\frac{1}{\delta}\int_{\mathbb{S}^1_{x,+}}\left|H_\epsilon(x,v)-H_\epsilon(x,v^{\mathrm{in}})\right|\psi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\phi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\mathrm{d}v\to 0 \qquad \text{as } \delta\to 0 \text{ uniformly in } x.$$

Therefore, for each $x\in\overline{\Omega}$,

$$\lim_{\delta\to 0}G_{\epsilon,\delta}(x) = H_\epsilon(x,v^{\mathrm{in}})\left(\frac{1}{\delta}\lim_{\delta\to 0}\int_{\mathbb{S}^1_{x,+}}\psi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\phi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\mathrm{d}v\right)\to C_{\psi_0,\phi_0}H_\epsilon(x,v^{\mathrm{in}}),$$

where the constant C_{ψ_0,ϕ_0} is given by

$$C_{\psi_0,\phi_0} = \frac{1}{\delta}\lim_{\delta\to 0}\int_{\mathbb{S}^1_{x,+}}\psi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\phi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\mathrm{d}v = \int_{\mathbb{R}}\phi_0(r)\psi_0(r)\mathrm{d}r.$$

Applying the Lebesgue Dominated Convergence theorem we obtain

$$\begin{aligned}
&\lim_{\delta\to 0} M_\psi(f_1)\\
&= \frac{C_{\psi_0,\phi_0}}{\epsilon}\int_{\partial\Omega}\psi_0(x-x^{\text{out}})\,H_\epsilon(x,v^{\text{in}})\mathrm{d}S_x\\
&= \frac{C_{\psi_0,\phi_0}}{\epsilon}\int_{\partial\Omega}\psi_0(x-x^{\text{out}})\,\mathrm{e}^{-\int_0^{\tau_-(x,v^{\text{in}})}\sigma_a(x-sv^{\text{in}})\mathrm{d}s}\phi_0\left(\frac{\left|x-\tau_-(x,v^{\text{in}})v^{\text{in}}-x^{\text{in}}\right|}{\epsilon}\right)n(x)\cdot v^{\text{in}}\mathrm{d}S_x.
\end{aligned}$$

Furthermore if we make the change of variables using

$$y = x_-(x,v^{\text{in}}) = x-\tau_-(x,v^{\text{in}})v^{\text{in}},$$

then by the non-degeneracy in (2.17), the mapping is invertible and we claim that

$$\left|n(y)\cdot v^{\text{in}}\right|\mathrm{d}S_y = -n(y)\cdot v^{\text{in}}\mathrm{d}S_y = \left|n(x)\cdot v^{\text{in}}\right|\mathrm{d}S_x = n(x)\cdot v^{\text{in}}\mathrm{d}S_x. \tag{2.18}$$

This relation can be justified through the physical meanings of $\left|n(x)\cdot v^{\text{in}}\right|\mathrm{d}S_x$ and $\left|n(y)\cdot v^{\text{in}}\right|\mathrm{d}S_y$ as the effective fluxes into and out of the boundary. The mathematical proof for (2.18) is given in appendix B. Making such change of variables, we obtain that

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0} M_\psi(f_1) \tag{2.19}$$

$$\begin{aligned}
&= \lim_{\epsilon\to 0}\frac{C_{\psi_0,\phi_0}}{\epsilon}\int_{\partial\Omega}\psi_0(x(y)-x^{\text{out}})\,\mathrm{e}^{-\int_0^{\tau_-(x(y),v^{\text{in}})}\sigma_a(x(y)-sv^{\text{in}})\mathrm{d}s}\phi_0\left(\frac{\left|y-x^{\text{in}}\right|}{\epsilon}\right)\left|n(y)\cdot v^{\text{in}}\right|\mathrm{d}S_y\\
&= C_{\psi_0,\phi_0}\left|n(x^{\text{in}})\cdot v^{\text{in}}\right|\mathrm{e}^{-\int_0^{\tau_-(x^{\text{out}},v^{\text{in}})}\sigma_a(x^{\text{out}}-sv^{\text{in}})\mathrm{d}s}\\
&= C_{\psi_0,\phi_0}\left|n(x^{\text{in}})\cdot v^{\text{out}}\right|\mathrm{e}^{-\int_0^{\tau_-(x^{\text{out}},v^{\text{out}})}\sigma_a(x^{\text{out}}-sv^{\text{out}})\mathrm{d}s},
\end{aligned} \tag{2.20}$$

where we have applied the differential relation $\mathrm{d}S_y = \mathrm{d}|y-x^{\text{in}}|$ and $\psi_0(0)=1$. Note that the last term involves the x-ray transformation of σ_a.

Step 2: limit of $M_\psi(f_2)$ By (2.11), the contribution of f_2 toward the measurement is

$$M_\psi(f_2) = \int_{\mathbb{S}^1}\int_{\partial\Omega_v^+}\int_0^{\tau_-(x,v)}\psi(x,v)\mathrm{e}^{-\int_0^s\sigma_a(x-\tau v)\,\mathrm{d}\tau}F_f(x-sv)n(x)\cdot v\mathrm{d}s\mathrm{d}S_x\mathrm{d}v. \tag{2.21}$$

Make a change of variables in the above integral with

$$y = x-sv, \qquad \text{for} \quad x\in\partial\Omega_v^+ \quad \text{and} \quad s\in(0,\tau_-(x,v)). \tag{2.22}$$

Note that $y\to(x,s)$ is a one-to-one mapping with the relation (verified in appendix B)

$$\mathrm{d}y = n(x)\cdot v\mathrm{d}s\mathrm{d}S_x \tag{2.23}$$

and the inverse map is

$$s=\tau_+(y,v), \qquad x = y+\tau_+(y,v)v.$$

Hence one can rewrite the integral in (2.21) as

$$M_\psi(f_2) = \int_{\mathbb{S}^1}\int_\Omega \psi(y+\tau_+(y,v)v, v)\mathrm{e}^{-\int_0^{\tau_+(y,v)} \sigma_a(y+\tau_+(y,v)v-\tau v)\,\mathrm{d}\tau} F_f(y)\mathrm{d}y\mathrm{d}v.$$

With the definition of ψ and the bound for F_f in (1.4), we obtain that

$$\begin{aligned}|M_\psi(f_2)| &\leqslant \int_{\mathbb{S}^1}\int_\Omega \psi_0\left(\frac{v-v^{\mathrm{in}}}{\delta}\right)|F_f(y)|\,\mathrm{d}y\mathrm{d}v\\ &= \left(\int_{\mathbb{S}^1}\psi_0\left(\frac{v-v^{\mathrm{in}}}{\delta}\right)\mathrm{d}v\right)\left(\int_\Omega |F_f(y)|\,\mathrm{d}y\right)\\ &\leqslant C_\Omega\delta\,||F_f\,||_{L^p(\Omega)} \leqslant C_{\Omega,F}\delta\,||\phi\,||_{L^p(\Gamma_-)}\end{aligned} \tag{2.24}$$

where the constant $C_{\Omega,F}$ is independent of ϕ. The L^p-norm of ϕ can be estimated using its definition:

$$\begin{aligned}||\phi\,||^p_{L^p(\Gamma_-)} &= \int\int_{\Gamma_-}\phi^p(x,v)\,|n(x)\cdot v|\,\mathrm{d}S_x\mathrm{d}v\\ &= \frac{1}{\epsilon^p\delta^p}\int\int_{\Gamma_-}\phi_0^p\left(\frac{|x-x^{\mathrm{in}}|}{\epsilon}\right)\phi_0^p\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)|n(x)\cdot v|\,\mathrm{d}S_x\mathrm{d}v\\ &\leqslant \frac{1}{\epsilon^p\delta^p}\left(\int_{\partial\Omega}\phi_0\left(\frac{|x-x^{\mathrm{in}}|}{\epsilon}\right)\mathrm{d}S_x\right)\left(\int_{\mathbb{S}^1}\phi_0\left(\frac{|v-v^{\mathrm{in}}|}{\delta}\right)\mathrm{d}v\right)\\ &\leqslant C_{\phi_0}\epsilon^{-(p-1)}\delta^{-(p-1)}.\end{aligned}$$

Plugging such bound back in (2.24) we obtain

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}|M_\psi(f_2)| \leqslant C_{\Omega,\phi_0}\lim_{\epsilon\to 0}\lim_{\delta\to 0}\epsilon^{-\frac{p-1}{p}}\delta^{\frac{1}{p}} = 0. \tag{2.25}$$

Finally, by combining (2.20) and (2.25) we have

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}M_\psi(f) = \iint_{\Gamma_+}\psi(x,v)f(x,v)\,\mathrm{d}\Gamma_+ = C_{\phi_0,\psi_0}\left|n(x^{\mathrm{in}})\cdot v^{\mathrm{in}}\right|\mathrm{e}^{-\int_0^{\tau_-(x^{\mathrm{out}},v^{\mathrm{out}})}\sigma_a(x^{\mathrm{out}}-sv^{\mathrm{out}})\mathrm{d}s}.$$

Therefore, the x-ray transformation of σ_a is uniquely determined by the measurement, which in turn implies that σ_a is uniquely recoverable by the measurement. Moreover, since C_{ϕ_0,ψ_0} is independent of the particular form of F_f and the step in showing $M_\psi(f_2)$ only relies on the bound (1.4), we conclude that σ_a is also independent of F_f as long as (1.4) holds. □

2.1. Examples

Theorem 1.1 is rather general and one only needs to verify two conditions in order to apply it: the well-posedness of the forward problem and the bound (1.4). For many kinetic equations these conditions follow from energy methods. Below we give two examples.

The first example is the classical linear RTE with $F_f = \sigma_s\langle f\rangle$ and the equation reads

$$v\cdot\nabla_x f = -\sigma_a f + \sigma_s\langle f\rangle\,. \tag{2.26}$$

The statement of the unique solvability of σ_a is

Theorem 2.1. *Suppose Ω is a strictly convex and bounded domain with a C^2 boundary. Suppose there exists a constant $\sigma_0 > 0$ such that*

$$\sigma_a \in C(\overline{\Omega}), \qquad \sigma_a \geqslant \sigma_s \geqslant \sigma_0 > 0.$$

Then with proper choices of the incoming data, the absorption coefficient σ_a can be uniquely recovered from the measurement of the outgoing data.

This is the example studied in the original singular decomposition work [16] where the subcritical case with $\sigma_a - \sigma_s > 0$ is considered. We are now able to treat the critical and subcritical cases with $\sigma_a \geqslant \sigma_s$ in a unified way.

Proof. Let ϕ be a nonnegative incoming data such that $\phi \in L^2(\Gamma_-)$. Then the positivity of f follows from the maximum principle of the linear RTE and the unique solvability is classical [18]. In equation (2.26) we have $F_f(x) = \sigma_s \langle f \rangle$. To obtain an L^2-bound of F_f, multiply (2.26) by $2f$ and integrate in (x, v). This gives

$$\int_\Omega \int_{\mathbb{S}^1} v \cdot \nabla_x f^2 = -2 \int_\Omega \int_{\mathbb{S}^1} \sigma_s \left(f - \langle f \rangle\right)^2 - 2 \int_\Omega \int_{\mathbb{S}^1} (\sigma_a - \sigma_s) f^2 \leqslant -2\sigma_0 \, ||f - \langle f \rangle\,||^2_{L^2(\Omega \times \mathbb{S}^1)}.$$

By integration by parts, the left-hand satisfies

$$\int_\Omega \int_{\mathbb{S}^1} v \cdot \nabla_x f^2 = \int_{\partial\Omega} \int_{\mathbb{S}^1} (n(x) \cdot v) f^2 \geqslant \iint_{\Gamma_-} (n(x) \cdot v) f^2 = -\,||\phi\,||^2_{L^2(\Gamma_-)}.$$

Combining the above two inequalities we have

$$||f - \langle f \rangle\,||^2_{L^2(\Omega \times \mathbb{S}^1)} \leqslant \frac{1}{2\sigma_0}\, ||\phi\,||^2_{L^2(\Gamma_-)}. \tag{2.27}$$

Denote $g = f - \langle f \rangle$. Since $f \geqslant 0$, we have

$$v \cdot \nabla f \leqslant -\sigma_s g. \tag{2.28}$$

Solving along charateristics, we have

$$f(x + tv, v) \leqslant \phi(x, v) - \int_0^t \sigma_s(x + \tau v) g(x + \tau v, v)\, \mathrm{d}\tau, \qquad (x, v) \in \Gamma_-, \;\; t \in [0, \tau_+(x, v)].$$

Hence, for any $(x, v) \in \Gamma_-$ and $t \in [0, \tau_+(x, v)]$, it holds that

$$\begin{aligned} f^2(x + tv, v) &\leqslant 2\phi^2(x, v) + 2 \left(\int_0^{\tau_+(x,v)} \sigma_s(x + \tau v)\, |g(x + \tau v, v)|\, \mathrm{d}\tau \right)^2 \\ &\leqslant 2\phi^2(x, v) + 2\,(\mathrm{diam}(\Omega))\, ||\sigma_s\,||^2_{L^\infty(\Omega)} \int_0^{\tau_+(x,v)} g^2(x + \tau v, v)\, \mathrm{d}\tau. \end{aligned}$$

Integrating in $(x, v) \in \Gamma_-$ and $t \in [0, \tau_+(x, v)]$, we obtain that

$$\int\int_{\Gamma_-}\int_0^{\tau_+(x,v)} f^2(x+tv, v)\mathrm{d}t\mathrm{d}\Gamma_- \leqslant 2\,(\mathrm{diam}(\Omega))\,||\phi\,||^2_{L^2(\Gamma_-)} + 2\,(\mathrm{diam}(\Omega))^2\,||\sigma_s\,||^2_{L^\infty(\Omega)}\int\int_{\Gamma_-}\int_0^{\tau_+(x,v)} g^2(x+\tau v, v)\,\mathrm{d}\tau\mathrm{d}\Gamma_-.$$

Using a similar changing of variables as in (2.22) by letting $z = x + \tau v$, we then derive that

$$||f\,||^2_{L^2(\Omega\times\mathbb{S}^1)} \leqslant C_\Omega\,||\phi\,||^2_{L^2(\Gamma_-)} + C_\Omega\,||g\,||^2_{L^2(\Omega\times\mathbb{S}^1)} \leqslant C_\Omega\,||\phi\,||^2_{L^2(\Gamma_-)},$$

where the last inequality follows from (2.27). Hence, we derive that

$$||F_f\,||_{L^2(\Omega\times\mathbb{S}^1)} = ||\sigma_s\,\langle f\rangle\,||_{L^2(\Omega\times\mathbb{S}^1)} \leqslant ||\sigma_s\,||_{L^\infty(\Omega)}\,||f\,||_{L^2(\Omega\times\mathbb{S}^1)} \leqslant C_\Omega\,||\phi\,||_{L^2(\Gamma_-)},$$

which combined with theorem 1.1 gives the desired unique solvability of σ_a. □

In the second example we consider a nonlinear RTE, which couples the temperature and the intensity of the rays. The equation has the form [26]:

$$\begin{aligned} v\cdot\nabla_x I &= -\sigma_a I + \sigma_a T^4, \\ \Delta_x T &= \sigma_a T^4 - \sigma_a\,\langle I\rangle\,. \end{aligned}$$

In our proof of reconstruction, we will use an incoming condition for I and a zero-boundary condition for T. The final system reads

$$v\cdot\nabla_x I = -\sigma_a I + \sigma_a T^4, \qquad I\big|_{\Gamma_-} = \phi(x, v), \tag{2.29}$$

$$\Delta_x T = \sigma_a T^4 - \sigma_a\,\langle I\rangle\,. \qquad T\big|_{\partial\Omega} = 0. \tag{2.30}$$

For a given I, the temperature T is uniquely determined in terms of $\langle I\rangle$ by solving (2.30). This defines a valid functional T on $\langle I\rangle$.

The statement of the unique solvability of σ_a in (2.29) and (2.30) is

Theorem 2.2. *Suppose Ω is a strictly convex and bounded domain with a C^2 boundary. Suppose there exists a constant $\sigma_0 > 0$ such that*

$$\sigma_a \geqslant \sigma_0 > 0, \qquad \sigma_a \in C(\overline{\Omega}),$$

Then with proper choices of the incoming data, the absorption coefficient σ_a can be uniquely recovered from the measurement of the outgoing data.

Proof. Given an incoming data ϕ for I and a zero boundary condition T_B for T, we show the well-posedness of (2.29) and (2.30) in appendix A. The non-negativity of I follows directly from the observation that $\sigma_a T^4 \geqslant 0$. Now we have $F_f = \sigma_a T^4$ and we want to show that there exists a constant C_0 such that

$$\left|\left|\sigma_a T^4\,\right|\right|_{L^2(\Omega)} \leqslant C_0\,||\phi\,||_{L^2(\Gamma_-)}. \tag{2.31}$$

Such L^2-bound can be obtained by the energy method along a similar line as in [26]. For the convenience of the reader we include the details here. The full equation with the boundary conditions reads

$$v \cdot \nabla_x I = -\sigma_a I + \sigma_a T^4, \qquad I\big|_{\Gamma_-} = \phi(x, v), \tag{2.32}$$

$$\Delta_x T = \sigma_a T^4 - \sigma_a \langle I \rangle, \qquad T\big|_{\partial\Omega} = 0. \tag{2.33}$$

Multiply (2.32) by I and (2.33) by T^4. Then integrate both equations in (x, v) and take their difference. By rearranging terms we get

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega}\int_{\mathbb{S}^1}(n(x)\cdot v)I^2 + 4\int_\Omega T^3\left|\nabla_x T\right|^2 &= -\int_\Omega\int_{\mathbb{S}^1}\sigma_a I^2 + 2\int_\Omega\int_{\mathbb{S}^1}\sigma_a \langle I\rangle T^4 - \int_\Omega\int_{\mathbb{S}^1}\sigma_a T^8.\\ &= -\int_\Omega\int_{\mathbb{S}^1}\sigma_a\left(\langle I\rangle - T^4\right)^2 - \int_\Omega\int_{\mathbb{S}^1}\sigma_a\left(I^2 - \langle I\rangle^2\right).\end{aligned}$$

where it has been shown in theorem A.1 that $T \geqslant 0$ given ϕ non-negative. Dropping the term involving T^3, we have

$$\sigma_0 \left|\left| \langle I\rangle - T^4 \right|\right|^2_{L^2(\Omega)} \leqslant \int_\Omega\int_{\mathbb{S}^1}\sigma_a\left(\langle I\rangle - T^4\right)^2 \leqslant \frac{1}{2}\left|\left|\phi\right|\right|^2_{\Gamma_-} \tag{2.34}$$

and

$$\sigma_0 \left|\left|I - \langle I\rangle\right|\right|^2_{L^2(\Omega)} \leqslant \int_\Omega\int_{\mathbb{S}^1}\sigma_a\left(I - \langle I\rangle\right)^2 = \int_\Omega\int_{\mathbb{S}^1}\sigma_a\left(I^2 - \langle I\rangle^2\right) \leqslant \frac{1}{2}\left|\left|\phi\right|\right|^2_{\Gamma_-}. \tag{2.35}$$

Combining (2.34) with (2.35), we obtain that

$$\left|\left|\sigma_a(I - T^4)\right|\right|_{L^2(\Omega\times\mathbb{S}^1)} \leqslant \left|\left|\sigma_a\right|\right|_{L^\infty}\left(\left|\left|I - \langle I\rangle\right|\right|_{L^2(\Omega\times\mathbb{S}^1)} + \left|\left|\langle I\rangle - T^4\right|\right|_{L^2(\Omega\times\mathbb{S}^1)}\right) \leqslant \frac{\left|\left|\sigma_a\right|\right|_{L^\infty}}{\sigma_0}\left|\left|\phi\right|\right|^2_{\Gamma_-}.$$

Since $\sigma_a(T^4 - I)$ is simply the forcing term in (2.32), we can apply the bound for (2.28) to derive that

$$\left|\left|I\right|\right|_{L^2(\Omega\times\mathbb{S}^1)} \leqslant C_\Omega\left|\left|\phi\right|\right|^2_{\Gamma_-}.$$

This implies that

$$\left|\left|\sigma_a T^4\right|\right|_{L^2(\Omega)} \leqslant \left|\left|\sigma_a\right|\right|_{L^\infty}\left(\left|\left|I\right|\right|_{L^2(\Omega\times\mathbb{S}^1)} + \left|\left|I - T^4\right|\right|_{L^2(\Omega\times\mathbb{S}^1)}\right) \leqslant C_\Omega(\left|\left|\sigma_a\right|\right|_{L^\infty}, \sigma_0)\left|\left|\phi\right|\right|^2_{\Gamma_-},$$

which is the desired bound in (2.31). The unique solvability of σ_a then follows from theorem 1.1. □

3. Recovery of the scattering coefficient: averaging lemma

In this section, we show how to use the celebrated averaging lemma for kinetic equations to recover the scattering coefficient. We will work out a specific example as an illustration. The equation under consideration is (2.26), which we recall as

$$v \cdot \nabla_x f = -\sigma_a f + \sigma_s \langle f\rangle. \tag{3.1}$$

Since σ_a has been found by theorem 2.1, in what follows we assume that σ_a is given and focus on finding σ_s.

First we recall the statement of the averaging lemma. For the purpose of the current work, we only need the most basic version which is stated as

Theorem 3.1 ([11, 19, 23]). *Suppose* $0 < \sigma_0 \leqslant \sigma_s \leqslant \sigma_a$ *with* $\sigma_a \in C(\overline{\Omega})$ *where* Ω *is open and bounded. Suppose* $\phi \in L^p(\Gamma_-)$ *and* $g \in L^p(\Omega \times \mathbb{S}^1)$ *for some* $p > 1$ *and* f *satisfies the equation*

$$v \cdot \nabla_x f = -\sigma_a f + \sigma_s \langle f \rangle + g, \qquad f\big|_{\Gamma_-} = \phi(x, v). \tag{3.2}$$

Then for any $\gamma \leqslant \inf\{\frac{1}{p}, 1 - \frac{1}{p}\}$, *the velocity average of* f *satisfies* $\langle f \rangle \in W^{\gamma,p}(\Omega)$ *with the bound*

$$|| \langle f \rangle ||_{W^{\gamma,p}(\Omega)} \leqslant C_0 \left(||\phi||_{L^p(\Gamma_-)} + ||g||_{L^p(\Omega\times\mathbb{S}^1)} \right).$$

We also recall the basic L^p energy estimate [20] for equation (3.2):

Theorem 3.2 ([20]). *Suppose* $\phi \in L^p(\Gamma_-)$ *and* $g \in L^p(\Omega \times \mathbb{S}^1)$ *for some* $p \in [1, \infty]$. *Then* $f \in L^p(\Omega \times \mathbb{S}^1)$ *with the bound*

$$||f||_{L^p(\Omega\times\mathbb{S}^1)} \leqslant C_0 \left(||\phi||_{L^p(\Gamma_-)} + ||g||_{L^p(\Omega\times\mathbb{S}^1)} \right).$$

Our main result in this part is

Theorem 3.3. *Let* $\Omega \subseteq \mathbb{R}^2$ *be a strictly convex and bounded domain with a* C^2 *boundary. Suppose* $0 < \sigma_0 \leqslant \sigma_s \leqslant \sigma_a$ *with* $\sigma_a \in C(\overline{\Omega})$ *given. Then with proper choices of the incoming data, the scattering coefficient* σ_s *in (3.1) can be uniquely recovered from the measurement of the outgoing data.*

Proof. For any given ϕ, let f be the solution to (3.2). Decompose it into three parts: $f = f_1 + f_2 + f_3$, where

$$v \cdot \nabla_x f_1 = -\sigma_a f_1, \qquad f_1\big|_{\Gamma_-} = \phi(x, v), \tag{3.3}$$

$$v \cdot \nabla_x f_2 = -\sigma_a f_2 + \sigma_s \langle f_1 \rangle, \qquad f_2\big|_{\Gamma_-} = 0, \tag{3.4}$$

$$v \cdot \nabla_x f_3 = -\sigma_a f_3 + \sigma_s \langle f_2 \rangle + \sigma_s \langle f_3 \rangle, \qquad f_3\big|_{\Gamma_-} = 0. \tag{3.5}$$

Note that given σ_a, σ_s, the first two functions f_1, f_2 are explicitly solvable. The idea of the proof is to show f_3 is more regular than f_2, which in turn more regular than f_1, using the averaging lemma. By posing the correct geometry for the incoming and measuring functions, one can show f_2 dominates the data, and is used to reconstruct σ_s.

Incoming and Measurement First we need to specify the incoming data ϕ and the measurement function ψ. Fix $(x^{\text{in}}, v^{\text{in}}) \in \Gamma_-$ and $(x^{\text{out}}, v^{\text{out}}) \in \Gamma_+$ such that

$$v^{\text{in}} \nparallel v^{\text{out}}, \qquad v^{\text{in}} \cdot v^{\text{out}} > 0. \tag{3.6}$$

Let ℓ_1 be the ray initiated at x^{in} along the direction v^{in} and ℓ_2 the ray initiated at x^{out} along the direction $-v^{\text{out}}$. Since $v^{\text{in}} \nparallel v^{\text{out}}$, the two rays ℓ_1 and ℓ_2 have a unique intersection inside Ω, which we denote as x_0. For later use, let $s_0 > 0$ be the exit time associated with x_0 in the direc-

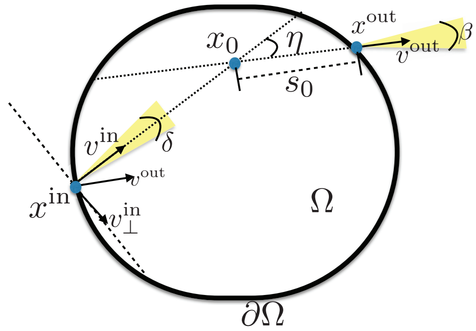

Figure 1. Geometry and some physical quantities.

tion of v^{out}, or more explicitly,

$$x_0 = x^{\text{out}} - s_0 v^{\text{out}} = x^{\text{in}} + s_0' v^{\text{in}}. \tag{3.7}$$

The main goal is to find $\sigma_s(x_0)$. Define $v_\perp^{\text{in}}$ as the unit vector such that

$$v_\perp^{\text{in}} \cdot v^{\text{in}} = 0, \quad \text{and} \quad \eta := v_\perp^{\text{in}} \cdot v^{\text{out}} > 0. \tag{3.8}$$

For the illustration of the geometry, see figure 1.

Let ϕ_0 be a smooth even function on $\mathbb{R}$ such that

$$0 \leqslant \phi_0(r) \leqslant 1, \qquad \overline{\text{Supp}\phi_0} = [-1,1], \qquad \phi_0(0) = 1, \qquad \int_{\mathbb{R}} \phi_0(r)\mathrm{d}r = 1.$$

Let ψ_0 be the same smooth function defined in (2.9) with $\overline{\text{Supp}\psi_0} = [-1,1]$. We choose the incoming data ϕ and the measurement function ψ as

$$\phi(x,v) = \frac{1}{\epsilon\delta}\phi_0\left(\frac{(x-x^{\text{in}})\cdot v_\perp^{\text{in}}}{\epsilon\eta}\right)\phi_0\left(\frac{|v-v^{\text{in}}|}{\delta}\right), \qquad (x,v)\in\Gamma_-,$$
$$\psi(x,v) = \frac{1}{\theta\beta}\psi_0\left(\frac{|x-x^{\text{out}}|}{\theta}\right)\psi_0\left(\frac{|v-v^{\text{out}}|}{\beta}\right), \qquad (x,v)\in\Gamma_+.$$

Quickly, we have

$$\mathcal{M}_\psi(f) = \int\int_{\Gamma_-} \psi f \mathrm{d}\Gamma_- = \mathcal{M}_\psi(f_1) + \mathcal{M}_\psi(f_2) + \mathcal{M}_\psi(f_3).$$

The essence of the proof is to show that $\mathcal{M}_\psi(f_1)$ and $\mathcal{M}_\psi(f_3)$ are negligible while $\mathcal{M}_\psi(f_2)$ is used to reconstruct $\sigma_s(x_0)$. The estimate for $\mathcal{M}_\psi(f_3)$ relies on the averaging lemma, and the estimate for $\mathcal{M}_\psi(f_1)$ follows from a basic geometric argument.

As a preparation, we first give an estimate of L^r-bound of ϕ (with r to be determined later):

$$\begin{aligned}\int\int_{\Gamma_-}\phi^r(x,v)|v\cdot n|\mathrm{d}S_x\mathrm{d}v &= \frac{1}{\epsilon^r\delta^r}\int_{\partial\Omega}\int_{\mathbb{S}^1_{x,+}}\phi_0^r\left(\frac{(x-x^{\text{in}})\cdot v_\perp^{\text{in}}}{\epsilon\eta}\right)\phi_0^r\left(\frac{|v-v^{\text{in}}|}{\delta}\right)|v\cdot n|\mathrm{d}S_x\mathrm{d}v\\ &\leqslant \left(\frac{1}{\delta^r}\int_{\mathbb{S}^1}\phi_0^r\left(\frac{|v-v^{\text{in}}|}{\delta}\right)\mathrm{d}v\right)\left(\frac{1}{\epsilon^r}\int_{\partial\Omega}\phi_0^r\left(\frac{(x-x^{\text{in}})\cdot v_\perp^{\text{in}}}{\epsilon\eta}\right)\mathrm{d}S_x\right),\end{aligned}$$

where the v-integral is bounded as

$$\frac{1}{\delta^r}\int_{\mathbb{S}^1}\phi_0^r\left(\frac{|v-v^{\rm in}|}{\delta}\right)\mathrm{d}v=\frac{1}{\delta^r}\int_0^{2\pi}\phi_0^r\left(\frac{|\sin\omega/2|}{\delta/2}\right)\mathrm{d}\omega\leqslant\frac{1}{\delta^r}\int_0^{2\pi}\phi_0\left(\frac{|\sin\omega/2|}{\delta/2}\right)\mathrm{d}\omega\leqslant c_0\delta^{-(r-1)}.$$

In order to estimate the boundary integral, we take $v_\perp^{\rm in}$ as the horizontal axis and take $\epsilon\eta$ small enough such that $\partial\Omega$ is a graph parametrized by

$$x_2=f(x_1),\qquad x_1\in(x_1^{\rm in}-\epsilon\eta,\ x_1^{\rm in}+\epsilon\eta),\qquad x=(x_1,x_2),$$

where $f\in C^1[x_1^{\rm in}-h_0,\ x_1^{\rm in}+h_0]$ for some fixed h_0. Then the boundary integral satisfies

$$\begin{aligned}\frac{1}{\epsilon^r}\int_{\partial\Omega}\phi_0^r\left(\frac{(x-x^{\rm in})\cdot v_\perp^{\rm in}}{\epsilon\eta}\right)\mathrm{d}S_x&=\frac{1}{\epsilon^r}\int_{x_1^{\rm in}-\epsilon\eta}^{x_1^{\rm in}+\epsilon\eta}\phi_0^r\left(\frac{x_1-x_1^{\rm in}}{\epsilon\eta}\right)\sqrt{1+|f'(x_1)|^2}\mathrm{d}x_1\\&\leqslant\frac{c_0}{\epsilon^r}\int_{x_1^{\rm in}-\epsilon\eta}^{x_1^{\rm in}+\epsilon\eta}\phi_0\left(\frac{x_1-x_1^{\rm in}}{\epsilon\eta}\right)\mathrm{d}x_1\leqslant c_0\epsilon^{-(r-1)}\eta,\end{aligned}$$

where c_0 depends on the C^1-norm of f, which is assumed to be bounded since $\partial\Omega$ is C^1. Note that such bound is independent of $x^{\rm in}$ since $\partial\Omega$ is compact. Combining the two integrals, we have

$$||\phi||_{L^r(\Gamma_-)}\leqslant\left(\int\int_{\Gamma_-}\phi^r(x,v)|v\cdot n|\mathrm{d}S_x\mathrm{d}v\right)^{1/r}\leqslant c_0\epsilon^{-\frac{r-1}{r}}\delta^{-\frac{r-1}{r}}\eta^{\frac{1}{r}},\qquad r>1.$$

Averaging lemma Now we apply the L^r-energy bound and the averaging lemma to obtain a bound for $\langle f_1\rangle$, $\langle f_2\rangle$, and f_3. First, a direct application of theorem 3.1 gives

$$||\langle f_1\rangle||_{W^{s_0,r}}\leqslant c_0||\phi||_{L^r(\Gamma_-)}\leqslant c_0\epsilon^{-\frac{r-1}{r}}\delta^{-\frac{r-1}{r}}\eta^{\frac{1}{r}},$$

where $s_0=\inf\{\frac{1}{r},1-\frac{1}{r}\}$. By the Sobolev embedding, we have

$$||\langle f_1\rangle||_{L^{p_1}(\Omega)}\leqslant c_0||\langle f_1\rangle||_{W^{s_0,r}}\leqslant c_0\,\epsilon^{-\frac{r-1}{r}}\delta^{-\frac{r-1}{r}}\eta^{\frac{1}{r}},\qquad\forall p_1\leqslant\frac{1}{\frac{1}{r}-\frac{s_0}{2}}.$$

Since $\langle f_1\rangle$ is the source term in the equation for f_2, we apply the averaging lemma again and get

$$||\langle f_2\rangle||_{L^{p_2}(\Omega)}\leqslant c_0||\langle f_2\rangle||_{W^{s_1,p_1}(\Omega)}\leqslant c_0||\langle f_1\rangle||_{L^{p_1}(\Omega\times\mathbb{S}^1)}\leqslant c_0\epsilon^{-\frac{r-1}{r}}\delta^{-\frac{r-1}{r}}\eta^{\frac{1}{r}},\tag{3.9}$$

where the exponents satisfy that

$$s_1=\inf\left\{\frac{1}{p_1},\ 1-\frac{1}{p_1}\right\},\qquad p_2\leqslant\frac{1}{\frac{1}{p_1}-\frac{s_1}{2}}.$$

By theorem 3.2, we also have

$$||\langle f_3\rangle||_{L^{p_2}(\Omega)}\leqslant||f_3||_{L^{p_2}(\Omega\times\mathbb{S}^1)}\leqslant c_0||\langle f_2\rangle||_{L^{p_2}(\Omega)}.\tag{3.10}$$

Contribution from f_3 Using the change of variables in (2.23), we obtain the contribution of f_3 to the measurement of the outgoing data as

$$
\begin{aligned}
\left|\int\int_{\Gamma_+}\psi(x,v)f_3(x,v)\,\mathrm{d}\Gamma_+\right| &\leqslant ||\sigma_s||_{L^\infty}\int_{\mathbb{S}^1}\int_\Omega \psi(y+\tau_+(y,v)v,v)\,|\langle f_2\rangle(y)+\langle f_3\rangle(y)|\,\mathrm{d}y\mathrm{d}v\\
&\leqslant c_0\int_{\mathbb{S}^1}\int_\Omega \frac{1}{\theta\beta}\psi_0\left(\frac{|y+\tau_+(y,v)v-x^{\text{out}}|}{\theta}\right)\psi_0\left(\frac{|v-v^{\text{out}}|}{\beta}\right)|\langle f_2\rangle(y)+\langle f_3\rangle(y)|\,\mathrm{d}y\mathrm{d}v\\
&= c_0\int_\Omega\left(\int_{\mathbb{S}^1}\frac{1}{\theta\beta}\psi_0\left(\frac{|y+\tau_+(y,v)v-x^{\text{out}}|}{\theta}\right)\psi_0\left(\frac{|v-v^{\text{out}}|}{\beta}\right)\mathrm{d}v\right)|\langle f_2\rangle(y)+\langle f_3\rangle(y)|\,\mathrm{d}y\\
&\leqslant c_0\underbrace{\left(\int_\Omega\left(\int_{\mathbb{S}^1}\frac{1}{\theta\beta}\psi_0\left(\frac{|y+\tau_+(y,v)v-x^{\text{out}}|}{\theta}\right)\psi_0\left(\frac{|v-v^{\text{out}}|}{\beta}\right)\mathrm{d}v\right)^{p_2'}\mathrm{d}y\right)^{\frac{1}{p_2'}}}_{T}||\langle f_2\rangle||_{L^{p_2}(\Omega)}.
\end{aligned}
$$

where $\frac{1}{p_2'}+\frac{1}{p_2}=1$ and the last step follows from Hölder inequality and (3.10). The factor T is estimated as follows.

$$
\begin{aligned}
T^{p_2'} &= \int_\Omega\left(\int_{\mathbb{S}^1}\frac{1}{\theta\beta}\psi_0\left(\frac{|y+\tau_+(y,v)v-x^{\text{out}}|}{\theta}\right)\psi_0\left(\frac{|v-v^{\text{out}}|}{\beta}\right)\mathrm{d}v\right)^{p_2'}\mathrm{d}y\\
&\leqslant \left(\int_{\mathbb{S}^1}\left(\int_\Omega\frac{1}{\theta^{p_2'}}\psi_0^{p_2'}\left(\frac{|y+\tau_+(y,v)v-x^{\text{out}}|}{\theta}\right)\mathrm{d}y\right)\frac{1}{\beta}\psi_0\left(\frac{|v-v^{\text{out}}|}{\beta}\right)\mathrm{d}v\right)\left(\int_{\mathbb{S}^1}\frac{1}{\beta}\psi_0\left(\frac{|v-v^{\text{out}}|}{\beta}\right)\mathrm{d}v\right)^{\frac{p_2'}{p_2}}\\
&\leqslant c_0\int_{\mathbb{S}^1}\underbrace{\left(\int_\Omega\frac{1}{\theta^{p_2'}}\psi_0^{p_2'}\left(\frac{|y+\tau_+(y,v)v-x^{\text{out}}|}{\theta}\right)\mathrm{d}y\right)}_{T_1}\frac{1}{\beta}\psi_0\left(\frac{|v-v^{\text{out}}|}{\beta}\right)\mathrm{d}v.
\end{aligned}
$$

For each $v\in\mathbb{S}^1$, if we apply the change of variables

$$x = y+\tau_+(y,v)v\in\partial\Omega_v^+,$$

with $\partial\Omega_v^+$ defined in (2.12), then T_1 satisfies

$$T_1=\int_{\partial\Omega_v^+}\int_0^{\tau_-(x,v)}\frac{1}{\theta^{p_2'}}\psi_0^{p_2'}\left(\frac{|x-x^{\text{out}}|}{\theta}\right)\mathrm{d}s\mathrm{d}x\leqslant(\mathrm{diam}(\Omega))\int_{\partial\Omega_v^+}\frac{1}{\theta^{p_2'}}\psi_0^{p_2'}\left(\frac{|x-x^{\text{out}}|}{\theta}\right)\mathrm{d}x\leqslant c_0\theta^{-(p_2'-1)}.$$

Therefore, T is uniformly bounded in v with the bound

$$T^{p_2'}\leqslant c_0\theta^{-(p_2'-1)}\int_{\mathbb{S}^1}\frac{1}{\beta}\psi_0\left(\frac{|v-v^{\text{out}}|}{\beta}\right)\mathrm{d}v\leqslant c_0\theta^{-(p_2'-1)}.$$

Inserting the estimate for T back into $\mathcal{M}_\psi(f_3)$ and using (3.9) and (3.10), we have

$$|\mathcal{M}_\psi(f_3)|=\left|\int\int_{\Gamma_+}\psi(x,v)f_3(x,v)\,\mathrm{d}\Gamma_+\right|\leqslant c_0\theta^{-\frac{p_2'-1}{p_2'}}\epsilon^{-\frac{r-1}{r}}\delta^{-\frac{r-1}{r}}\eta^{\frac{1}{r}}=c_0\theta^{-\frac{1}{p_2}}\epsilon^{-\frac{r-1}{r}}\delta^{-\frac{r-1}{r}}\eta^{\frac{1}{r}}.$$

We will choose the parameter properly later to make $\mathcal{M}_\psi(f_3)$ a negligible term, namely, we will choose parameters so that

$$\theta^{-\frac{1}{p_2}}\epsilon^{-\frac{r-1}{r}}\delta^{-\frac{r-1}{r}}\eta^{\frac{1}{r}}\ll 1 \tag{3.11}$$

Contribution from f_1 We show in this part that by properly choosing the parameters, the contribution from f_1 is zero. The formula under consideration is

$$\mathcal{M}_\psi(f_1) = \iint_{\Gamma_+} \psi(x,v) f_1(x,v)\, \mathrm{d}\Gamma_+,$$

where we solve equation (3.3) to obtain

$$f_1(x,v) = \mathrm{e}^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv)\mathrm{d}s} \phi(x - \tau_-(x,v)v, v), \qquad (x,v) \in \overline{\Omega} \times \mathbb{S}^1.$$

Definitions of ψ and ϕ give

$$\mathcal{M}_\psi(f_1) = \frac{1}{\epsilon\delta}\frac{1}{\theta\beta} \int\int_{\Gamma_+} \mathrm{e}^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv)\mathrm{d}s} \psi_0\left(\frac{|x - x^{\mathrm{out}}|}{\theta}\right) \psi_0\left(\frac{|v - v^{\mathrm{out}}|}{\beta}\right) \times \phi_0\left(\frac{(x - \tau_-(x,v)v - x^{\mathrm{in}}) \cdot v_\perp^{\mathrm{in}}}{\epsilon\eta}\right) \phi_0\left(\frac{|v - v^{\mathrm{in}}|}{\delta}\right) \mathrm{d}\Gamma_+.$$

The sufficient condition for $\mathcal{M}_\psi(f_1)$ to vanish is

$$\mathrm{Supp}\left(\psi_0\left(\frac{|v - v^{\mathrm{out}}|}{\beta}\right)\right) \cap \mathrm{Supp}\left(\phi_0\left(\frac{|v - v^{\mathrm{in}}|}{\delta}\right)\right) = \emptyset. \tag{3.12}$$

One sufficient condition for (3.12) to hold is

$$\left|v^{\mathrm{out}} - v^{\mathrm{in}}\right| > \beta + \delta, \tag{3.13}$$

since then there does not exist any v satisfying that

$$\left|v - v^{\mathrm{out}}\right| \leqslant \beta \qquad \text{and} \qquad \left|v - v^{\mathrm{in}}\right| \leqslant \delta.$$

Recall that η is defined in (3.8) as

$$\eta = v^{\mathrm{out}} \cdot v_\perp^{\mathrm{in}} > 0.$$

Therefore, by (3.6), we have

$$v^{\mathrm{out}} \cdot v^{\mathrm{in}} = \sqrt{1 - \eta^2}.$$

This gives

$$\left|v^{\mathrm{out}} - v^{\mathrm{in}}\right|^2 = 2 - 2v^{\mathrm{out}} \cdot v^{\mathrm{in}} = 2 - 2\sqrt{1 - \eta^2}.$$

Hence we have the estimate

$$\eta \leqslant \left|v^{\mathrm{out}} - v^{\mathrm{in}}\right| \leqslant 2\eta. \tag{3.14}$$

It is then clear that a sufficient condition for (3.13) (and thus (3.12)) to hold is

$$\eta > \beta + \delta. \tag{3.15}$$

Such condition gives that $\mathcal{M}_\psi(f_1) = 0$.

Contribution from f_2 The main contribution to the measurement comes from f_2, which we compute below. Denote such contribution as $\mathcal{M}_\psi(f_2)$. Then for any $(x,v) \in \overline{\Omega} \times \mathbb{S}^1$, we have

$$
\begin{aligned}
\mathcal{M}_\psi(f_2) &= \iint_{\Gamma_+}\int_0^{\tau_-(x,v)} \psi(x,v)\mathrm{e}^{-\int_0^s \sigma_a(x-\tau v)\,\mathrm{d}\tau}\sigma_s(x-sv)\,\langle f_1\rangle\,(x-sv)\mathrm{d}s\,\mathrm{d}\Gamma_+\\
&= \iint_{\Gamma_+}\int_0^{\tau_-(x,v)}\int_{\mathbb{S}^1} \psi(x,v)\mathrm{e}^{-\int_0^s \sigma_a(x-\tau v)\,\mathrm{d}\tau}\sigma_s(x-sv)f_1(x-sv,w)\mathrm{d}w\mathrm{d}s\,\mathrm{d}\Gamma_+\\
&= \iint_{\Gamma_+}\int_0^{\tau_-(x,v)}\int_{\mathbb{S}^1} \psi(x,v)\mathrm{e}^{-\int_0^s \sigma_a(x-\tau v)\,\mathrm{d}\tau}\mathrm{e}^{-\int_0^{\tau_-(x-sv,w)}\sigma_a(x-sv-\tau w)\,\mathrm{d}\tau}\\
&\qquad\qquad\qquad\qquad\times\sigma_s(x-sv)\phi((x-sv)'_w,w)\mathrm{d}w\mathrm{d}s\,\mathrm{d}\Gamma_+,
\end{aligned}
$$

where $(x-sv)'_w$ is the entry point of $x-sv$ along the direction w. To simplify the notation, we denote

$$
H(s,x,v,w) = \mathrm{e}^{-\int_0^s \sigma_a(x-\tau v)\,\mathrm{d}\tau}\mathrm{e}^{-\int_0^{\tau_-(x-sv,w)}\sigma_a(x-sv-\tau w)\,\mathrm{d}\tau}\sigma_s(x-sv).
$$

Separate $\mathcal{M}_\psi(f_2)$ into two parts as

$$
\begin{aligned}
\mathcal{M}_\psi(f_2) &= \iint_{\Gamma_+}\int_0^{\tau_-(x,v)}\int_{\mathbb{S}^1} \psi(x,v)H(s_0,x^{\mathrm{out}},v^{\mathrm{out}},v^{\mathrm{in}})\phi((x-sv)'_w,w)\mathrm{d}w\mathrm{d}s\,\mathrm{d}\Gamma_+\\
&\quad+\iint_{\Gamma_+}\int_0^{\tau_-(x,v)}\int_{\mathbb{S}^1} \psi(x,v)\left(H(s,x,v,w)-H(s_0,x^{\mathrm{out}},v^{\mathrm{out}},v^{\mathrm{in}})\right)\phi((x-sv)'_w,w)\mathrm{d}w\mathrm{d}s\,\mathrm{d}\Gamma_+\\
&\triangleq \mathcal{M}_{2,1}+\mathcal{M}_{2,2}.
\end{aligned}
$$

To treat the first term $\mathcal{M}_{2,1}$ we insert the definitions of ϕ,ψ into $\mathcal{M}_{2,1}$ and obtain

$$
\begin{aligned}
\mathcal{M}_{2,1} = \frac{H(s_0,x^{\mathrm{out}},v^{\mathrm{out}},v^{\mathrm{in}})}{\epsilon\delta}\frac{1}{\theta\beta}\iint_{\Gamma_+}\int_0^{\tau_-(x,v)}\int_{\mathbb{S}^1}\psi_0\left(\frac{|x-x^{\mathrm{out}}|}{\theta}\right)\psi_0\left(\frac{|v-v^{\mathrm{out}}|}{\beta}\right)\phi_0\left(\frac{|w-v^{\mathrm{in}}|}{\delta}\right)\\
\times\phi_0\left(\frac{((x-sv)'_w-x^{\mathrm{in}})\cdot v^{\mathrm{in}}_\perp}{\epsilon\eta}\right)\mathrm{d}w\mathrm{d}s\,\mathrm{d}\Gamma_+.
\end{aligned}
$$

Now we reformulate the second ϕ_0-term, whose argument satisfies

$$
\begin{aligned}
(x-sv)'_w &= (x-sv)-\tau_-(x-sv,w)w\\
&= (x^{\mathrm{out}}-sv^{\mathrm{out}})-\tau_-(x-sv,w)v^{\mathrm{in}}+R(x,v,s,w),
\end{aligned}
\tag{3.16}
$$

where the remainder term R is

$$
R(x,v,s,w) = (x-x^{\mathrm{out}})-s(v-v^{\mathrm{out}})-\tau_-(x-sv,w)\left(w-v^{\mathrm{in}}\right).
$$

By corollary C.1, we have that $\nabla_x\tau_-(\cdot,w)$ is uniformly bounded in w if we choose

$$
\theta+\delta+\beta<\gamma_*.
$$

Then by using (3.7) again, we have

$$
\frac{((x-sv)'_w-x^{\mathrm{in}})\cdot v^{\mathrm{in}}_\perp}{\epsilon\eta} = \frac{\left(x^{\mathrm{out}}-sv^{\mathrm{out}}-x^{\mathrm{in}}+R\right)\cdot v^{\mathrm{in}}_\perp}{\epsilon\eta} = \frac{s_0-s+\frac{1}{\eta}R\cdot v^{\mathrm{in}}_\perp}{\epsilon}.
$$

Let z be the new variable given by

$$z = s - \frac{1}{\eta} R \cdot v_\perp^{\text{in}}.$$

Then

$$\frac{\partial z}{\partial s} = 1 - \frac{1}{\eta}\frac{\partial R}{\partial s} \cdot v_\perp^{\text{in}} = 1 + \frac{1}{\eta}(v - v^{\text{out}}) \cdot v_\perp^{\text{in}} - \frac{1}{\eta}\big(v \cdot \nabla_x \tau_-(x - sv, w)\big)(w - v^{\text{in}}) \cdot v_\perp^{\text{in}}.$$

Due to the compact supports of ϕ_0 and ψ_0, the variables (x, v, w) in R satisfy that

$$\left|x - x^{\text{out}}\right| \leqslant \theta, \qquad \left|v - v^{\text{out}}\right| \leqslant \beta, \qquad \left|w - v^{\text{in}}\right| \leqslant \delta.$$

If we impose that

$$\eta \gg \beta + \delta, \tag{3.17}$$

then $\partial z/\partial s > 1/2$ and we can make the change of variable from s to z. Denote $I = z^{-1}(0, \tau_-(x, v))$. Then

$$\lim_{\epsilon,\theta \to 0} \lim_{\substack{\eta \to 0 \\ \eta \gg \beta + \delta}} \mathcal{M}_{2,1} = \lim_{\epsilon,\theta \to 0} \lim_{\substack{\eta \to 0 \\ \eta \gg \beta + \delta}} \frac{H(s_0, x^{\text{out}}, v^{\text{out}}, v^{\text{in}})}{\epsilon\delta} \frac{1}{\theta\beta} \iint_{\Gamma_+} \iint_{I\,\mathbb{S}^1} \psi_0\left(\frac{|x - x^{\text{out}}|}{\theta}\right) \psi_0\left(\frac{|v - v^{\text{out}}|}{\beta}\right)$$
$$\times\, \phi_0\left(\frac{s_0 - z}{\epsilon}\right) \phi_0\left(\frac{|w - v^{\text{in}}|}{\delta}\right) \frac{\partial z}{\partial s} \mathrm{d}w \mathrm{d}s\, \mathrm{d}\Gamma_+.$$

Since s_0 is an interior point by corollary C.2, we have

$$\lim_{\epsilon,\theta \to 0} \lim_{\substack{\eta \to 0 \\ \eta \gg \beta + \delta}} \mathcal{M}_{2,1} = H(s_0, x_0^{\text{out}}, v^{\text{in}}, v^{\text{in}}).$$

where $x^{\text{out}}, v^{\text{out}}$ are replaced by $x_0^{\text{out}}, v^{\text{in}}$ in the limit $\eta \to 0$. Meanwhile, by the continuity of τ_- and σ_a, the second term $\mathcal{M}_{2,2}$ will vanish in the limit.

Consider that under conditions (3.11) and (3.15), assuming $\delta^{-\frac{r-1}{r}} \eta^{\frac{1}{r}} \to 0$, then $\mathcal{M}_\psi(f_1) = 0$ and $\mathcal{M}_\psi(f_3) \to 0$, overall we have

$$\sigma_s(x_0) = \mathrm{e}^{\int_0^{s_0} \sigma_a(x^{\text{out}} - \tau v^{\text{out}})\,\mathrm{d}\tau} \mathrm{e}^{\int_0^{\tau_-(x_0, v^{\text{in}})} \sigma_a(x_0 - \tau v^{\text{in}})\,\mathrm{d}\tau} \lim_{\epsilon,\theta \to 0} \lim_{\substack{\eta \to 0 \\ \eta \gg \beta + \delta}} \mathcal{M}_\psi(f)$$
$$= \mathrm{e}^{\int_0^{s_0} \sigma_a(x_0^{\text{out}} - \tau v^{\text{in}})\,\mathrm{d}\tau} \mathrm{e}^{\int_0^{\tau_-(x_0, v^{\text{in}})} \sigma_a(x_0 - \tau v^{\text{in}})\,\mathrm{d}\tau} \lim_{\epsilon,\theta \to 0} \lim_{\substack{\eta \to 0 \\ \eta \gg \beta + \delta}} \mathcal{M}_\psi(f).$$

Choice of the parameters We now collect all requirements on the parameters, namely equation (3.11), (3.15) and (3.17). Choose $\theta \to 0$ and $\epsilon \to 0$ independent of η, these requirements reduce to:

$$\delta^{-\frac{r-1}{r}} \eta^{\frac{1}{r}} \ll 1, \qquad \beta + \delta \ll \eta. \tag{3.18}$$

In the borderline case where $\delta = \eta$, the sufficient condition for the first inequality in (3.18) to hold is

$$\frac{r-1}{r} < \frac{1}{r} \Rightarrow r < 2.$$

This suggests that we can find proper parameters by letting $\theta \to 0$ and $\epsilon \to 0$ independent of η and setting

$$\beta = \delta = \eta^{1+\beta_0}.$$

with β_0 small enough, then (3.18) holds for $r \in (1, 2)$. $\square$

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Appendix A. Well-posedness of the nonlinear RTE

In this appendix we use the classical monotonicity method combined with the Schauder fixed-point argument to show that the nonlinear RTE given in (2.32) and (2.33) is well-posed. Recall that the equations are given by

$$v \cdot \nabla_x I = -\sigma_a I + \sigma_a T^4, \qquad I\big|_{\Gamma_-} = \phi(x, v), \tag{A.1}$$

$$\Delta_x T = \sigma_a T^4 - \sigma_a \langle I \rangle, \qquad T\big|_{\partial\Omega} = 0 \tag{A.2}$$

where $\phi \geqslant 0$ and $\phi \in L^\infty(\Gamma_-)$. The statement of the well-posedness result is

Theorem A.1. *Suppose $\phi \in L^\infty(\Gamma_-)$ and $\phi \geqslant 0$. Then (A.1) and (A.2) has a unique solution.*

Proof. Let $\mathcal{D}$ be the solution set given by

$$\mathcal{D} = \{T\big|\ 0 \leqslant T \leqslant ||\phi||_{L^\infty}\}.$$

Take $H \in \mathcal{D}$. We want to construct a map $\mathcal{F}$ and show that $\mathcal{F}(H) \in \mathcal{D}$. Let I_H be the solution such that

$$v \cdot \nabla_x I_H = -\sigma_a I_H + \sigma_a H^4, \qquad I_H\big|_{\Gamma_-} = \phi(x, v).$$

Such I_H exists by a direct integration along the characteristics. Since $H^4 \geqslant 0$ and $\phi \geqslant 0$, we have $I_H \geqslant 0$. Moreover, if we consider $I_H - ||\phi||_{L^\infty}$, then it satisfies

$$v \cdot \nabla_x (I_H - ||\phi||_{L^\infty}) = -\sigma_a (I_H - ||\phi||_{L^\infty}) + \sigma_a \left(H^4 - ||\phi||_{L^\infty}\right), \qquad (I_H - ||\phi||_{L^\infty})\big|_{\Gamma_-} \leqslant 0.$$

Since $H^4 - ||\phi||_{L^\infty} \leqslant 0$, we have $I_H \leqslant ||\phi||_{L^\infty}$. Define $\mathcal{F}(H) = T$ where T is the solution to the equation

$$\Delta_x T = \sigma_a T^4 - \sigma_a \langle I_H \rangle, \qquad T\big|_{\partial\Omega} = 0.$$

or equivalently,

$$-\Delta_x T = -\sigma_a T^4 + \sigma_a \langle I_H \rangle , \qquad T\big|_{\partial\Omega} = 0. \tag{A.3}$$

We use the classical monotonicity method for semilinear elliptic equations to show that such T exists and is unique. First, let $\underline{T} = 0$ and $\overline{T}$ be the unique solution to the equation

$$-\Delta_x \overline{T} = \sigma_a \langle I_H \rangle , \qquad \overline{T}|_{\partial\Omega} = 0.$$

Since it holds that

$$-\Delta_x \underline{T} - \sigma_a \langle I_H \rangle \leqslant 0 = -\sigma_a \underline{T}^4, \qquad \underline{T}\big|_{\partial\Omega} = 0,$$

and

$$-\Delta_x \overline{T} - \sigma_a \langle I_H \rangle = 0 \geqslant -\sigma_a \overline{T}^4, \qquad \overline{T}\big|_{\partial\Omega} = 0,$$

the functions $\underline{T}$ and $\overline{T}$ serve as the sub- and super-solutions of (A.3). Moreover, we have

$$0 \leqslant \underline{T} \leqslant \overline{T}.$$

We use an inductive argument to build an increasing sequence as follows. Fix a constant λ which satisfies

$$\lambda > 4\, ||\sigma_a||_{L^\infty}\, ||\phi||_{L^\infty}^{3/4}.$$

This guarantees that the function $f(x) = \lambda x - \sigma_a x^4$ is increasing for any $x \in (0, ||\phi||_{L^\infty})$. Initialize the sequence at $T_0 = \underline{T}$ and suppose at the inductive step that

$$0 \leqslant T_k \leqslant ||\phi||_{L^\infty}^{1/4}.$$

Define T_{k+1} as the unique solution to the equation

$$-\Delta_x T_{k+1} + \lambda T_{k+1} = \lambda T_k - \sigma_a T_k^4 + \sigma_a \langle I_H \rangle , \qquad T_{k+1}\big|_{\partial\Omega} = 0. \tag{A.4}$$

Note that $T_{k+1} \geqslant 0$ since by the choice of λ and the assumption of T_k the right-hand side satisfies

$$\lambda T_k - \sigma_a T_k^4 + \sigma_a \langle I_H \rangle \geqslant \sigma_a \langle I_H \rangle \geqslant 0.$$

Moreover, $T_{k+1} \leqslant ||\phi||_{L^\infty}^{1/4}$ since we have

$$-\Delta_x T_{k+1} + \lambda T_{k+1} \leqslant \lambda T_k,$$

which implies that

$$\max T_{k+1} \leqslant \max T_k \leqslant ||\phi||_{L^\infty}^{1/4}.$$

Now we show that $T_{k+1} \geqslant T_k$ for all $k \geqslant 0$. First, $T_1 \geqslant T_0 = 0$ since we have shown that $T_k \geqslant 0$ for all k. Next, the difference $T_{k+1} - T_k$ satisfies the equation

$$-\Delta_x (T_{k+1} - T_k) + \lambda (T_{k+1} - T_k) = f(T_k) - f(T_{k-1}) \geqslant 0, \qquad (T_{k+1} - T_k)\big|_{\partial\Omega} = 0.$$

where recall that $f(x) = \lambda x - x^4$. Hence

$$\min_{\overline{\Omega}}(T_{k+1} - T_k) = \min_{\partial\Omega}(T_{k+1} - T_k) = 0,$$

which implies that $T_{k+1} \geqslant T_k$. We thereby have constructed an increasing sequence. Lastly we want to show that $T_k \leqslant \overline{T}$ for all $k \geqslant 0$. This is done by considering the equation for $T_k - \overline{T}$ which reads

$$-\Delta_x \left(T_k - \overline{T}\right) + \lambda \left(T_k - \overline{T}\right) = f(T_{k-1}) - \lambda\overline{T}, \qquad (T_k - \overline{T})\big|_{\partial\Omega} = 0.$$

By the induction assumption at k such that $T_{k-1} \leqslant \overline{T}$, the right-hand side of the equation satisfies

$$f(T_{k-1}) - \lambda\overline{T} \leqslant f(T_{k-1}) - \left(\lambda\overline{T} - \sigma_a\overline{T}^4\right) \leqslant 0.$$

Hence by the maximum principle, we have

$$\max_{\overline{\Omega}}(T_{k+1} - \overline{T}) = \max_{\partial\Omega}(T_{k+1} - \overline{T}) = 0,$$

which gives that $T_{k+1} \leqslant \overline{T}$. Overall, we have

$$0 = \underline{T} = T_0 \leqslant T_1 \leqslant \cdots \leqslant T_k \leqslant \cdots \leqslant \overline{T}.$$

Together with the L^∞ bound of T_k, we have that there exists $T \in L^\infty(\Omega)$ such that

$$T_k \to T \quad \text{pointwise and in } L^4.$$

Passing $k \to \infty$ in (A.4) shows T is a weak solution of (A.3) and $||T||_{L^\infty} \leqslant ||\phi||_{L^\infty}^{1/4}$. The L^∞-bounds of T and I_H shows that $T \in W^{2,\infty}(\Omega)$. Hence the mapping $\mathcal{F}$ is compact and we can then apply the Schauder fixed-point theorem to obtain a strong solution to (A.1) and (A.2). The uniqueness can be shown by directly taking the difference of two potential solutions and using the energy estimate. □

Appendix B. Geometry

In this appendix, we show the proofs for two geometric relations (2.18) and (2.23). First we prove (2.18).

Proof of (2.18). Suppose that in a small neighborhood of $x \in \partial\Omega$, the boundary $\partial\Omega$ is parametrized as

$$x = x(u), \qquad u \in (u_0, u_1).$$

Then the corresponding small neighborhood of y, given that y is the exit point of x, is also parametrized by u through the relation

$$y = y(u) = x(u) - \tau_-(x(u), v^{\text{in}})v^{\text{in}}, \qquad u \in (u_0, u_1).$$

Therefore, $\frac{\mathrm{d}x}{\mathrm{d}u}$ and $\frac{\mathrm{d}y}{\mathrm{d}u}$ are both along the tangential direction. Moreover,

$$\mathrm{d}S_x = \left|\frac{\mathrm{d}x}{\mathrm{d}u}\right| \mathrm{d}u, \qquad \mathrm{d}S_y = \left|\frac{\mathrm{d}y}{\mathrm{d}u}\right| \mathrm{d}u.$$

which gives

$$\frac{\mathrm{d}S_x}{\mathrm{d}S_y} = \frac{|\mathrm{d}x/\mathrm{d}u|}{|\mathrm{d}y/\mathrm{d}u|}.$$

Note that for any unit vectors α, β, we have

$$\left|\alpha \cdot \beta^{\perp}\right| = \left|\alpha^{\perp} \cdot \beta\right|. \tag{B.1}$$

Therefore, if we denote T_x and T_y as the unit tangential directions at x and y respectively, then

$$\left|n(x) \cdot v^{\mathrm{in}}\right| = \left|T_x^{\perp} \cdot v^{\mathrm{in}}\right| = \left|T_x \cdot \left(v^{\mathrm{in}}\right)^{\perp}\right| = \left|\frac{\mathrm{d}x}{\mathrm{d}u} \cdot \left(v^{\mathrm{in}}\right)^{\perp}\right| \frac{1}{\left|\frac{\mathrm{d}x}{\mathrm{d}u}\right|}$$

Similarly,

$$\left|n(y) \cdot v^{\mathrm{in}}\right| = \left|\frac{\mathrm{d}y}{\mathrm{d}u} \cdot \left(v^{\mathrm{in}}\right)^{\perp}\right| \frac{1}{\left|\frac{\mathrm{d}y}{\mathrm{d}u}\right|}.$$

Therefore,

$$\frac{\left|n(y) \cdot v^{\mathrm{in}}\right|}{\left|n(x) \cdot v^{\mathrm{in}}\right|} = \frac{|\mathrm{d}x/\mathrm{d}u|}{|\mathrm{d}y/\mathrm{d}u|} \frac{\left|\frac{\mathrm{d}y}{\mathrm{d}u} \cdot \left(v^{\mathrm{in}}\right)^{\perp}\right|}{\left|\frac{\mathrm{d}x}{\mathrm{d}u} \cdot \left(v^{\mathrm{in}}\right)^{\perp}\right|}.$$

Observe that by the definition of y, we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}x}{\mathrm{d}u} - \frac{\mathrm{d}\tau_-(x(u), v^{\mathrm{in}})}{\mathrm{d}u} v^{\mathrm{in}}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}u} \cdot \left(v^{\mathrm{in}}\right)^{\perp} = \left(\frac{\mathrm{d}x}{\mathrm{d}u} - \frac{\mathrm{d}\tau_-(x(u), v^{\mathrm{in}})}{\mathrm{d}u} v^{\mathrm{in}}\right) \cdot \left(v^{\mathrm{in}}\right)^{\perp} = \frac{\mathrm{d}x}{\mathrm{d}u} \cdot \left(v^{\mathrm{in}}\right)^{\perp}.$$

Therefore,

$$\frac{\left|n(y) \cdot v^{\mathrm{in}}\right|}{\left|n(x) \cdot v^{\mathrm{in}}\right|} = \frac{|\mathrm{d}x/\mathrm{d}u|}{|\mathrm{d}y/\mathrm{d}u|} = \frac{\mathrm{d}S_x}{\mathrm{d}S_y},$$

which is equivalent to (2.18). □

Next we verify (2.23).

Proof of (2.23). Fix $x \in \partial\Omega_v^{+}$. Suppose the neighborhood of x (in $\partial\Omega$) is a curve parametrized as

$$x = x(u), \qquad u \in (u_0, u_1).$$

Then

$$y(u,s) = x(u) - sv.$$

The Jacobian of the mapping $y \to (u,s)$ is

$$\left|\det\begin{pmatrix} \frac{\mathrm{d}y_1}{\mathrm{d}u} & \frac{\mathrm{d}y_2}{\mathrm{d}u} \\ \frac{\mathrm{d}y_1}{\mathrm{d}s} & \frac{\mathrm{d}y_2}{\mathrm{d}s}\end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}u} & \frac{\mathrm{d}x_2}{\mathrm{d}u} \\ -v_1 & -v_2\end{pmatrix}\right| = \left|-v_2\frac{\mathrm{d}x_1}{\mathrm{d}u} + v_1\frac{\mathrm{d}x_2}{\mathrm{d}u}\right| = \left|v^\perp \cdot \nabla_u x\right| = \left|v^\perp \cdot T_x\right| \left|\nabla_u x\right|.$$

where T_x is the tangent direction at x. By (B.1), we have

$$\left|\det\begin{pmatrix} \frac{\mathrm{d}y_1}{\mathrm{d}u} & \frac{\mathrm{d}y_2}{\mathrm{d}u} \\ \frac{\mathrm{d}y_1}{\mathrm{d}s} & \frac{\mathrm{d}y_2}{\mathrm{d}s}\end{pmatrix}\right| = |v \cdot n(x)| \left|\nabla_u x\right|.$$

Therefore (2.23) holds since

$$\mathrm{d}y = \left|\frac{\partial(y_1, y_2)}{\partial(u,s)}\right| \mathrm{d}u\mathrm{d}s = |v \cdot n(x)| \left|\nabla_u x\right| \mathrm{d}u\mathrm{d}s = n(x) \cdot v\mathrm{d}S_x\mathrm{d}s,$$

where we can remove the absolute value sign since $n(x) \cdot v > 0$. □

Appendix C. Some technical lemmas

This appendix is devoted to showing several technical results used in the proof of theorem 3.3. The notations $x^{\mathrm{in}}, x_0^{\mathrm{out}}, v^{\mathrm{in}}, s_0, x_0, v^{\mathrm{out}}$ represent the same quantities as in the theorem.

Lemma C.1. *There exists γ_0 small enough such that $\tau_-(x - sv, w)$ is C^1 in (x, v, s, w) over the domain*

$$\left|x - x_0^{\mathrm{out}}\right| + \left|w - v^{\mathrm{in}}\right| + \left|v - v^{\mathrm{in}}\right| < \gamma_0, \qquad s \in (0, \tau_-(x,v)), \qquad (x,v) \in \Gamma_+. \tag{C.1}$$

Moreover, the bound $||\nabla_x \tau_-(\cdot, w)||_{L^\infty}$ is independent of w over the region (C.1).

Proof. By lemma 2.1, we only need to show that there exists a constant $c_{0,1} > 0$ such that

$$w \cdot n((x - sv)_-) < -c_{0,1} < 0 \tag{C.2}$$

for any (x, v, s, w) satisying (C.1), recalling that $(x - sv)_-$ is the backward exit point of $x - sv$. The idea is to show that $(x - sv)_-$ is close to x^{in} when γ_0 is small. Then by the continuity of the outward normal n, we obtain (C.2) from the non-degeneracy condition at $(x^{\mathrm{in}}, v^{\mathrm{in}})$. The closeness of $(x - sv)_-$ to x^{in} is fairly evident from the geometry shown in figure C1.

For a rigorous proof, we first assume, via a proper rotation and translation, that v^{in} is along the positive y-axis and x^{in} and x_0^{out} are both on the y-axis. Since Ω is convex and $v^{\mathrm{in}} \cdot n(x^{\mathrm{in}}) \neq 0$, we have

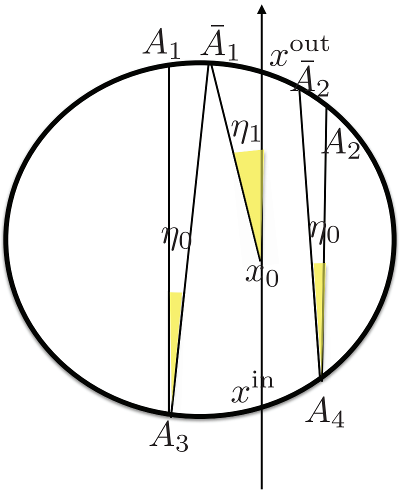

Figure C1. Geometry for non-degeneracy.

$$v^{\rm in} \cdot n(x_0^{\rm out}) > 0.$$

Take small neighborhoods $\mathcal{N}(x^{\rm in}), \mathcal{N}(x_0^{\rm out})$ around $x^{\rm in}$ and $x_0^{\rm out}$ on $\partial\Omega$ such that

$$v^{\rm in} \cdot n(x) < \frac{1}{2} v^{\rm in} \cdot n(x^{\rm in}) < 0, \qquad \forall\, x \in \mathcal{N}(x^{\rm in}),$$
$$v^{\rm in} \cdot n(x) > \frac{1}{2} v^{\rm in} \cdot n(x_0^{\rm out}) > 0, \qquad \forall\, x \in \mathcal{N}(x_0^{\rm out}),$$

Denote the boundary vertices of $\mathcal{N}(x^{\rm in}), \mathcal{N}(x_0^{\rm out})$ as A_1, A_2, A_3, A_4. By adjusting the sizes of $\mathcal{N}(x^{\rm in}), \mathcal{N}(x_0^{\rm out})$ we can choose these vertices in the way such that

$$A_1A_3 \,/\!/\, A_2A_4 \,/\!/\, y-\text{axis}.$$

Choose $\overline{A}_1$ and $\overline{A}_2$ as two points on $\text{arc}(A_1x_0^{\rm out})$ and $\text{arc}(A_2x_0^{\rm out})$ respectively such that

$$\angle A_1A_3\overline{A}_1 = \angle A_2A_4\overline{A}_2 =: \eta_0.$$

Denote the region bounded by the line segments $\overline{A}_1A_3$, $\overline{A}_2A_4$ and the two arcs $\text{arc}(\overline{A}_1\overline{A}_2)$, $\text{arc}(A_3A_4)$ as D_0. Then for any $(x, v) \in \Gamma_+$ with $\cos^{-1}(v \cdot v^{\rm in}) < \eta_0$ and any $s \in (0, \tau_-(x, v))$, we have

$$(x - sv)_- \in \mathcal{N}(x^{\rm in}).$$

Hence, for such (x, v, s) we have

$$v^{\rm in} \cdot n((x - sv)_-) < \frac{1}{2} v^{\rm in} \cdot n(x^{\rm in}) < 0.$$

Take γ_0 small enough such that

$$\gamma_0 < \min\left\{\frac{1}{2}\eta_0,\ \frac{1}{4}v^{\text{in}}\cdot n(x^{\text{in}}),\ \left|\overline{A}_1 x_0^{\text{out}}\right|,\ \left|\overline{A}_2 x_0^{\text{out}}\right|\right\}.$$

Then

$$w\cdot n((x-sv)_-) < \frac{1}{4}v^{\text{in}}\cdot n(x^{\text{in}}) < 0$$

for any (x,v,s,w) satisfying (C.1). Hence τ_- is C^1 over the region (C.1). The explicit formula for $\nabla_x\tau_-$ in lemma 2.1 shows that $||\nabla_x\tau_-(\cdot,w)||_{L^\infty}$ is uniformly bounded in w. $\square$

Two immediate consequences follow.

Corollary C.1. *There exist η_*,γ_* such that if η in theorem 3.3 satisfies $\eta<\eta_*$, then $\tau_-(x-sv,w)$ is C^1 in (x,v,s,w) over the domain*

$$\left|x-x^{\text{out}}\right|+\left|w-v^{\text{in}}\right|+\left|v-v^{\text{out}}\right|<\gamma_*,\qquad s\in(0,\tau_-(x,v)),\qquad (x,v)\in\Gamma_+. \tag{C.3}$$

Moreover, the bound $||\nabla_x\tau_-(\cdot,w)||_{L^\infty}$ is independent of w over the region (C.1).

Proof. By lemma C.1, we only need to show that x^{out} is close to x_0^{out} and v^{out} is close to v^{in} by taking η_* small. By (3.14), if we taking $\eta_*<\frac{1}{8}\gamma_0$, then

$$\left|v^{\text{out}}-v^{\text{in}}\right|\leqslant 2\eta<\frac{1}{4}\gamma_0.$$

Denote the angle $\angle\overline{A}_1x_0x_0^{\text{out}}$ as $\overline{\eta}$. Then for $\eta_*<\overline{\eta}$, the point x^{out} is on arc$(\overline{A}_1x_0^{\text{out}})$. Since

$$\lim_{\overline{\eta}\to 0}\left|\overline{A}_1-x_0^{\text{out}}\right|=0,$$

by choosing η_* small enough, we have

$$\left|x^{\text{out}}-x_0^{\text{out}}\right|<\frac{1}{4}\gamma_0.$$

Hence if we let $\gamma_*=\frac{1}{2}\gamma_0$, then for any (x,v,s,w) in the region (C.3), they also satisfy that

$$\begin{aligned}
&\left|x-x_0^{\text{out}}\right|+\left|w-v^{\text{in}}\right|+\left|v-v^{\text{in}}\right|\\
&\leqslant\left|x-x^{\text{out}}\right|+\left|w-v^{\text{in}}\right|+\left|v-v^{\text{out}}\right|+\left|x^{\text{out}}-x_0^{\text{out}}\right|+\left|v^{\text{out}}-v^{\text{in}}\right|\\
&<\frac{1}{2}\gamma_0+\frac{1}{4}\gamma_0+\frac{1}{4}\gamma_0=\gamma_0,
\end{aligned}$$

whereby lemma C.1 applies. $\square$

Corollary C.2. *Let γ_* be the upper bound such that τ_- is C^1 in the domain (C.3). Then for γ_* small enough, s_0 is always an interior point in $(0,\tau_-(x,v))$ whenever (x,v) satisfies (C.3).*

Proof. First recall that $s_0\in(0,\tau_-(x^{\text{out}},v^{\text{out}}))$. Then

$$\sigma_0:=\tau_-(x^{\text{out}},v^{\text{out}})-s_0>0.$$

By corollary C.1, the backward exist time $\tau_-(x, v)$ is continuous for (x, v) in the closure of the domain dictated by (C.3). Hence if γ_* is small enough, then

$$\left|\tau_-(x, v) - \tau_-(x^{\text{out}}, v^{\text{out}})\right| < \frac{1}{2}\sigma_0.$$

Therefore,

$$\tau_-(x, v) - s_0 > \tau_-(x^{\text{out}}, v^{\text{out}}) - \frac{1}{2}\sigma_0 - s_0 = \sigma_0 > 0,$$

which shows $s \in (0, \tau_-(x, v))$. □

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