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Real centrifugal forces in relativistic rotating spacetimes: a simple introduction

Elmo Benedetto[1,2,*], **Marco Di Mauro**[3], **Antonio Feoli**[2], **Antonella Lucia Iannella**[2] and **Adele Naddeo**[4]

[1] Department of Computer Science, University of Salerno, Via Giovanni Paolo II, 132, I-84084 Fisciano (SA), Italy
[2] Department of Engineering, University of Sannio, Piazza Roma 21, I-82100, Benevento, Italy
[3] Department of Physics, University of Trento, Via Sommarive 14, I-38123, Povo (TN), Italy
[4] INFN, Via Cinthia, I-80126 Naples, Italy

E-mail: elmobenedetto@libero.it

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Abstract
During typical general relativity courses, the so-called frame-dragging effect is explained by emphasizing the presence of a gravitational Coriolis-like force term. The key difference is that, unlike the usual Coriolis force, this is not a fictitious force but agravitational force caused by the rotating body. In general, textbooks do not discuss also the possibility of a gravitational centrifugal-like force. In this paper, which has a didactic aim, we analyze this further gravitational term. The analysis we perform can be valuable in undergraduate courses of general relativity.

Keywords: centrifugal force, lense-thirring metric, gravitomagnetism

(Some figures may appear in colour only in the online journal)

1. Introduction

Rotational motions in relativistic spacetimes, and the closely connected investigation of the gravitational fields generated by rotating bodies, have been the subject of in-depth studies for

[*] Author to whom any correspondence should be addressed.

more than 100 years [1–16]. Einstein himself, in the first part of his long route in developing general relativity, where he explored the physical properties of gravitational fields using the equivalence principle, often used reference frames rotating with constant angular velocity [17]. In particular, the famous thought experiment of the rotating disc was a fundamental step for his crucial intuition that gravity involved the curvature of spacetime. A closely related issue is that of the gravitational fields generated by rotating objects, and the physical effects, such as frame-dragging, that occur in such fields. In the weak field case, the motion of a test particle in such a field can be described effectively in terms of a velocity-dependent force, which may be viewed as a gravitational analogue of the magnetic force and which has the same form as the (also velocity-dependent) Coriolis force which is felt by a particle moving in a rotating reference frame. The difference is, of course, that while the Coriolis force is a fictitious inertial force, which disappears when the motion is described in an inertial reference frame, the same is not true for this gravitational force, which is a consequence of the real gravitational field generated by the rotating object. The relation between the two forces is analogous to that between the real gravitational force due to a homogeneous gravitational field and the inertial force felt by a particle in a reference frame with constant linear acceleration. However, it is well-known from classical mechanics, that an object in a rotating reference frame, besides the Coriolis force, also feels a centrifugal force, which unlike the former is independent of velocity, hence it is felt also by objects at rest with respect to the rotating frame. In this paper, we show by a simple calculation that an analogue of this force, effectively acting as a repulsive gravitational force, exists in the vicinity of a rotating mass. Before doing this, as warm-ups we consider rotating frames in special relativity, and in the spacetime describing a non-rotating mass. Test particles at rest with respect to such frames feel a centrifugal force, which by the equivalence principle is analogous to a radial outgoing gravitational field, and contrasts with the usual gravitational field which attracts the particle towards the centre.

The main topic considered in this paper is the study of the forces felt by test particles in the gravitational field of a rotating object, in the weak field case. This field is described, as well-known, by the Lense-Thirring metric. We consider both the cases in which the test particle is at rest with respect to the reference frame identified by the coordinates in which the metric is written (this should describe the situation in which the test particle is at rest with respect to the center of mass of the central body), and in which the test particle is at rest with respect to a reference frame which rotates around the center of mass of the central body.

2. Rotating frames in special relativity

In this section, we consider the case of flat, Minkowski spacetime, i.e. special relativity. In this case, of course, we only have an observer executing a rotational motion, and there is no object generating any gravitational field. Let us start from the Minkowski metric in cylindrical coordinates [18]

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - dz^2. \tag{1}$$

If we consider a plane system ($dz = 0$) rotating with an angular velocity ω, the angle transforms as

$$\theta = \theta' + \omega t \Rightarrow d\theta = d\theta' + \omega dt \tag{2}$$

getting the so-called Langevin metric [19]

$$ds^2 = \left(1 - \frac{\omega^2 r^2}{c^2}\right) c^2 dt^2 - dr^2 - r^2 d\theta'^2 - 2\omega r^2 d\theta' dt. \tag{3}$$

In relations (1) and (3) the time coordinate is the same and that is the time measured by the inertial observer. The Langevin metric is stationary with

$$\begin{cases} g_{00} = 1 - \dfrac{\omega^2 r^2}{c^2} \\ g_{11} = -1 \\ g_{22} = -r^2 \\ g_{02} = g_{20} = -\dfrac{\omega r^2}{c} \end{cases} \tag{4}$$

Obviously, given the tensor nature of the curvature, this metric represents a flat spacetime in curvilinear coordinates. Instead, following relations (84.6) and (84.7) of chapter X of [18], it is possible to deduce that the spatial metric is hyperbolic, and we have

$$\mathrm{d}l^2 = \mathrm{d}r^2 + \frac{r^2\mathrm{d}\theta'^2}{1 - \frac{\omega^2 r^2}{c^2}}. \tag{5}$$

For an observer at rest with respect to the rotating system, the time is

$$\mathrm{d}\tau = \sqrt{1 - \frac{\omega^2 r^2}{c^2}}\,\mathrm{d}t \tag{6}$$

The relation (6) is not symmetrical, as it happens in transformations between inertial frames, and, in the light of the equivalence principle, the observer on the rotating frame can interpret it as an effect of the following centrifugal gravitational potential

$$V = -\frac{\omega^2 r^2}{2} \tag{7}$$

with

$$g_{00} = 1 + \frac{2V}{c^2} \tag{8}$$

obtaining the gravitational time delay in a weak field [18]

$$\mathrm{d}\tau = \sqrt{1 + g_{00}}\,\mathrm{d}t. \tag{9}$$

Furthermore, if the observer moves with respect to the rotating system, he can also be affected by a fictitious Coriolis gravitational force and, if v_r is the relative velocity, we have the following Coriolis potential [20–22]

$$V = (\omega \wedge r) \cdot v_r. \tag{10}$$

3. Rotating frame in Schwarzschild spacetime

Now we consider some cases in which there is a gravitational field. Let us consider, indeed, a rotating platform with a mass at its center. In other words, we have a rotating system immersed in the following metric

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{11}$$

where, as it is well-known, θ and ϕ are the standard polar and azimuthal angles, t is the time measured by a stationary clock located infinitely far from the massive body M, $r = C/2\pi$ where C is the circumference of a circle centered on the massive body. Finally, obviously, G is the Newtonian gravitational constant and c is the speed of light. In the case of simple circular motion, considering $\theta = \frac{\pi}{2}$, it becomes

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - r^2 d\phi^2. \tag{12}$$

If we consider a rotating coordinate in Schwarzschild spacetime, with the same calculation obtained in the flat spacetime, it is easy to get the Langevin metric in this curved background [23]

$$ds^2 = \left(1 - \frac{2GM}{c^2 r} - \frac{\omega^2 r^2}{c^2}\right)c^2 dt^2 - r^2 d\phi'^2 - 2\omega r^2 d\phi' dt \tag{13}$$

Therefore, for an observer at rest on the rotating platform, we have

$$ds^2 = \left(1 - \frac{2GM}{c^2 r} - \frac{\omega^2 r^2}{c^2}\right)c^2 dt^2 \tag{14}$$

and

$$d\tau = \sqrt{1 - \frac{2GM}{c^2 r} - \frac{\omega^2 r^2}{c^2}}\, dt. \tag{15}$$

In the weak field case (i.e. at a large distance from the central mass compared with its Schwarzschild radius) we have the sum of two gravitational potentials, one is real and the other is fictitious

$$V = -\frac{GM}{r} - \frac{\omega^2 r^2}{2}. \tag{16}$$

The real gravitational force pulls a test particle towards the center, while the fictitious one generates a force oriented in the opposite direction. Also in this metric, if the observer moves with respect to the rotating system, it can also be affected by a fictitious Coriolis gravitational force.

4. Lense-Thirring spacetime

A very interesting case is that in which the central mass M is rotating around an axis, with angular momentum J. If the rotation is slow, the gravitational field generated by it is described by the well-known Lense-Thirring metric [24]

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 - \frac{4GMa}{cr} d\phi dt. \tag{17}$$

where $a = \frac{J}{Mc}$. It is useful to remember that the metric (17) is a limit of the Kerr metric which is an exact solution of Einstein's equations. It is generally shown that a body moving with respect to these coordinates can be subject to a gravitational and non-fictitious Coriolis force. In order not to burden the calculations, having this paper an educational goal, we limit ourselves to considering motions on the equatorial plane, i.e. $\theta = \frac{\pi}{2}$ and circular orbits (i.e. $\mathrm{d}r = 0$). The metric becomes:

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - r^2\mathrm{d}\phi^2 - \frac{4GMa}{cr}\mathrm{d}\phi\mathrm{d}t. \tag{18}$$

We immediately note that (18) is Langevin like-type

$$\mathrm{d}s^2 = \left(1 + \frac{2V}{c^2}\right)c^2\mathrm{d}t^2 - r^2\mathrm{d}\theta^2 - 2\omega r^2\mathrm{d}\theta\mathrm{d}t. \tag{19}$$

Relations (18) and (19) are in fact identical if

$$\omega r^2 = \frac{2GJ}{c^2 r} \tag{20}$$

from which

$$\omega = \frac{2GJ}{c^2 r^3}. \tag{21}$$

An observer at rest with respect to these coordinates is seen, from infinity, as if he were at rest on a rotating platform at a distance r from the centre. The key difference is that now she feels only a gravitational force directed towards the center with potential $-\frac{GM}{r}$ and no centrifugal potential. As already mentioned, if he moves with respect to this reference frame, he also feels a real gravitational Coriolis force. Now let us consider an observer immersed in the Lense-Thirring metric but at rest with respect to an observer at infinity. We can write

$$d\theta = d\Phi - \omega t \tag{22}$$

obtaining

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - r^2(\mathrm{d}\Phi^2 + \omega^2\mathrm{d}t^2 - 2\omega\mathrm{d}\Phi\mathrm{d}t) - 2\omega r^2(\mathrm{d}\Phi - \omega\mathrm{d}t)\mathrm{d}t \tag{23}$$

$$= \left(1 - \frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - r^2\mathrm{d}\Phi^2 - \omega^2 r^2\mathrm{d}t^2 + 2\omega r^2\mathrm{d}\Phi\mathrm{d}t \tag{24}$$

$$- 2\omega r^2\mathrm{d}\Phi\mathrm{d}t + 2\omega^2 r^2\mathrm{d}t^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - r^2\mathrm{d}\Phi^2 + \omega^2 r^2\mathrm{d}t^2 \tag{25}$$

Therefore

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r} + \frac{\omega^2 r^2}{c^2}\right)c^2\mathrm{d}t^2 - r^2\mathrm{d}\Phi^2. \tag{26}$$

Thanks to (21), finally we have the following gravitational potential

$$V = -\frac{GM}{r} + \frac{\omega^2 r^2}{2} = -\frac{GM}{r} + \frac{2J^2G^2}{c^4 r^4}. \tag{27}$$

In general relativity, the effective potential for radial part of motion is defined by the relation

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \widetilde{V}^2 = \widetilde{E}^2 \tag{28}$$

where $\widetilde{E}$ is the energy at infinity per unit rest mass. Following Misner, Thorne and Wheeler [25], in the Newtonian limit we get

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V = \frac{\widetilde{E}^2 - 1}{2}. \tag{29}$$

By considering the derivative with respect to τ, we can write

$$\frac{\mathrm{d}r}{\mathrm{d}\tau}a + \frac{\mathrm{d}V}{\mathrm{d}\tau} = 0. \tag{30}$$

We write the previous relation as follows

$$\frac{\mathrm{d}r}{\mathrm{d}\tau}a + \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0 \tag{31}$$

from which

$$a = -\frac{\mathrm{d}V}{\mathrm{d}r}. \tag{32}$$

Therefore the body is subject to two gravitational forces of opposite directions, with a resulting acceleration

$$a = \frac{GM}{r^2} - \frac{8J^2G^2}{c^4r^5}. \tag{33}$$

The repulsive force wins if

$$J > c^2\sqrt{\frac{Mr^3}{8G}}. \tag{34}$$

Quite interestingly, we notice that if we set $r = \frac{2GM}{c^2}$, we get

$$J > c^2\sqrt{\frac{G^2M^4}{c^6}} = \frac{GM^2}{c}. \tag{35}$$

This is exactly the same inequality which marks the appearance of a naked singularity in a Kerr black hole, i.e. the value of the angular momentum over which the event horizon and the ergosphere disappear. It is well-known that for $J = \frac{GM^2}{c}$, the black hole becomes extremal. We want to mention that when the inequality (35) is satisfied by a parent star a black hole does not form since centrifugal forces prevent collapse, in line with our discussion. Of course, our computation is only valid in the regime of slow rotation and weak field, where the Lense-Thirring metric can be used, hence the angular momentum in our case will always be well below this critical value. However, an interesting parallel can be drawn with the computation of the Schwarzschild radius in terms of escape velocity. In that case, in fact, although the framework used involves a regime of weak field (which is therefore very far from the situation in which an actual event horizon can occur), the result is quantitatively correct (not only dimensionally, but also the prefactor comes out right). An analogous situation occurs for our inequality (35), where a computation valid only in the approximation of low angular momentum and weak field, nevertheless gives the correct result. Obviously, the effect is very small, which is a common feature of weak-field general relativistic effects in the Solar System. For example, as the angular momentum of the Earth is $J \approx 7.2 \times 10^{33}$ Kgm^2 s^{-1}, if the Moon were forced not to follow the frame-dragging, considering that the distance is about $3.8 \cdot 10^8$ m, it would experience a centrifugal acceleration

$$a \approx 3 \times 10^{-29}\mathrm{m\ s}^{-2}.$$

Even for a body near the Earth's surface, the effect is negligible in fact, in this case, it would feel a repulsive acceleration

$$a \approx 10^{-21}\ \mathrm{m\ s^{-2}}.$$

Despite its smallness, the effect is nevertheless a prediction of the theory.

5. Conclusions

In this paper, we have analyzed the forces felt by test particles in the gravitational field of a rotating object, in the weak field case. We consider both the cases in which the test particle is at rest with respect to the Lense-Thirring reference frame and in which the test particle is at rest with respect to an observer at infinity. In the latter case, the test particle is subject to two gravitational forces of opposite directions and therefore we have also a repulsive gravitational force caused by the rotating body. The repulsive force is however always smaller than the attractive one, hence the net force is always attractive. This is to be expected since if the central body rotates too fast it cannot be held together by its gravity, so it cannot exist at all.

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Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Elmo Benedetto https://orcid.org/0000-0002-5920-0221
Marco Di Mauro https://orcid.org/0000-0002-1532-8274
Antonio Feoli https://orcid.org/0000-0001-8227-1303

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