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Anamorphic quasiperiodic universes in modified and Einstein gravity with loop quantum gravity corrections

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Abstract

The goal of this work is to elaborate on new geometric methods of constructing exact and parametric quasiperiodic solutions for anamorphic cosmology models in modified gravity theories, MGTs, and general relativity, GR. There exist previously studied generic off-diagonal and diagonalizable cosmological metrics encoding gravitational and matter fields with quasicrystal like structures, QC, and holonomy corrections from loop quantum gravity, LQG. We apply the anholonomic frame deformation method, AFDM, in order to decouple the (modified) gravitational and matter field equations in general form. This allows us to find integral varieties of cosmological solutions determined by generating functions, effective sources, integration functions and constants. The coefficients of metrics and connections for such cosmological configurations depend, in general, on all spacetime coordinates and can be chosen to generate observable (quasi)-periodic/aperiodic/fractal/stochastic/(super) cluster/filament/polymer like (continuous, stochastic, fractal and/or discrete structures) in MGTs and/or GR. In this work, we study new classes of solutions for anamorphic cosmology with LQG holonomy corrections. Such

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solutions are characterized by nonlinear symmetries of generating functions for generic off-diagonal cosmological metrics and generalized connections, with possible nonholonomic constraints to Levi–Civita configurations and diagonalizable metrics depending only on a time like coordinate. We argue that anamorphic quasiperiodic cosmological models integrate the concept of quantum discrete spacetime, with certain gravitational QC-like vacuum and nonvacuum structures. And, that of a contracting universe that homogenizes, isotropizes and flattens without introducing initial conditions or multiverse problems.

Keywords: mathematical cosmology, geometry of nonholonomic spacetimes, modified gravity theories, post modern inflation paradigm, loop quantum gravity and cosmology, quasiperiodic cosmological structures, ekpyrotic universes

1. Introduction and motivation

It is thought that near the Planck limit any quantum gravity theory is characterized by discrete degrees of freedom, respective of quantum minimal length and quantum symmetries, and anisotropic and inhomogeneous fluctuating/random configurations. On the other hand, observations show that the accelerating Universe is flat, smooth and scale free at large-scale distances when the spectrum of primordial curvature perturbations is nearly scale-invariant, adiabatic and Gaussian [1, 2]. We cite papers [3–10] for recent reviews, discussions, critique and new results on postmodern inflation scenarios developed and advocated by prominent theorists in relation to the Planck 2013 and Planck 2015 cosmological data [11–17]. Here we note that for meta-galactic and galactic distances, the Planck 2015 and WMAP, ACT and SPT teams' observation and theoretical results[5] on spacetime anisotropy and topology, dark energy, and constraints on inflation and accelerating cosmology parameters. Such works conclude on the existence of mixed aperiodic and quasiperiodic structures (for gravitational, dark matter and standard matter) described as net-works for the first group- and (super) cluster-scale, strong gravitational lensing/light filaments/polymer and quasicrystal, QC, like configurations.

In our partner works [18–20], we proved that Starobinsky-like inflation [21] and various dark energy, DE, and dark matter, DM, effects in a Universe with quasiperiodic (super) cluster and filament configurations can be determined by a nontrivial QC spacetime structure. We cite see [22–38] for important works and references on the physics and mathematics of QCs in condensed matter physics but also with possible connections to cosmology. Various F-modified (for instance, $F(R) = R + \alpha R^2$) cosmological models[6] can be with singularities and encode inhomogeneous and locally anisotropic properties. For reviews on modified gravity theories, MGTs, readers may consider [9, 39–45, 46–54]. In papers [55, 56], a detailed analytical and numerical study of possible holonomy corrections from LQG to $f(R)$ gravity was performed. It was shown that, as a result of such quantum corrections (and various generic off-diagonal, nonholonomic and/or QC contributions investigated in [18–20, 49–52]) the dynamics may

[5] Consistency and implications for inflationary, ekpyrotic and anamorphic bouncing cosmologies, and other type cosmological models, are discussed in [4, 17].

[6] In more general contexts, one considers various modified gravity theories, $F(R, \mathcal{T})$, $F(T)$, ... determined by functionals on Ricci scalars, energy-momentum and/or torsion tensors etc; in various papers, such functionals are denoted also as $f(R)$, $f(T)$, ...

change substantially and, for certain well defined conditions, one obtains better predictions for the inflationary phase as compared with current observations. Various approaches to LQG and spin network theories, see also constructions on loop quantum cosmology, LQC, are reviewed in [57–61]. In the past, certain criticism against LQG (see, for instance, [62]) was motivated in the bulk by arguments that the mathematical formalism is not that which is familiar for the particle physicists working with perturbation theory, Fock spaces, background fields etc; see reply and discussion in [63].

The main objectives of this work are to study how quasiperiodic and/or aperiodic QC like structures with possible holonomy LQG corrections modify inflation and acceleration cosmology scenarios in MGTs and GR; to analyse if such effects can be modeled in the framework of the Einstein gravity theory; and to show how such generic off-diagonal cosmological solutions can be constructed and treated in anamorphic cosmology. The extensions of cosmological models to spacetimes with nontrivial quasiperiodic/aperiodic and general anistoropic structures is not a trivial task. It is necessary to elaborate on new classes of exact and/or parametric solutions of gravitational and matter field cosmological equations which, in general, depend on all spacetime variables via generating and integration functions with mixed smooth and discrete degree of freedom and anisotropically polarized physical constants. We emphasize that it is not possible to describe, for instance, growth of any QC structure and compute certain cosmological effects determined by non-perturbative and nonlinear gravitational interactions if we restrict our models to only diagonal homogeneous and isotropic metrics like the Friedmann–Lamaî tre–Roberstson–Worker, FLRW, one and possible generalizations with Lie group/algebroid symmetries [49–52]. In such cases, the cosmological solutions are determined by some integration and/or structure group constants, and depend only on a time like coordinate. We can not describe in a realistic form quasiperiodic/aperiodic spacetime structures, and their evolution, using only time-depending functions and FLRW metrics. In order to formulate and develop an unified geometric approach for all observational data on (super) cluster and extra long cosmological distances, we have to work with 'non-diagonalizable' metrics[7] and generalized connections, and apply new numeric and analytic methods for constructing more general classes of solutions in MGTs and cosmology models with quasiperiodic structure, inhomogeneities and local anisotropies. The new classes of cosmological solutions incorporate generating functions and integration functions, with various integration constants and parameters, which allow more opportunities to compare with experimental data. Even some subclasses of solutions can be parameterized by effective diagonal metrics[8], the diagonal coefficients contain various physical data of nonlinear classical and quantum interactions encoded via generating functions and effective sources.

In contrast to the general purpose of unification of physical interactions and development of fundamental and geometric principles of quantization (for instance, in string theory and deformation quantization), the approaches based on LQG and spin networks were performed originally just as theories of quantum gravity combining the general relativity (GR) and quantum mechanics. The main principle was to provide a non-perturbative formulation when the background independence (the key feature of Einstein's theory) is preserved. At the present time, LQG is supposed to have a clear conceptual and logical setup following from physical considerations and supported by a rigorous mathematical formulation. In this work, we study a toy cosmological model with LQG contributions, whilst keeping in mind that such

[7] Which can not be diagonalized by coordinate transforms, in a local or infinite spacetime region.
[8] For certain limits with small off-diagonal corrections and/or nonholonomically constrained configurations, for instance, incorporating anomorphic smoothing phases.

constructions will be expanded on for spin network models and further generalizations to QC configurations. Here, in addition to the references presented above, we cite some fundamental works [64–69] on LQG for also considering developments in loop quantum cosmology and possible extensions, for example, to deformation quantization. We emphasize that we analyze examples with a special class of holonomy corrections from LQG in order to prove that possible quantum modifications do not affect the main results on anamorphic cosmological models with QC structure.

With respect to our toy LQC model, we also note that we restrict our study to quantum gravity quasiperiodic effects in anamorphic cosmology by considering a special class of holonomy corrections from LQG in order to distinguish possible non-perturbative and background independent modifications. In this approach, quantization can be performed in certain forms preserving the Lorentz local invariance in the continuous limit. Here we note that if the quantization formalism is developed on (co)tangent bundles, one gets quantum corrections and respective cosmological terms violating this local symmetry [72]. In a more general context, such an approach involves reformulation of the LQG in nonholonomic variables with double $2 + 2$ and $3 + 1$ fibrations considered in [69, 73]. Details on the so-called ADM, i.e. Arnowitt–Deser–Misner, formalism in GR can be found, for instance, in [57–59, 74]. In order to construct new classes of cosmological solutions, we shall apply the anholonomic frame deformation method, AFDM (see details and examples for accelerating cosmology and DE and DM physics in [49, 52, 75–77]).

The paper is organized as follows: In section 2, we outline the most important formulas on nonholonomic variables, frame, linear and nonlinear connection deformations used for constructing (in general) generic off-diagonal cosmological solutions depending on all spacetime coordinates. It is shown how using such constructions we can decouple the gravitational and matter field equations in accelerating cosmology if the Einstein gravity and various $f(R)$ modifications, with LQG corrections. In nonholonomic variables we formulate the criteria for anamorphic cosmological phases and analyze possible small parametric deformations in terms of quasi-FLRW metrics for nonholonomic Friedmann equations.

Section 3 is devoted to the study of geometric properties of new classes of generic off-diagonal cosmological solutions modeling QC like structures in MGTs with LQG sources. In this section the conditions on generating and integration functions and integration constants when such configurations encode quasiperiodic/aperiodic structure of possible different origin (induced by F-modifications, gravitational like polarization of mass like constants, anamorphic phases with effective polarization of the cosmological constant, and LQC sources) are formulated. Four such classes of solutions are constructed in explicit form and the criteria for anamorphic QC phases are formulated. Here, we also provide solutions for nonlinear superpositions resulting in hierarchies with new anamorphic QC like cosmological solutions.

In section 4, we consider small parametric decompositions for quasi-FLRW metrics encoding QC like structures. It is proven that in such cases the cosmological solutions with gravitationally polarized cosmological constants and the criteria for anamorphic phases can be written in certain forms similar to homogeneous cosmological configurations. In such cases, QC and LQG modified Friedmann equations can be derived in explicit form.

We discuss the results in section 5. Appendix provides a summary on geometric methods for constructing off-diagonal and diagonal cosmological solutions.

2. Nonholonomic variables and anamorphic cosmology

To be able to construct, in explicit form, exact and parametric quasiperiodic cosmological solutions in MGTs with quantum corrections we have to re-write the fundamental gravitational and matter field equations in such nonholonomic variables when a decoupling and general integration of corresponding systems of nonlinear partial differential equations, PDEs, are possible. Readers are referred to [49, 52, 72, 73, 75–77] for details on the geometry and applications of the AFDM as a method of constructing exact solutions in gravity and Ricci flow theories. In this section, we show how such nonholonomic variables can be introduced in MGT and GR theory and formulate a geometric approach to anamorphic cosmology [1–6]. The constructions will be used in the next section for decoupling the fundamental cosmological PDEs with matter field sources and LQG corrections parameterized as in [55, 56, 69–71, 78, 79].

2.1. N-adapted frames and connection deformations in MGTs

We presume that the metric properties of a four dimensional, 4d, cosmological spacetime manifold $\mathbf{V}$ are defined by a metric $\mathbf{g}$ of pseudo-Riemannian signature $(+++-)$ which can be parameterized as a distinguished metric, *d*-metric,

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}(u)\mathbf{e}^{\alpha}\otimes\mathbf{e}^{\beta} = g_i(x^k)\mathrm{d}x^i\otimes\mathrm{d}x^i + g_a(x^k,y^b)\mathbf{e}^a\otimes\mathbf{e}^b$$
$$= \mathbf{g}_{\alpha'\beta'}(u)\mathbf{e}^{\alpha'}\otimes\mathbf{e}^{\beta'}, \text{ for } \mathbf{g}_{\alpha'\beta'}(u) = \mathbf{g}_{\alpha\beta}\mathbf{e}^{\alpha}_{\ \alpha'}\mathbf{e}^{\beta}_{\ \beta'}. \quad (1)$$

In these formulas, we use *N*-adapted frames, $\mathbf{e}_{\alpha} = (\mathbf{e}_i, e_a)$, and dual frames, $\mathbf{e}^{\alpha} = (x^i, \mathbf{e}^a)$,

$$\mathbf{e}_i = \partial/\partial x^i - N_i^a(u)\partial/\partial y^a, e_a = \partial_a = \partial/\partial y^a,$$
$$\mathrm{e}^{\mathrm{i}} = \mathrm{d}x^i, \mathbf{e}^a = \mathrm{d}y^a + N_i^a(u^{\gamma})\mathrm{d}x^i \text{ and } \mathbf{e}^{\alpha} = \mathbf{e}^{\alpha}_{\ \alpha'}(u)\mathrm{d}u^{\alpha'}. \quad (2)$$

The local coordinates on $\mathbf{V}$ are labeled $u^{\gamma} = (x^k, y^c)$, or $u = (x, y)$, when indices run corresponding values $i, j, k, ... = 1, 2$ and $a, b, c, ... = 3, 4$ (for nonholonomic $2+2$ splitting, for $u^4 = y^4 = t$ being a time like coordinate and $u^{\grave{\imath}} = (x^i, y^3)$ considered as spacelike coordinates endowed with indices $\grave{\imath}, \grave{\jmath}, \grave{k}, ... = 1, 2, 3$. We note that a local basis[9] $\mathbf{e}_{\alpha}$ is nonholonomic (equivalently, non-integrable, or anholonomic) if the commutators

$$\mathbf{e}_{[\alpha}\mathbf{e}_{\beta]} := \mathbf{e}_{\alpha}\mathbf{e}_{\beta} - \mathbf{e}_{\beta}\mathbf{e}_{\alpha} = C^{\gamma}_{\alpha\beta}(u)\mathbf{e}_{\gamma} \quad (3)$$

contain nontrivial anholonomy coefficients $C^{\gamma}_{\alpha\beta} = \{C^b_{ia} = \partial_a N^b_i, C^a_{ji} = \mathbf{e}_j N^a_i - \mathbf{e}_i N^a_j\}$.

A value $\mathbf{N} = \{N^a_i\} = N^a_i \frac{\partial}{\partial y^a}\otimes\mathrm{d}x^i$ determined by frame coefficients in (2) defines a nonlinear connection, *N*-connection, structure as an *N*-adapted decomposition of the tangent bundle

$$T\mathbf{V} = hT\mathbf{V}\oplus vT\mathbf{V} \quad (4)$$

into conventional horizontal, h, and vertical, v, subspaces. On a 4d metric-affine manifold $\mathbf{V}$, this states an equivalent fibred structure with nonholonomic $2+2$ spacetime decomposition (splitting). In particular, such a h-v-splitting states a double, h and v, diadic frame structure on any (pseudo) Riemannian spacetime. We shall use boldface symbols for geometric/physical objects on a spacetime manifold $\mathbf{V}$ endowed with geometric objects $(\mathbf{g}, \mathbf{N}, \mathbf{D})$. The values $\mathbf{D}$ is a distinguished connection, *d*-connection, $\mathbf{D} = (hD, vD)$ defined as a linear connection, i.e. a metric-affine one, preserving the *N*-connection splitting (4) under parallel transports. We denote by $\mathcal{T} = \{\mathbf{T}^{\alpha}_{\ \beta\gamma}\}$ the torsion of $\mathbf{D}$, which can be computed in standard form, see geometric preliminaries in [49, 52, 69, 72, 73, 75–77].

[9] In literature, one uses equivalent terms like frame, tetrad, vierbein systems.

On a nonholonomic spacetime manifold $\mathbf{V}$, we can work equivalently with two linear connections defined by the same metric structure $\mathbf{g}$:

$$(\mathbf{g},\mathbf{N}) \rightarrow \begin{cases} \nabla : & \nabla \mathbf{g}=0;\ {}^{\nabla}\mathcal{T}=0, \text{ for the Levi–Civita, LC, }-\text{connection} \\ \widehat{\mathbf{D}} : & \widehat{\mathbf{D}}\mathbf{g}=0;\ h\widehat{\mathcal{T}}=0, v\widehat{\mathcal{T}}=0, hv\widehat{\mathcal{T}}\neq 0, \text{ for the canonical } d-\text{connection.} \end{cases} \tag{5}$$

As a result, it is possible to formulate equivalent models of pseudo-Riemannian geometry and/or Riemann–Cartan geometry with nonholonomically induced torsion[10], see a summary of most important formulas in the appendix.

2.2. Anamorphic cosmology in nonholonomic variables

Based on invariant criteria, authors [1–6] attempted to develop a complete scenario explaining the smoothness and flatness of the universe on large scales with a smoothing phase that acts like a contracting universe. In this section, we develop a model of anamorphic cosmology in the framework of MGTs with quasiperiodic/aperiodic structures and LQC-corrections. The approach relies on having time-varying masses for particles and certain Weyl-invariant values that define certain aspects of contracting and/or expanding cosmological backgrounds. For off-diagonal cosmological models with nontrivial vacuum structures, the variation of masses and physical constants have a natural explanation via gravitational polarization functions [18–20, 49–52]. Let us denote such variations of a particle mass $m \rightarrow \check{m}(x^i,t) \simeq \check{m}(t)$ and of Planck mass $M_P \rightarrow \check{M}_P(x^i,t) \simeq \check{M}_P(t)$, which depends on the type of generating functions we consider. The actions for particle motion and modified gravity are written respectively as

$${}^{p}\mathcal{S}=\int \frac{\check{m}}{\check{M}_P}\mathrm{d}s \text{ and} \tag{6}$$

$$\mathcal{S} = \int \mathrm{d}^4u\sqrt{|\mathbf{g}|}[\mathbf{F}(\widehat{\mathbf{R}}) + {}^{m}\mathcal{L}(\phi)] \tag{7}$$

$$= \int \mathrm{d}^4u\sqrt{|\mathbf{g}|}[\frac{1}{2}\check{M}_P^2(\phi)\widehat{\mathbf{R}} - \frac{1}{2}\kappa(\phi)\mathbf{g}^{\alpha\beta}(\mathbf{e}_\alpha\phi)(\mathbf{e}_\beta\phi) - {}^{J}V(\phi) + {}^{m}\mathcal{L}(\phi)], \tag{8}$$

where $\check{M}_P^2(\phi) := M_{Pl}^0\sqrt{f(\phi)}$ is positive definite (we can work in a system of coordinates when $M_{Pl}^0 = 1$). Above actions are written for a d-metric $\mathbf{g}_{\alpha\beta}$ (1), $\kappa(\phi)$ is the nonlinear kinetic coupling function and $\widehat{\mathbf{R}}$ is the scalar curvature of $\widehat{\mathbf{D}}$. In our works, we use left labels in order to

[10] It should be emphasized that the canonical distortion relation $\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}$, where the distortion distinguished tensor, d-tensor, $\widehat{\mathbf{Z}} = \{\widehat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}[\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma}]\}$, is an algebraic combination of the coefficients of the corresponding torsion d-tensor $\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma}\}$ of $\widehat{\mathbf{D}}$. The curvature tensors of both linear connections are computed in standard forms, $\widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta}\}$ and ${}^{\nabla}\mathcal{R} = \{R^{\alpha}{}_{\beta\gamma\delta}\}$ (respectively, for $\widehat{\mathbf{D}}$ and ∇). This allows us to introduce the corresponding Ricci tensors, $\widehat{\mathcal{R}}ic = \{\widehat{\mathbf{R}}_{\ \beta\gamma} := \widehat{\mathbf{R}}^{\gamma}{}_{\alpha\beta\gamma}\}$ and $Ric = \{R_{\ \beta\gamma} := R^{\gamma}{}_{\alpha\beta\gamma}\}$. The value $\widehat{\mathcal{R}}ic$ is characterized by h-v N-adapted coefficients, $\widehat{\mathbf{R}}_{\alpha\beta} = \{\widehat{R}_{ij} := \widehat{R}^{k}{}_{ijk},\ \widehat{R}_{ia} := -\widehat{R}^{k}{}_{ika},\ \widehat{R}_{ai} := \widehat{R}^{b}{}_{aib},\ \widehat{R}_{ab} := \widehat{R}^{c}{}_{abc}\}$. There are also two different scalar curvatures, $R := \mathbf{g}^{\alpha\beta}R_{\alpha\beta}$ and $\widehat{\mathbf{R}} := \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\alpha\beta} = g^{ij}\widehat{R}_{ij} + g^{ab}\widehat{R}_{ab}$. We can also consider additional constraints resulting in zero values for the canonical d-torsion, $\widehat{\mathcal{T}} = 0$, considering some limits $\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}\rightarrow 0} = \nabla$.

denote, for instance, that ${}^{m}\mathcal{L}$ is for matter fields (for this label, m is from 'mass') and ${}^{J}V$ is for the Jordan frame representation[11]

Here we note that $\mathbf{F}(\widehat{\mathbf{R}}) = \mathbf{F}[\widehat{\mathbf{R}}(\mathbf{g}, \widehat{\mathbf{D}}, \phi)]$ is also a functional of the scalar field ϕ but we use simplified notations using the assumption that $\widehat{\mathbf{R}}(\mathbf{g},\widehat{\mathbf{D}})$ are related to ϕ by a source term for modified Einstein equations with such a nonlinear scalar field.

The gravitational field equations for MGT with functional $\mathbf{F}(\widehat{\mathbf{R}})$ in (7) can be derived by a N-adapted variational calculus, see details in [18–20, 49–52] and references therein. We obtain a system of nonlinear PDEs which can be represented in effective Einstein form,

$$\widehat{\mathbf{R}}_{\mu\nu} = \mathbf{\Upsilon}_{\mu\nu}, \tag{9}$$

where the right effective source is parameterized

$$\mathbf{\Upsilon}_{\mu\nu} = {}^{F}\mathbf{\Upsilon}_{\mu\nu} + {}^{m}\mathbf{\Upsilon}_{\mu\nu} + \overline{\mathbf{\Upsilon}}_{\mu\nu}. \tag{10}$$

Let us explain how three terms in this source are defined. The functionals $\mathbf{F}(\widehat{\mathbf{R}})$ and ${}^{1}\mathbf{F}(\widehat{\mathbf{R}}) := \mathrm{d}\mathbf{F}(\widehat{\mathbf{R}})/\mathrm{d}\widehat{\mathbf{R}}$ determine an energy-momentum tensor,

$${}^{F}\mathbf{\Upsilon}_{\mu\nu} = \left(\frac{\mathbf{F}}{2\ {}^{1}\mathbf{F}} - \frac{\widehat{\mathbf{D}}^{2}\ {}^{1}\mathbf{F}}{{}^{1}\mathbf{F}}\right)\mathbf{g}_{\mu\nu} + \frac{\widehat{\mathbf{D}}_{\mu}\widehat{\mathbf{D}}_{\nu}\ {}^{1}\mathbf{F}}{{}^{1}\mathbf{F}}. \tag{11}$$

The source for the scalar matter fields can be computed in standard form,

$${}^{m}\mathbf{\Upsilon}_{\mu\nu} = \frac{1}{2M_P^2}\ {}^{m}\mathbf{T}_{\alpha\beta}, \tag{12}$$

and the holonomic contributions from LQG, $\overline{\mathbf{\Upsilon}}_{\mu\nu}$ (20), will be defined in section 2.5. We shall be able to find, in explicit form, exact solutions for the system (9) for any source (10), which via frame transforms $\mathbf{\Upsilon}_{\mu\nu} = e^{\mu'}_{\ \mu} e^{\nu'}_{\ \nu} \mathbf{\Upsilon}_{\mu'\nu'}$ can be parameterized into N-adapted diagonalized form as

$$\mathbf{\Upsilon}^{\mu}_{\ \nu} = \mathrm{diag}[\ {}_{h}\mathbf{\Upsilon}(x^{i}),\ {}_{h}\mathbf{\Upsilon}(x^{i}),\ \mathbf{\Upsilon}(x^{i},t),\ \mathbf{\Upsilon}(x^{i},t)]. \tag{13}$$

In these formulas, the generating source functions ${}_{h}\Upsilon(x^{i})$ and $\Upsilon(x^{i},t)$ have to be prescribed in some forms which will generate exact solutions compatible with observational/ experimental data.

[11] If in (7) and (8) $\mathbf{F}(\widehat{\mathbf{R}}) = \widehat{\mathbf{R}}^{2}$, ${}^{J}V(\phi) = -\Lambda$ and ${}^{m}\mathcal{L} = 0$, we obtain a quadratic action for nonholonomic MGTs studied in [18–20, 76, 77], when $\mathcal{S} = \int \mathrm{d}^{4}u\sqrt{|\mathbf{g}|}[\widehat{\mathbf{R}}^{2} + {}^{m}\mathcal{L}]$. The equivalence of such actions to nonholonomic deformations of the Einstein gravity with scalar field sources can be derived from the invariance (both for ∇ and $\widehat{\mathbf{D}}$) under global dilatation symmetry with a constant σ, $g_{\mu\nu} \to \mathrm{e}^{-2\sigma} g_{\mu\nu}, \phi \to \mathrm{e}^{2\sigma}\widetilde{\phi}$. We can re-define the physical values from the Jordan to the Einstein, E, frame using $\phi = \sqrt{3/2}\ln|2\widetilde{\phi}|$, when ${}^{E}\mathcal{S} = \int \mathrm{d}^{4}u\sqrt{|g|}\left(\frac{1}{2}\widehat{\mathbf{R}} - \frac{1}{2}\mathbf{e}_{\mu}\phi\ \mathbf{e}^{\mu}\phi - 2\Lambda\right)$. The field equations derived from ${}^{E}\mathcal{S}$ are

$$\widehat{\mathbf{R}}_{\mu\nu} - \mathbf{e}_{\mu}\phi\ \mathbf{e}_{\nu}\phi - 2\Lambda\mathbf{g}_{\mu\nu} = 0 \text{ and } \widehat{\mathbf{D}}^{2}\phi = 0.$$

To find explicit solutions we can consider $\mathbf{\Upsilon}_{\mu\nu} \sim diag[0, 0,\ \mathbf{\Upsilon},\ \mathbf{\Upsilon}]$, where $\mathbf{\Upsilon}(t)$ will be determined by scalar fields in anamorphic QC phase and possible holonomic corrections to the Hubble constant. We obtain the Einstein gravity theory if $\mathbf{F}(\widehat{\mathbf{R}}) = R$ for $\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}\to 0} = \nabla$. For simplicity, we can consider matter actions ${}^{m}\mathcal{S} = \int \mathrm{d}^{4}u\sqrt{|\mathbf{g}|}\ {}^{m}\mathcal{L}$ for matter field Lagrange densities ${}^{m}\mathcal{L}$ depending only on coefficients of a metric field and do not depend on their derivatives when

$${}^{m}\mathbf{T}_{\alpha\beta} := -\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|}\ {}^{m}\mathcal{L})}{\delta\mathbf{g}^{\alpha\beta}} = {}^{m}\mathcal{L}\mathbf{g}^{\alpha\beta} + 2\frac{\delta({}^{m}\mathcal{L})}{\delta\mathbf{g}_{\alpha\beta}}.$$

.

2.3. Small parametric deformations for quasi-FLRW metrics

In N-adapted bases, the models of locally anisotropic and inhomogeneous anamorphic cosmology are characterized by three essential properties during the smoothing phase:

1. masses are polarized with a certain dependence on time and space like coordinates $m \to \check{m}(x^i, t)$ and/or $m \to \check{m}(t)$;
2. necessary type combinations of N-adapted Weyl-invariant signatures incorporating aspects of contracting and expanding locally anisotropic backgrounds;
3. using nonlinear symmetries of generic off-diagonal solutions $\mathbf{g}_{\alpha\beta}$ (1) and considering nonholonomic deformations on a small parameter ε, we can express, via frame transforms, the cosmological solutions of (9), with prescribed sources $[\ _h\boldsymbol{\Upsilon},\ \boldsymbol{\Upsilon}]$, in such a quasi-FLRW form[12]

$$\mathrm{d}s^2 = \widehat{a}^2(x^k, t)\mathbf{e}_{\grave{\imath}}\mathbf{e}^{\grave{\imath}} - e_4\mathbf{e}^4, \tag{14}$$

for $N_j^3 = n_j(x^k, t)$ and $N_j^4 = w_j(x^k, t)$, where

$$\begin{aligned} \mathbf{e}^{\grave{\imath}} &= (\mathrm{d}x^i, \mathbf{e}^3 = \mathrm{d}y^3 + n_j(x^k, t)\mathrm{d}x^j) \simeq (\mathrm{d}x^i, \mathrm{d}y^3 + \varepsilon\chi_j^3(x^k, t)\mathrm{d}x^j), \\ \mathbf{e}^4 &= \mathrm{d}t + w_j(x^k, t)\mathrm{d}x^j \simeq \mathrm{d}t + \varepsilon\chi_i^4(x^k, t)\mathrm{d}x^j. \end{aligned} \tag{15}$$

The locally anisotropic scale coefficient can be considered as isotropic in certain limits (for additional assumptions on homogeneity), $\widehat{a}^2(x^k, t) \simeq \widehat{a}^2(t)$ and computed together with effective polarization functions χ_j^3 and χ_i^4 all encoding data on possible nonlinear generic off-diagonal interactions, QC and/or LQG contributions. In next section, we shall prove how such values can be computed for certain classes of generic off-diagonal exact solutions in MGTs and GR.

Using the effective scale factor $\widehat{a}^2$ from (14), we can introduce the respective effective and locally anisotropically polarized Hubble parameter,

$$\widehat{H} := e_4(\ln\widehat{a}) = \partial_t(\ln\widehat{a}) = (\ln\widehat{a})^*. \tag{16}$$

Considering a new time like coordinate $\check{t}$, for $t = t(x^i, \check{t})$ and transforming $\sqrt{|h_4|}\partial t/\partial\check{t}$ into a scale factor $\widehat{a}(x^i, \check{t})$, we represent (15) in the form

$$\begin{aligned} &\mathrm{d}s^2 = \check{a}^2(x^i, \check{t})[\eta_i(x^k, \check{t})(\mathrm{d}x^i)^2 + \check{h}_3(x^k, \check{t})(\mathbf{e}^3)^2 - (\check{\mathbf{e}}^4)^2], \\ &\text{where } \eta_i = \check{a}^{-2}\mathrm{e}^{\psi}, \check{a}^2\check{h}_3 = h_3, \mathbf{e}^3 = \mathrm{d}y^3 + \partial_k n\, \mathrm{d}x^k, \check{\mathbf{e}}^4 = \mathrm{d}\check{t} + \sqrt{|h_4|}(\partial_i t + w_i). \end{aligned} \tag{17}$$

For a small parameter ε, with $0 \leqslant \varepsilon < 1$, we the off-diagonal deformations are given by effective polarization functions

$$\eta_i \simeq 1 + \varepsilon\chi_i(x^k, \widehat{t}), \partial_k n \simeq \varepsilon\widehat{n}_i(x^k), \sqrt{|h_4|}\ w_i \simeq \varepsilon\widehat{w}_i(x^k, \widehat{t}).$$

We can work, for convenience, with both types of nonholonomic ε-deformations of FLRW metrics (nonholonomic FLRW models). Such approximations can be considered after a generic off-diagonal cosmological solution was constructed in a general form.

[12] This term means that for $\varepsilon \to 0$ and any approximation $\widehat{a}^2(t)$ a standard FLRW metric is generated.

2.4. Effective FLRW geometry for nonholonomic MGTs

Following a N-adapted variational calculus for MGTs Lagrangians resulting in respective dynamical equations (see similar holonomic variants in [55, 56]), we can construct various models of locally anisotropic spacetimes [18–20, 49–52][13]. For $\mathbf{\Upsilon}_{\mu\nu} = {}^F\mathbf{\Upsilon}_{\mu\nu}$ in (9) and a d-metric (1) with diagonal homogeneous approximations, we obtain from (7) that in the Einstein frame

$$
{}^F\mathcal{S} = M_P^2 \int d^4u \sqrt{|\mathbf{g}|}\mathbf{F}(\widehat{\mathbf{R}}) \rightarrow {}^{EF}\mathcal{S} = M_P^2 \int d^4u \, {}^{EF}\mathcal{L},
$$
$$
\text{for } {}^{EF}\mathcal{L} = \overline{a}^3 \left[\frac{1}{2}\overline{\mathbf{R}} + \frac{1}{2}\left(\frac{\partial\overline{\phi}}{\partial\overline{t}}\right)^2 - V(\overline{\phi}) \right], \text{ where } \overline{\mathbf{R}} = 6\frac{\partial\overline{H}}{\partial\overline{t}} + 12\overline{H}^2.
$$

In these formulas, $V(\overline{\phi})$ is an effective potential and $\overline{a}$ and $\overline{\phi}$ are independent variables defined correspondingly by

$$
\overline{a} := \sqrt{{}^1\mathbf{F}(\widehat{\mathbf{R}})}\widehat{a}, \; d\overline{t} := \sqrt{{}^1\mathbf{F}(\widehat{\mathbf{R}})}dt, \; \frac{\partial}{\partial\overline{t}} := \overline{\partial};
$$
$$
\overline{\phi} := \sqrt{\frac{3}{2}} \ln|{}^1\mathbf{F}(\widehat{\mathbf{R}})|, V(\overline{\phi}) = \frac{1}{2}\left[\frac{\widehat{\mathbf{R}}}{{}^1\mathbf{F}(\widehat{\mathbf{R}})} - \frac{\mathbf{F}(\widehat{\mathbf{R}})}{\left({}^1\mathbf{F}(\widehat{\mathbf{R}})\right)^2} \right]. \tag{18}
$$

Using above variables for the Hamiltonian constraint ${}^{EF}\mathcal{H} := \overline{\partial}\overline{a}\frac{\partial({}^{EF}\mathcal{L})}{\partial\overline{\partial}\overline{a}} + \overline{\partial}\overline{\phi}\frac{(\partial\, {}^{EF}\mathcal{L})}{\partial\overline{\partial}\overline{\phi}} - {}^{EF}\mathcal{L}$ and effective density

$$
\overline{\rho} := \frac{1}{2}(\overline{\partial}\overline{\phi})^2 + V(\overline{\phi}), \tag{19}
$$

we express the effective Friedmann equation (in the Einstein frame, it is a constraint) $3\overline{H}^2 = \overline{\rho}$ when the dynamics is given by the conservation law $\overline{\partial}\overline{\rho} = -3\overline{H}(\overline{\partial}\overline{\phi})^2$. This dynamics is encoded also in an effective Raychaudhury equation $2\overline{\partial}\overline{H} = -(\overline{\partial}\overline{\phi})^2$, with $(\overline{\partial}\overline{\rho})^2 = 3\overline{\rho}(\overline{\partial}\overline{\phi})^2$.

2.5. LQC extensions of MGTs

LQC corrections to MGTs have been studied in series of works [55, 56, 70, 71, 78, 79]. As standard variables (we follow our notations (18)), we use $\overline{\beta} := \overline{\gamma}\overline{H}$, where $\overline{\gamma}$ is the Barbero–Immirzi parameter [66–68], and the volume $\overline{V} := \overline{a}^3$. For diagonal configurations, the holonomy corrections to the Friedemann equations are of type

$$
\overline{H}^2 = \frac{\overline{\rho}}{3}(1 - \frac{\overline{\rho}}{\overline{\rho}_c}), \tag{20}
$$

[13] In our works, we have to elaborate more 'sophisticate' systems of notations because such geometric modeling of cosmological scenarios and methods of constructing solutions of PDEs should include various terms with h-v-splitting; discrete and continuous classical and quantum corrections, diagonal and off-diagonal terms, different types of connections which were not considered in other works by other authors. The most important conventions on our notations are that we use boldface symbols for the spaces and geometric objects endowed with N-connection structure and that left labels are abstract ones associated to some classes of geometric/physical objects. Right Latin and Greek indices can be abstract ones or transformed into coordinate indices with possible h- and v-splitting. Unfortunately, it is not possible to simplify such a system of notations if we follow multiple purposes related to geometric methods of constructing exact solutions in gravity and cosmology theories, analysing different phases of anamoprhic cosmology with generic off-diagonal terms etc.

where the critical density $\overline{\rho}_c := 2/\sqrt{3}\gamma^3$ is computed in EF (see [60] for a status report on different approaches to LCQ). This formula can be applied for small deformations with respect to N-adapted frames taking, for simplicity, a function $\overline{\rho}(t)$ determining the component $\overline{\Upsilon}_{\mu\nu}$ in (9) and (10).

In a more general context, we can consider locally anisotropic configurations with $\overline{\rho}(x^i,t)$ associated to any ${}^{EF}\mathcal{H}[\overline{\beta}(x^i,t),\overline{V}(x^i,t)]$, with conjugated Poisson bracket $\{\overline{\beta}(x^i,t),\overline{V}(x^i,t)\} = \overline{\gamma}/2$, when $\overline{H}(x^i,t) = \frac{\sin(\sqrt{2\sqrt{3}\overline{\gamma}}\overline{\beta}(x^i,t))}{\sqrt{2\sqrt{3}\overline{\gamma}}}$, for a re-scaling in order to have a well-defined quantum theory. We note that there were formulated different models and inequivalent approaches to LQG and LQC, see a variant [69] which is compatible with deformation quantization. For simplicity, we shall add the term

$$\overline{\Upsilon} = -\frac{\overline{\rho}^2}{3\overline{\rho}_c} \tag{21}$$

in N-adapted $\overline{\Upsilon}_{\mu\nu} = \mathrm{diag}[\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon},\overline{\Upsilon}]$, see below the formula (27), as an additional LQG contribution in the right part of certain generalized Friedmann equations with a nonlinear redefinition of scalar field effective density $\frac{\overline{\rho}}{3} \to \frac{\overline{\rho}}{3}(1-\frac{\overline{\rho}}{\overline{\rho}_c})$.

2.6. Nonholonomic Friedmann eqs in anamorphic cosmology with LQG corrections

The cosmological models with generic off-diagonal metrics parameterized in N-adapted form with respect to bases (15) are characterized by two dimensionless quantities (being Weyl-invariant if the homogeneity conditions are imposed),

$$\begin{aligned} {}^{m}\Theta &:= (\widehat{H}+\overline{H}+\frac{\check{m}^*}{\check{m}})\check{M}_P^{-1} = \frac{\check{\alpha}_m^*}{\check{\alpha}_m \check{M}_P} \text{ for } \check{\alpha}_m := \widehat{a}\check{m}/M_P^0; \\ {}^{Pl}\Theta &:= (\widehat{H}+\overline{H}+\frac{\check{M}_P^*}{\check{M}_P})\check{M}_P^{-1} = \frac{\check{\alpha}_{Pl}^*}{\check{\alpha}_{Pl}\check{M}_{Pl}} \text{ for } \check{\alpha}_{Pl} := \widehat{a}\check{M}_P/M_P^0 \end{aligned} \tag{22}$$

for M_P^0 being the value of the reduced Planck mass in the frame where it does not depend on time. These values distinguish respectively such cosmological models (see details in [1, 2] but for holonomic structures):

$$\begin{array}{llll} & \text{anamorphosis} & \text{inflation} & \text{ekpyrosis} \\ & & & \\ {}^{m}\Theta\ (\text{background}) & <0\ (\text{contracts}) & >0\ (\text{expands}) & <0\ (\text{contracts}) \\ {}^{Pl}\Theta\ (\text{curvaturepert.}) & >0\ (\text{grow}) & >0\ (\text{grow}) & >0\ (\text{decay}). \end{array} \tag{23}$$

Here we note that the priority of the AFDM is that we can consider any cosmological solution in a MGT or GR and than to write it in N-adapted form with ε-deformations. This allows us to compute all physical important values like ${}^{m}\Theta$ and ${}^{Pl}\Theta$ and analyse if and when an anamporhic phase is possible. We note that ${}^{m}\Theta$ is negative, for instance, as in modified ekpyrotic models, but ${}^{Pl}\Theta$ is positive as in locally anisotropic inflationary models. In such theories, the effective $\check{m}$ and $\check{M}_P$ are determined by certain QC and/or LQC configurations.

Reproducing in N-adapted frames for d-metrics of type (14) the calculus presented in appendix A (with Einstein and Jordan nonholonomic frame representations) of [1], we obtain respectively such a version of locally anisotropic and inhomogeneous first and second Friedmann equations,

$$3(\,^{m}\Theta)^2 = \left[\frac{\,^{A}\rho + \,^{m}\rho + \,^{A}\rho/\sqrt{\overline{\rho}_c}}{\check{M}_P^4} - (\frac{\check{m}}{\check{M}_P})^2\frac{\kappa}{\check{\alpha}_m^2} + (\frac{\check{m}}{\check{M}_P})^6\frac{\sigma^2}{\check{\alpha}_m^6}\right]\left[1 - \partial(\frac{\check{m}}{\check{M}_P})/\partial\ln\check{\alpha}_m\right]^{-2},$$
$$(\,^{Pl}\Theta)^* = -(\,^{A}\rho + \,^{m}\rho + \,^{A}\rho/\sqrt{\overline{\rho}_c})/2\check{M}_P^3. \tag{24}$$

In these formulas, $K(\phi) := [\frac{3}{2}(f_{,\phi})^2) + \kappa(\phi)f(\phi)]/f^2(\phi)$ and the values (energy density, pressure)

$$2(M_P^0)^4\,(\,^{A}\rho) := K(\phi)(\phi^*)^2 + \check{M}_P^4\,^{J}V(\phi)/f^2(\phi),\ 2(M_P^0)^4\,(\,^{A}p) := K(\phi)(\phi^*)^2 - \check{M}_P^4\,^{J}V(\phi)/f^2(\phi),$$

are determined by coefficients in (8), for $\kappa = (+1, 0, -1)$ being the spacial curvature, and the constant σ^2 should be considered if we try to limit the background cosmology to that described by a homogeneous and anisotropic Kasner-like metric (see formula (A.5) in [1]). For simplicity, we shall consider in this work $\sigma^2 = 1$ even MGTs can contain certain locally anisotropic configurations.

Finally, we note that we can identify $^{A}\rho$ with $\overline{\rho}$ (19) for F-modified gravity theories.

3. Off-diagonal anamorphic cosmology in MGT and LQG

Applying the anholonomic frame deformation method, AFDM, we can construct various classes of off-diagonal and diagonal cosmological solutions of (modified) gravitational field equations (9). After the metric, frame and connection structure, and the effective sources (10), have been parameterized in N-adapted form, we can select necessary type diagonal or off-diagonal configurations, consider small parameter decompositions, and approximate the generating/integration functions to some constant values compatible with observational data. We do not repeat that geometric formalism and refer readers to [49, 52, 72, 75, 76] for details on AFDM and applications in modern cosmology. The purpose of this section is to state the conditions for the generating functions and (effective) sources and quantum corrections which describe quasiperiodic/aperiodic quasicrystal, QC, like cosmological structures. There are used necessary type quadratic line elements for general solutions found in in the mentioned references and the partner papers [18–20].

3.1. Generating functions encoding QC like MGT and LQG corrections

The metrics for off-diagonal locally anisotropic and inhomogeneous cosmological spacetimes are defined as solutions, with nonholonomically induced torsion and Killing symmetry on $\partial/\partial y^3$[14]. Via nonholonomic frame transforms, such metrics can be always written in a coordinate basis, $\mathbf{g} = g_{\alpha\beta}(x^k, t)\mathrm{d}u^\alpha \otimes \mathrm{d}u^\beta$, and/or in N-adapted form (1),

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j + \{h_3[\mathrm{d}y^3 + (\,_1n_k + \,_2n_k\int \mathrm{d}t\frac{(\partial_t\Psi)^2}{\Upsilon^2|h_3|^{5/2}})\mathrm{d}x^k]^2 - (\frac{\partial_t\Psi}{2\Upsilon|h_3|^{1/2}})^2\,[\mathrm{d}t + \frac{\partial_i\Psi}{\partial_t\Psi}\mathrm{d}x^i]^2\}, \tag{25}$$

$$h_3 = -\partial_t(\Psi^2)/\Upsilon^2\left(h_3^{[0]}(x^k) - \int \mathrm{d}t\partial_t(\Psi^2)/4\Upsilon\right). \tag{26}$$

[14] For simplicity, in this work we do not consider more general classes of solutions with generic dependence on all spacetime coordinates and do analyze the details how Levi–Civita, LC, configurations can be extracted by solving additional nonholonomic constraints, see [49, 52, 72, 75, 76] and references therein.

In this formula, $g_{ij} = \delta_{ij}\mathrm{e}^{\,\psi(x^k)}$ and ${}_1n_k(x^i)$, ${}_2n_k(x^i)$ and $h_a^{[0]}(x^k)$ are integration functions, see details in appendix. The coefficient h_3, or $\Psi(x^i,t)$, is the generating function[15] and the generating h- and v-sources (see (10) and (13)) are given by terms of effective gravity modifications, matter field and LQG contributions,

$$ {}_h\Upsilon(x^i) = {}_h^F\boldsymbol{\Upsilon}(x^i) + {}_h^m\boldsymbol{\Upsilon}(x^i) + {}_h\overline{\boldsymbol{\Upsilon}}(x^i) \text{ and } \boldsymbol{\Upsilon}(x^i,t) = {}^F\boldsymbol{\Upsilon}(x^i,t) + {}^m\boldsymbol{\Upsilon}(x^i,t) + \overline{\boldsymbol{\Upsilon}}(x^i,t). \tag{27} $$

Off-diagonal metrics of type (25) posses an important nonlinear symmetry, which allows us to re-define the generating function and generating source

$$ \begin{aligned} &(\Psi,\ \Upsilon) \leftrightarrow (\Phi, \Lambda = \mathrm{const}), \text{ when } \Upsilon(x^k,t) \to \ \Lambda, \text{ for} \\ &\Phi^2 = \Lambda \int \mathrm{d}t \Upsilon^{-1}\partial_t(\Psi^2) \text{ and } \Psi^2 = \Lambda^{-1}\int \mathrm{d}t\Upsilon\partial_t(\Phi^2), \end{aligned} \tag{28} $$

by introducing an effective cosmological constant Λ as a source and the functional $\Phi(\Lambda,\Psi,\ \Upsilon)$ as a new generating function. This property can be proven by considering the relation $\Lambda\partial_t(\Psi^2) = \Upsilon\partial_t(\Phi^2)$ in above formulas for the d-metric. We can consider that nonlinear generic off-diagonal interactions on MGTs may induce an effective cosmological constant with splitting, $\Lambda = {}^F\Lambda + {}^m\Lambda + \overline{\Lambda}$. The terms of this sum are determined respectively by modifications of GR resulting in ${}^F\Lambda$; by nonlinear interactions of matter (i.e. scalar field ϕ) resulting in ${}^m\Lambda$; and by an effective $\overline{\Lambda}$ associated to holonomy modifications from LQG. Technically, it is more convenient to work with some data (Φ,Λ) for generating solutions and then to redefine the formulas in terms of generating function and generalized source $(\Psi,\ \Upsilon)$. We can also extract torsionless cosmological configurations[16].

The generating functions and/or sources can be chosen in such forms that the cosmological spacetime solutions encode nontrivial gravitational and/or matter field quasicrystal like, QC, configurations and possible additional LQG effects. We use an additional $3+1$ decomposition with spacelike coordinates $x^{\grave{\imath}}$ (for $\grave{\imath} = 1,2,3$), time like coordinate $y^4 = t$, being adapted to another $2+2$ decomposition with a fibration by 3d hypersurfaces $\widehat{\Xi}_t$, see details in [73, 74]. For such configurations, we can consider a canonical nonholonomically deformed Laplace

[15] We note that such solutions are defined in explicit form by coefficients of (1) computed in this form (see sketch of proofs in appendix):

$$ g_i = \mathrm{e}^{\,\psi(x^k)} \text{ is a solution of } \psi^{\bullet\bullet} + \psi'' = 2\ {}_h\Upsilon; $$
$$ g_3 = h_3 = -\partial_t(\Psi^2)/\Upsilon^2\left(h_3^{[0]}(x^k) - \int \mathrm{d}t\partial_t(\Psi^2)/4\Upsilon\right);\ g_4 = h_4^{[0]}(x^k) - \int \mathrm{d}t\partial_t(\Psi^2)/\,4\Upsilon; $$
$$ N_k^3 = n_k(x^i,t) = {}_1n_k(x^i) + {}_2n_k(x^i)\int \mathrm{d}t(\partial_t\Psi)^2/\Upsilon^2\left|h_3^{[0]}(x^i) - \int \mathrm{d}t\ \partial_t(\Psi^2)/4\Upsilon\right|^{\frac{5}{2}};\ N_i^4 = w_i(x^k,t) = \frac{\partial_i\Psi}{\partial_t\Psi}. $$

[16] The nonholonomically induced torsion of solutions (25) can be constrained to be zero by choosing certain subclasses of generating functions and sources. We have to consider a subclass of generating functions and sources when for $\Psi = \check{\Psi}(x^i,t), \partial_t(\partial_i\check{\Psi}) = \partial_i(\partial_t\check{\Psi})$ and $\Upsilon(x^i,t) = \Upsilon[\check{\Psi}] = \check{\Upsilon}$, or $\Upsilon = \mathrm{const}$. Then, we can introduce functions $\check{A}(x^i,t)$ and $n(x^k)$ subjected to the conditions that $w_i = \check{w}_i = \partial_i\check{\Psi}/\partial_t\check{\Psi} = \partial_i\check{A}$ and $n_k = \check{n}_k = \partial_k n(x^i)$. Such assumptions are considered in order to simplify the formulas for cosmological solutions (see details in [49, 52, 72, 75, 76], where the AFDM is applied for generating more general classes of solutions depending on all spacetime coordinates. We obtain a quadratic line element defining generic off-diagonal LC-configurations (a proof is sketched at the end of appendix),

$$ \mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j + \{h_3[\mathrm{d}y^3 + (\partial_k n)\mathrm{d}x^k]^2 - \frac{1}{4h_3}\left[\frac{\partial_t\check{\Psi}}{\check{\Upsilon}}\right]^2\ [\mathrm{d}t + (\partial_i\check{A})\mathrm{d}x^i]^2\}. $$

operator ${}^{b}\widehat{\Delta} := ({}^{b}\widehat{D})^2 = b^{\grave{i}\grave{j}}\widehat{D}_{\grave{i}}\widehat{D}_{\grave{j}}$ (determined by the 3d part of d-metric) as a distortion of ${}^{b}\Delta := ({}^{b}\nabla)^2$. Such a value can be defined and computed on any $\widehat{\Xi}_t$ using a d-metric (1) and respective 3d space like projections/ restrictions of $\widehat{\mathbf{D}}$. We chose a subclass of generating functions $\Psi = \Pi$ subjected to the condition that it is a solution of an evolution equation (with conserved dynamics) of type

$$\frac{\partial \Pi}{\partial t} = {}^{b}\widehat{\Delta}\left[\frac{\delta \mathcal{F}}{\delta \Pi}\right] = -\,{}^{b}\widehat{\Delta}(\Theta\Pi + Q\Pi^2 - \Pi^3). \tag{29}$$

Such a nonlinear PDE can be derived for a functional defining an effective free energy

$$\mathcal{F}[\Pi] = \int \left[-\frac{1}{2}\Pi\Theta\Pi - \frac{Q}{3}\Pi^3 + \frac{1}{4}\Pi^4\right]\sqrt{b}\mathrm{d}x^1\mathrm{d}x^2\delta y^3, \tag{30}$$

where $b = \det|b_{\grave{i}\grave{j}}|$ is the determinant of the 3d spacelike metric, $\delta y^3 = \mathbf{e}^3$ and the operator Θ and parameter Q are defined in the partner works [18, 19]. Different choices of Θ and Q induce different classes of quasiperiodic, aperiodic and/or QC order of corresponding classes of gravitational solutions. We note that the functional (30) is of Lyapunov type considered in quasicrystal physics, see [30, 31, 38] and references therein, and for applications of geometric flows in modern cosmology and astrophysics, with generalized Lyapunov–Perelman functionals [73, 76, 77]. In this paper, we do not enter into details how certain QC structures and their quasiperiodic/aperiodic deformations can be reproduced in explicit form but consider that such configurations can always be modelled by some evolution equations derived for a respective free energy. The generating/integration functions and parameters should be chosen in certain forms which are compatible with experimental data.

Let us explain how the quadratic element (25) defines exact solutions of MGT field equation (9). We prescribe the generating function and sources with respective associated constants, i.e. certain data for $\Phi(x^i,t)$; ${}^{F}\Lambda$, ${}^{m}\Lambda$, $\overline{\Lambda}$ (defining their sum Λ); ${}^{F}_{h}\boldsymbol{\Upsilon}(x^i)$, ${}^{m}_{h}\boldsymbol{\Upsilon}(x^i)$, ${}_{h}\overline{\boldsymbol{\Upsilon}}(x^i)$ and ${}^{F}\boldsymbol{\Upsilon}(x^i,t)$, ${}^{m}\boldsymbol{\Upsilon}(x^i,t)$, $\overline{\boldsymbol{\Upsilon}}(x^i,t)$ (defining respective sums ${}_{h}\Upsilon$ and Υ). Using formulas (28), we compute the parametric functional dependence $\Pi = \Psi[\Phi;\ {}^{F}\Lambda,\ {}^{m}\Lambda,\ \overline{\Lambda};\ {}^{F}\boldsymbol{\Upsilon},\ {}^{m}\boldsymbol{\Upsilon},\ \overline{\boldsymbol{\Upsilon}}]$ from

$$\Pi^2 = ({}^{F}\Lambda + {}^{m}\Lambda + \overline{\Lambda})^{-1}\int \mathrm{d}t({}^{F}\boldsymbol{\Upsilon} + {}^{m}\boldsymbol{\Upsilon} + \overline{\boldsymbol{\Upsilon}})\partial_t(\Phi^2).$$

As a result, we can find, in explicit form, the coefficients of d-metric (1) parameterized in the form (17), for the class of generic off-diagonal solutions with Killing symmetry on ∂_3,

$$\begin{aligned}
g_i &= \breve{a}^2\eta_i = \mathrm{e}^{\,\psi(x^k)} \text{ is a solution of } \psi^{\bullet\bullet} + \psi'' = 2\,({}^{F}_{h}\boldsymbol{\Upsilon} + {}^{m}_{h}\boldsymbol{\Upsilon} + {}_{h}\overline{\boldsymbol{\Upsilon}});\\
g_3 &= h_3(x^i,t) = \breve{a}^2\breve{h}_3(x^k,\breve{t}) = -\frac{\partial_t(\Pi^2)}{({}^{F}\boldsymbol{\Upsilon} + {}^{m}\boldsymbol{\Upsilon} + \overline{\boldsymbol{\Upsilon}})^2\left(h_3^{[0]}(x^k) - \int \mathrm{d}t\frac{\partial_t(\Pi^2)}{4({}^{F}\boldsymbol{\Upsilon} + {}^{m}\boldsymbol{\Upsilon} + \overline{\boldsymbol{\Upsilon}})}\right)};\\
g_4 &= h_4(x^i,t) = -\breve{a}^2 = h_4^{[0]}(x^k) - \int \mathrm{d}t\frac{\partial_t(\Pi^2)}{4({}^{F}\boldsymbol{\Upsilon} + {}^{m}\boldsymbol{\Upsilon} + \overline{\boldsymbol{\Upsilon}})};\\
N_k^3 &= n_k(x^i,t) = {}_{1}n_k(x^i) + {}_{2}n_k(x^i)\int \mathrm{d}t\frac{(\partial_t\Pi)^2}{({}^{F}\boldsymbol{\Upsilon} + {}^{m}\boldsymbol{\Upsilon} + \overline{\boldsymbol{\Upsilon}})^2|h_3^{[0]}(x^i) - \int \mathrm{d}t\,\frac{\partial_t(\Pi^2)}{4({}^{F}\boldsymbol{\Upsilon} + {}^{m}\boldsymbol{\Upsilon} + \overline{\boldsymbol{\Upsilon}})}|^{\frac{5}{2}}};\\
N_i^4 &= w_i(x^k,t) = \partial_i\,\Pi/\partial_t\Pi.
\end{aligned} \tag{31}$$

We emphasize that these formulas allow, for instance, to 'switch off' the contributions from LQG if we fix $\overline{\Lambda} = 0$ and $\overline{\boldsymbol{\Upsilon}}$ but consider nontrivial values for ${}^{F}\Lambda + {}^{m}\Lambda$ and ${}^{F}_{h}\boldsymbol{\Upsilon} + {}^{m}_{h}\boldsymbol{\Upsilon}$.

The values $({}_h\Upsilon, \Upsilon)$ define certain nonholonomic constraints on the sources and dynamics of (effective) matter fields and quantum corrections which allows us to integrate a system of nonlinear PDEs in explicit form and with decoupled h- v-cosmological evolution in certain N-adapted systems of reference. In explicit form, we compute using coefficients of $\widehat{\mathbf{D}}$ for a class of solutions (25). At the next step, it is possible to compute ${}^F\mathbf{\Upsilon}_{\mu\nu}$ (11),${}^m\mathbf{\Upsilon}_{\mu\nu}$(12) and $\overline{\mathbf{\Upsilon}}_{\mu\nu}$ (21) for arbitrary physically motivated values of F-modifications and solutions for scalar field ϕ. For instance, we generate physically motivated solutions by considering ε-parametric deformations (14) of some well defined cosmological solutions in GR or other type MGT, see examples [49, 52, 72]. For such small off diagonal locally anisotropic deformations, we have to chose $\widehat{a}^2(x^k,t)$ and $\chi^a_j(x^k,t)$ to be compatible with experimental gravity and observation cosmology data.

Other important examples with redefinition and/or prescription of the generating function and source are those when the integration functions in a class of metrics (25) are stated to be some constants and, for instance, $\Phi(x^i,t) \simeq \Phi(t)$, which results in some data $(\Pi(t),\ \Upsilon(t))$ following formulas (28). It is also possible to work with ε-parametric data $(\Phi(\varepsilon,x^i,t),\Lambda)$, and respective $(\Pi(\varepsilon,x^i,t),\ \Upsilon(\varepsilon,x^i,t))$, resulting formulas (15) for quasi-FLRW metrics (14). Here, it should be emphasized that even some further diagonal approximations with $\widehat{a}^2(x^k,t) \simeq \widehat{a}^2(t)$ will be considered, we shall generate FLRW metrics encoding partially some data on nonlinear and/or off-diagonal interactions, MGT terms and LQG corrections. Such solutions can not be found if we introduce diagonal homogeneous cosmological ansatz which transform, from the very beginning, the nonlinear systems of PDEs into some ODEs (related to gravitational and matter field equations in respective theories of gravity and cosmology).

3.2. N-adapted Weyl-invariant quantities for anamorphic phases

For any generic off-diagonal solution (25), we can compute with respect to N-adapted the $\breve{\alpha}$-coefficients and values ${}^m\Theta$ and ${}^{Pl}\Theta$ in (22). Re-writing such solutions in the form (17), with re-defined time like and space coordinates and scaling factor $\widehat{a} = \breve{a}$. Here, we note that we can model nonlinear off-diagonal interactions of gravitational and (effective) matter field interactions in terms of conventional polarization functions of fundamental physical constants (such values are introduced by analogy with electromagnetic interactions in certain classical or quantum media). Let us denote

$$\breve{m} = m_0\breve{\eta}(x^i,t) \text{ and } \breve{M}_{Pl} = M^0_{Pl}\sqrt{f(\phi)} = M^0_{Pl}\eta_{Pl}(x^i,t), \tag{32}$$

where $\breve{\eta}(x^i,t)$ and $\eta_{Pl}(x^i,t)$ are respective polarization of a particle mass m_0 and Planck constant M^0_{Pl}. The $\breve{\alpha}$-coefficients in off-diagonal backgrounds are expressed $\breve{\alpha}_m := \widehat{a}\breve{\eta}m_0/M^0_P$ and $\breve{\alpha}_{Pl} := \widehat{a}\eta_{Pl}$.

The values for analyzing the conditions for anamorphic phases of (25) are computed

$${}^m\Theta[\Pi]\ M^0_{Pl}\eta_{Pl} := \widehat{H} + \overline{H} + \breve{\eta}^* = (\ln|\widehat{a}\breve{\eta}|)^* \text{ and } {}^{Pl}\Theta[\Pi]\ M^0_{Pl}\eta_{Pl} := \widehat{H} + \overline{H} + \eta^*_{Pl} = (\ln|\widehat{a}\eta_{Pl}|)^*, \tag{33}$$

where the Hubble functions, $\widehat{H}$ (16) and $\overline{H}$ (20) are considered for (31) with $h_4 = -\breve{a}^2$ and $\overline{\rho}$ (19),

$$\widehat{H} = (\ln\widehat{a})^* = \frac{1}{2}\left(\ln\left|h^{[0]}_4(x^k) - \int \mathrm{d}t\frac{\partial_t(\Pi^2)}{4(\ {}^F\mathbf{\Upsilon} +\ {}^m\mathbf{\Upsilon} + \overline{\mathbf{\Upsilon}})}\right|\right)^* \text{ and } \overline{H} = \sqrt{\left|\frac{\overline{\rho}}{3}(1-\frac{\overline{\rho}}{\overline{\rho}_c})\right|}.$$

A generating function $\Pi = \Psi[\Phi;\ {}^F\Lambda,\ {}^m\Lambda,\ \overline{\Lambda};{}^F\mathbf{\Upsilon},\ {}^m\mathbf{\Upsilon},\ \overline{\mathbf{\Upsilon}}]$ may induce anamorphic cosmological phases following the conditions (23) determined by the data for the integration function $h^{[0]}_4(x^k)$; effective sources ${}^F\mathbf{\Upsilon}$, ${}^m\mathbf{\Upsilon}$, $\overline{\mathbf{\Upsilon}}$ and $\overline{\rho}$ contained in the sum $\widehat{H} + \overline{H}$.

The polarizations $\check{\eta}(x^i,t)$ and $\eta_{Pl}(x^i,t)$ modify ${}^{m}\Theta[\Pi]$ and ${}^{Pl}\Theta[\Pi]$ as follow from (33). Such values can be used for characterizing locally anisotropic cosmological models, even the analogs of generalized Friedmann equations (24) for all types of generating functions[17]. We compute

	anamorphosis	inflation	ekpyrosis
$M_{Pl}^0\ {}^{m}\Theta[\Pi] = (\ln\lvert\sqrt{\lvert h_4[\Pi]\rvert}\check{\eta}\rvert)^*/\eta_{Pl}$	<0 (contracts)	>0 (expands)	<0 (contracts)
$M_{Pl}^0\ {}^{Pl}\Theta[\Pi] = (\ln\lvert\sqrt{\lvert h_4[\Pi]\rvert}\eta_{Pl}\rvert)^*/\eta_{Pl}$	>0 (grow)	>0 (grow)	>0 (decay).

Such conditions impose additional nonholonomic constraints on generating functions, sources and integration functions and constants which induce QC structures as follow from (30).

3.3. Cosmological QCs for (effective) matter fields and LQG

Quasiperiodic cosmological structures can be induced by nonholonomic distributions of (effective) matter fields sources and quantum corrections.

3.3.1. Effective QC matter fields from MGT. Let us consider an effective scalar field $\overline{\phi} := \sqrt{\frac{3}{2}}\ln\lvert\ {}^{1}\mathbf{F}(\widehat{\mathbf{R}})\rvert$ with nonlinear scalar potential $V(\overline{\phi}) = \frac{1}{2}[\widehat{\mathbf{R}}/\ {}^{1}\mathbf{F}(\widehat{\mathbf{R}}) - \mathbf{F}(\widehat{\mathbf{R}})/\left({}^{1}\mathbf{F}(\widehat{\mathbf{R}})\right)^2]$ determined by a modification of GR, see (18). This results in an effective matter density $\overline{\rho} := \frac{1}{2}(\overline{\partial\phi})^2 + V(\overline{\phi})$ and respective ${}^{EF}\mathcal{L}$. Considering that $V(\overline{\phi})$ is chosen in a form that $\overline{\phi} = {}^{qc}\phi$ (the label qc emphasises modeling a QC structure) is a solution of

$$\frac{\partial({}^{qc}\phi)}{\partial t} = {}^{b}\widehat{\Delta}\left[\frac{\delta({}^{qc}\mathcal{F})}{\delta({}^{qc}\phi)}\right] = -{}^{b}\widehat{\Delta}[\Theta\,{}^{qc}\phi + Q({}^{qc}\phi)^2 - ({}^{qc}\phi)^3]$$

with effective free energy ${}^{qc}\mathcal{F}[{}^{qc}\phi] = \int\left[-\frac{1}{2}({}^{qc}\phi)\Theta({}^{qc}\phi) - \frac{Q}{3}({}^{qc}\phi)^3 + \frac{1}{4}({}^{qc}\phi)^4\right]\sqrt{b}\mathrm{d}x^1\mathrm{d}x^2\delta y^3$. This induces an effective matter source of type (11), when ${}^{qc}\boldsymbol{\Upsilon}_{\mu\nu} = {}^{F}\boldsymbol{\Upsilon}_{\mu\nu}[{}^{qc}\phi] = \mathrm{diag}({}^{qc}_{h}\Upsilon, {}^{qc}\Upsilon)$ is taken for an energy momentum tensor ${}^{F}\mathbf{T}_{\mu\nu}$ computed in standard form for a QC-field ${}^{qc}\phi$.

We conclude that F-modifications of GR can induce QC locally anisotropic configurations via effective matter field sources if the scalar potential is determined by a corresponding class of nonlinear interactions and associated free energy ${}^{qc}\mathcal{F}$. Such cosmologies for QC-modified gravity are described by N-adapted coefficients

$$\begin{aligned}
g_i &= \check{a}^2\eta_i = \mathrm{e}^{\,\psi(x^k)} \text{ is a solution of } \psi^{\bullet\bullet} + \psi'' = 2\,{}^{qc}_{h}\boldsymbol{\Upsilon};\\
g_3 &= h_3(x^i,t) = \check{a}^2\check{h}_3(x^k,\check{t}) = -\frac{\partial_t[\Psi^2({}^{qc}\phi)]}{({}^{qc}\boldsymbol{\Upsilon})^2\left(h_3^{[0]}(x^k) - \int \mathrm{d}t\frac{\partial_t[\Psi^2({}^{qc}\phi)]}{4({}^{qc}\Upsilon)}\right)};\\
g_4 &= h_4(x^i,t) = -\check{a}^2 = h_4^{[0]}(x^k) - \int \mathrm{d}t\frac{\partial_t[\Psi^2({}^{qc}\phi)]}{4({}^{qc}\Upsilon)};\\
N_k^3 &= n_k(x^i,t) = {}_1n_k(x^i) + {}_2n_k(x^i)\int \mathrm{d}t\frac{[\partial_t\Psi({}^{qc}\phi)]^2}{({}^{qc}\boldsymbol{\Upsilon})^2\lvert h_3^{[0]}(x^i) - \int\mathrm{d}t\,\frac{\partial_t[\Psi^2({}^{qc}\phi)]}{4({}^{qc}\boldsymbol{\Upsilon})}\rvert^{\frac{5}{2}}};\\
N_i^4 &= w_i(x^k,t) = \partial_i\,\Psi[{}^{qc}\phi]/\partial_t\Psi[{}^{qc}\phi].
\end{aligned} \tag{34}$$

[17] We can consider a standard interpretation as in [1, 2] for small ε-deformations in section 4.

A d-metric (1) with such coefficients describes a cosmological spacetime encoding 'pure' modified gravity contributions. The functional $\Psi^2({}^{qc}\phi)$ has to be prescribed in a form reproducing observational data. Considering additional sources for matter fields and quantum corrections, we can model quasiperiodic and/or aperiodic structures of different scales and resulting from different sources.

The values necessary for analyzing the conditions for anamorphic phases induced by QC matter fields from MGT as cosmological spacetimes (34) are computed

$$ {}^{m}\Theta[\Psi({}^{qc}\phi)]\, M^0_{Pl}\eta_{Pl} := \widehat{H} + \check{\eta}^* = (\ln|\widehat{a}\check{\eta}|)^* \text{ and } {}^{Pl}\Theta[\Psi({}^{qc}\phi)]\, M^0_{Pl}\eta_{Pl} := \widehat{H} + \eta^*_{Pl} = (\ln|\widehat{a}\eta_{Pl}|)^* $$

$$ \text{where } \widehat{H} = \frac{1}{2}\left(\ln|h_4^{[0]}(x^k) - \int \mathrm{d}t \frac{\partial_t[\Psi^2({}^{qc}\phi)]}{4({}^{qc}\Upsilon)}|\right)^*. $$

A generating function $\Psi[{}^{qc}\phi]$ may induce anamorphic cosmological phases following the conditions

	anamorphosis	inflation	ekpyrosis
$M^0_{Pl}\,{}^{m}\Theta[{}^{qc}\phi] = (\ln|\sqrt{\lvert h_4[{}^{qc}\phi]\rvert}\check{\eta}|)^*/\eta_{Pl}$	<0 (contracts)	>0 (expands)	<0 (contracts)
$M^0_{Pl}\,{}^{Pl}\Theta[{}^{qc}\phi] = (\ln|\sqrt{\lvert h_4[{}^{qc}\phi]\rvert}\eta_{Pl}|)^*/\eta_{Pl}$	>0 (grow)	>0 (grow)	>0 (decay).

Such conditions impose additional nonholonomic constraints on modifications of gravity via F-functionals and generating function $\Psi[{}^{qc}\phi]$ and source ${}^{qc}\Upsilon$ and integration functions. We do not consider quantum contributions in generating QCs and the mass m_0 is taken by a point particle.

3.3.2. Nonhomogeneous QC like scalar fields. For interactions of a scalar field $\phi = {}^{m}\phi$ with mass m and ${}^{m}\boldsymbol{\Upsilon}_{\mu\nu} = (2M_P)^{-2}\,{}^{m}\mathbf{T}_{\alpha\beta}$ (12) parameterized in N-adapted form, ${}^{qm}\boldsymbol{\Upsilon}_{\mu\nu} = {}^{m}\boldsymbol{\Upsilon}_{\mu\nu}[{}^{m}\phi] = diag({}^{qm}_{h}\Upsilon, {}^{qm}\Upsilon)$, we can generate QC like configurations by this class of solutions,

$$ \begin{aligned} & g_i = \check{a}^2\eta_i = \mathrm{e}^{\,\psi(x^k)} \text{ is a solution of } \psi^{\bullet\bullet} + \psi'' = 2\,{}^{qm}_{h}\boldsymbol{\Upsilon}; \\ & g_3 = h_3(x^i,t) = \check{a}^2\check{h}_3(x^k,\check{t}) = -\frac{\partial_t[\Psi^2({}^{m}\phi)]}{({}^{qm}\boldsymbol{\Upsilon})^2\left(h_3^{[0]}(x^k) - \int \mathrm{d}t \frac{\partial_t[\Psi^2({}^{m}\phi)]}{4\Psi({}^{qm}\Upsilon)}\right)}; \\ & g_4 = h_4(x^i,t) = -\check{a}^2 = h_4^{[0]}(x^k) - \int \mathrm{d}t \frac{\partial_t[\Psi^2({}^{m}\phi)]}{4({}^{qm}\Upsilon)}; \\ & N^3_k = n_k(x^i,t) = {}_1n_k(x^i) + {}_2n_k(x^i)\int \mathrm{d}t \frac{[\partial_t\Psi({}^{m}\phi)]^2}{({}^{qm}\boldsymbol{\Upsilon})^2|h_3^{[0]}(x^i) - \int \mathrm{d}t\, \frac{\partial_t[\Psi^2({}^{m}\phi)]}{4\Psi({}^{qm}\boldsymbol{\Upsilon})}|^{\frac{5}{2}}}; \\ & N^4_i = w_i(x^k,t) = \partial_i\,\Psi[{}^{m}\phi]/\partial_t\Psi[{}^{m}\phi]. \end{aligned} \tag{35} $$

We can consider additional constraints for zero torsion configurations which results in cosmological solutions in GR. Such off-diagonal metrics are determined by QC like matter distributions if

$$ \frac{\partial({}^{m}\phi)}{\partial t} = {}^{b}\widehat{\Delta}\left[\frac{\delta({}^{qm}\mathcal{F})}{\delta({}^{m}\phi)}\right] = -\,{}^{b}\widehat{\Delta}[\Theta\,{}^{m}\phi + Q({}^{m}\phi)^2 - ({}^{m}\phi)^3] $$

with effective free energy ${}^{qm}\mathcal{F}[{}^{m}\phi] = \int\left[-\frac{1}{2}({}^{m}\phi)\Theta({}^{m}\phi) - \frac{Q}{3}({}^{m}\phi)^3 + \frac{1}{4}({}^{m}\phi)^4\right]\sqrt{b}\mathrm{d}x^1\mathrm{d}x^2\delta y^3$.

It is possible to model double QC configurations with $\overline{\phi} = {}^{qc}\phi + {}^{m}\phi$, for instance, considering ${}^{m}\phi$ as a small modification of ${}^{qc}\phi$ and effective $\mathcal{F} \simeq {}^{qc}\mathcal{F}[{}^{qc}\phi] + {}^{qm}\mathcal{F}[{}^{m}\phi]$. In general, we do not have an additive law of QC free energies for nonlinear MGT and matter field interactions. The functional $\Psi[{}^{m}\phi]$ is different from $\Psi[{}^{qc}\phi]$.

The values for anamorphic phases induced by QC matter fields from MGT as cosmological spacetimes (35) are computed

$$
\begin{aligned}
&{}^{m}\Theta[\Psi({}^{qc}\phi),\ {}^{qm}\Upsilon]\ M^{0}_{Pl}\eta_{Pl} := \widehat{H} + \check{\eta}^{*} = (\ln|\widehat{a}\check{\eta}|)^{*} \text{ and}\\
&{}^{Pl}\Theta[\Psi({}^{qc}\phi),\ {}^{qm}\Upsilon]\ M^{0}_{Pl}\eta_{Pl} := \widehat{H} + \eta^{*}_{Pl} = (\ln|\widehat{a}\eta_{Pl}|)^{*}
\end{aligned}
$$

where $\widehat{H} = \widehat{H} = \frac{1}{2}\left(h_4^{[0]}(x^k) - \int \mathrm{d}t \frac{\partial_t[\Psi^2({}^{m}\phi)]}{4({}^{qm}\Upsilon)}\right)^{*}$. A generating function $\Psi[{}^{qc}\phi]$ may induce anamorphic cosmological phases following the conditions

$$
\begin{array}{llll}
 & \text{anamorphosis} & \text{inflation} & \text{ekpyrosis}\\
M^{0}_{Pl}\,{}^{m}\Theta[\Psi({}^{qc}\phi),\ {}^{qm}\Upsilon] = (\ln|\sqrt{|h_4|}\check{\eta}|)^{*}/\eta_{Pl} & <0 \text{ (contracts)} & >0 \text{ (expands)} & <0 \text{ (contracts)}\\
M^{0}_{Pl}\,{}^{Pl}\Theta[\Psi({}^{qc}\phi),\ {}^{qm}\Upsilon] = (\ln|\sqrt{|h_4|}\eta_{Pl}|)^{*}/\eta_{Pl} & >0 \text{ (grow)} & >0 \text{ (grow)} & >0 \text{ (decay)}.
\end{array}
$$

These conditions impose additional nonholonomic constraints on generating function $\Psi[{}^{qc}\phi]$ and source ${}^{qm}\Upsilon$ and integration functions. Quantum contributions are not considered and the scalar field with QC configurations is with polarization of mass m_0.

3.3.3. QC configurations induced by LQG corrections. Quantum corrections may also result in quasiperiodic/aperiodic QC like structures, for instance, if LQG sources of type $\check{\Upsilon} = \overline{\mathbf{\Upsilon}} = -\overline{\rho}^2[{}^{q}\phi]/3\overline{\rho}_c$ (21) are considered for generating cosmological solutions. This defines an effective scalar field $\overline{\phi} = {}^{q}\phi$ (the label q emphasizes the quantum nature of such a field). For LQG and GR, such solutions are of type (see footnote 16)

$$
\begin{aligned}
\mathrm{d}s^2 =\ & g_{ij}\mathrm{d}x^i\mathrm{d}x^j + \{h_3[\mathrm{d}y^3 + (\partial_k n)\mathrm{d}x^k]^2 - \frac{9(\overline{\rho}_c)^2}{4h_3}\left[\partial_t\check{\Psi}({}^{q}\phi)\right]^2\ [\mathrm{d}t + (\partial_i\check{A})\mathrm{d}x^i]^2\},\\
h_3 =\ & -9(\overline{\rho}_c)^2\partial_t(\check{\Psi}^2)/\overline{\rho}^4[{}^{q}\phi]\left(h_3^{[0]}(x^k) + 3\overline{\rho}_c\int \mathrm{d}t\partial_t[\check{\Psi}^2({}^{q}\phi)]/4\overline{\rho}^2[{}^{q}\phi]\right).
\end{aligned}
\tag{36}
$$

The QC structure is generated if ${}^{q}\phi$ is subjected by the conditions

$$
\frac{\partial({}^{q}\phi)}{\partial t} = {}^{b}\widehat{\Delta}\left[\frac{\delta({}^{q}\mathcal{F})}{\delta({}^{q}\phi)}\right] = -{}^{b}\widehat{\Delta}[\Theta\,{}^{q}\phi + Q({}^{q}\phi)^2 - ({}^{q}\phi)^3]
$$

for effective free energy ${}^{q}\mathcal{F}[{}^{q}\phi] = \int\left[-\frac{1}{2}({}^{q}\phi)\Theta({}^{q}\phi) - \frac{Q}{3}({}^{q}\phi)^3 + \frac{1}{4}({}^{q}\phi)^4\right]\sqrt{b}\mathrm{d}x^1\mathrm{d}x^2\delta y^3$. This type of loop QC configurations can be generated from vacuum gravitational fields.

The values for anamorphic phases for QC structures determined by LQG corrections of matter fields from MGT as cosmological spacetimes (36) are computed

$$
\begin{aligned}
&{}^{m}\Theta[\check{\Psi}({}^{q}\phi)]\ M^{0}_{Pl}\eta_{Pl} := \widehat{H} + \check{\eta}^{*} = (\ln|\widehat{a}\check{\eta}|)^{*} \text{ and}\\
&{}^{Pl}\Theta[\check{\Psi}({}^{q}\phi)]\ M^{0}_{Pl}\eta_{Pl} := \widehat{H} + \eta^{*}_{Pl} = (\ln|\widehat{a}\eta_{Pl}|)^{*}
\end{aligned}
$$

where $\widehat{H} = \ln\widehat{a} = \ln|\frac{3\overline{\rho}_c}{2h_3}\partial_t\check{\Psi}({}^{q}\phi)|$ is computed for

$$
h_4 = \overline{\rho}^4[{}^{q}\phi]\frac{\partial_t\check{\Psi}({}^{q}\phi)}{8\check{\Psi}}\left(h_3^{[0]}(x^k) + \frac{3\overline{\rho}_c}{4}\int \mathrm{d}t\frac{\partial_t[\check{\Psi}^2({}^{q}\phi)]}{\overline{\rho}^2[{}^{q}\phi]}/\right).
$$

Anamorphic cosmological phases are determined following the conditions

	anamorphosis	inflation	ekpyrosis
$M_{Pl}^{0}\ {}^{m}\Theta[\check{\Psi}(\ {}^{q}\phi)] = (\ln\lvert\sqrt{\lvert h_4\rvert\check{\eta}}\rvert)^{*}/\eta_{Pl}$	<0 (contracts)	>0 (expands)	<0 (contracts)
$M_{Pl}^{0}\ {}^{Pl}\Theta[\check{\Psi}(\ {}^{q}\phi)] = (\ln\lvert\sqrt{\lvert h_4\rvert\eta_{Pl}}\rvert)^{*}/\eta_{Pl}$	>0 (grow)	>0 (grow)	>0 (decay).

These conditions impose nonholonomic constraints on generating function $\check{\Psi}(\ {}^{q}\phi)$ for quantum contributions computed in LQG and for polarization of mass m_0 of a point particle.

3.4. Anamorphic off-diagonal cosmology with QC and LQG structures

Generic off-diagonal solutions (25) encoding parameterized form QC structures generated by different type sources considered in (34)–(36) can be written in the form similar to (17) with redefined time coordinate and scaling factor $\widehat{a} = \check{a}$. We obtain

$$\mathrm{d}s^2 = \widehat{a}^2(x^i,\check{t})[\eta_i(x^k,\check{t})(\mathrm{d}x^i)^2 + \check{h}_3(x^k,\check{t})(\mathbf{e}^3)^2 - (\check{\mathbf{e}}^4)^2],$$
$$\text{where } \eta_i = \check{a}^{-2}\mathrm{e}^{\psi}, \mathbf{e}^3 = \mathrm{d}y^3 + \partial_k n(x^i)\,\mathrm{d}x^k, \check{\mathbf{e}}^4 = \mathrm{d}\check{t} + \sqrt{\lvert h_4\rvert}(\partial_i t + w_i), \qquad (37)$$

$$\text{for } \check{h}_3 = -\partial_t(\Psi^2)/\check{a}^2(\ {}^{qc}\boldsymbol{\Upsilon} +\ {}^{qm}\Upsilon - \overline{\rho}^2[\ {}^{q}\phi]/3\overline{\rho}_c)^2(h_3^{[0]}(x^k) - \int \mathrm{d}t\frac{\partial_t(\Psi^2)}{4(\ {}^{qc}\boldsymbol{\Upsilon} +\ {}^{qm}\Upsilon - \overline{\rho}^2[\ {}^{q}\phi]/3\overline{\rho}_c)})$$
$$h_4 = -\widehat{a}^2(x^i,t) = h_4^{[0]}(x^k) - \int \mathrm{d}t\partial_t(\Psi^2)/4(\ {}^{qc}\boldsymbol{\Upsilon} +\ {}^{qm}\Upsilon - \overline{\rho}^2[\ {}^{q}\phi]/3\overline{\rho}_c),$$
$$w_i = \partial_i\,\Psi/\partial_t\Psi,$$

for a functional $\Psi = \Psi[\ {}^{qc}\phi,\ {}^{m}\phi,\ {}^{q}\phi]$. For a hierarchy of coupled three QC cosmological structures, we can subject such a functional of effective sources to conditions of type

$$\frac{\partial\Psi}{\partial t} = {}^{b}\widehat{\Delta}\left[\frac{\delta\mathcal{F}}{\delta\Psi}\right] = -\ {}^{b}\widehat{\Delta}(\Theta\Psi + Q\Psi^2 - \Psi^3),$$

with a functional for effective free energy $\mathcal{F}[\Psi] = \int\left[-\frac{1}{2}\Psi\Theta\Psi - \frac{Q}{3}\Psi^3 + \frac{1}{4}\Psi^4\right]\sqrt{b}\mathrm{d}x^1\mathrm{d}x^2\delta y^3$, written in conventional integro-functional forms.

The values characterizing anamorphic phases in QC cosmological spacetimes are computed

$${}^{m}\Theta\ M_{Pl}^{0}\eta_{Pl} := \widehat{H} + \overline{H} + \check{\eta}^{*} = (\ln\lvert\widehat{a}\check{\eta}\rvert)^{*} \text{ and } {}^{Pl}\Theta\ M_{Pl}^{0}\eta_{Pl} := \widehat{H} + \overline{H} + \eta_{Pl}^{*} = (\ln\lvert\widehat{a}\eta_{Pl}\rvert)^{*}$$

where the polarized Hubble functions, $\widehat{H}$ (16) and $\overline{H}$ (20), are taken for the quadratic element (37)

$$\widehat{H} = (\ln\widehat{a})^{*} = \frac{1}{2}\left(\ln\left\lvert h_4^{[0]}(x^k) - \int\mathrm{d}t\frac{\partial_t(\Psi^2)}{4(\ {}^{qc}\boldsymbol{\Upsilon} +\ {}^{qm}\Upsilon - \overline{\rho}^2[\ {}^{q}\phi]/3\overline{\rho}_c)}\right\rvert\right)^{*} \text{ and } \overline{H} = \sqrt{\left\lvert\frac{\overline{\rho}}{3}(1-\frac{\overline{\rho}}{\overline{\rho}_c})\right\rvert}.$$

A generating function $\Psi = \Psi[\Phi;\ {}^{F}\Lambda,\ {}^{m}\Lambda,\ \overline{\Lambda};{}^{F}\boldsymbol{\Upsilon},\ {}^{m}\boldsymbol{\Upsilon},\ \overline{\boldsymbol{\Upsilon}}]$ may induce anamorphic cosmological phases following the conditions (23). In the case of mixed 3 type QC structures, the Weyl type anamorphic characteristics are determined also by the data for the integration function $h_4^{[0]}(x^k)$; effective sources ${}^{F}\boldsymbol{\Upsilon},\ {}^{m}\boldsymbol{\Upsilon},\ \overline{\boldsymbol{\Upsilon}}$ and $\overline{\rho}$ contained in the sum $\widehat{H} + \overline{H}$. We compute

	anamorphosis	inflation	ekpyrosis
$M^0_{Pl}\ {}^m\Theta[\Psi,\ {}^{qc}\mathbf{\Upsilon},\ {}^{qm}\Upsilon,\overline{\rho}^2] = (\ln|\sqrt{|h_4|}\check{\eta}|)^*/\eta_{Pl}$	<0 (contracts)	>0 (expands)	<0 (contracts)
$M^0_{Pl}\ {}^{Pl}\Theta[\Psi,\ {}^{qc}\mathbf{\Upsilon},\ {}^{qm}\Upsilon,\overline{\rho}^2] = (\ln|\sqrt{|h_4|}\eta_{Pl}|)^*/\eta_{Pl}$	>0 (grow)	>0 (grow)	>0 (decay).

Such conditions impose additional nonholonomic constraints on generating functions and all types of sources and integration functions and constants which induce QC structures.

4. Small parametric anamorphic cosmological QC and LQG structures

The main goal of this paper is to prove that quasiperiodic and/or aperiodic (for instance, QC like) structures in MGT with LQG helicity contributions can be incorporated in a compatible way in the framework of the anamorphic cosmology [1–6]. For the classes of cosmological solutions constructed in general form in previous section, we can consider a procedure of small ε-deformations of d-metrics of type (25) with respective N-adapted frames and connections, see details in [49, 52, 72, 73, 75–77] and section 2.3. In this section, we show how using ε-deformations an off-diagonal 'prime' metric, $\mathring{\mathbf{g}}(x^i, y^3, t)$ (for applications in modern cosmolgoy, this metric can be diagonalizable under coordinate transforms[18]) into a 'target' metric, $\mathbf{g}(x^i, y^3, t)$.

4.1. N-adapted ε-deformations

We suppose that a 'prime' pseudo–Riemannian cosmological metric $\mathring{\mathbf{g}} = [\mathring{g}_i, \mathring{h}_a, \mathring{N}^j_b]$ can be parameterized in the form

$$\begin{aligned} ds^2 &= \mathring{g}_i(x^k,t)(dx^i)^2 + \mathring{h}_a(x^k,t)(\mathring{\mathbf{e}}^a)^2, \\ \mathring{\mathbf{e}}^3 &= dy^3 + \mathring{n}_i(x^k,t)dx^i, \mathring{\mathbf{e}}^4 = dt + \mathring{w}_i(x^k,t)dx^i. \end{aligned} \tag{38}$$

For instance, some data $(\mathring{g}_i, \mathring{h}_a)$ may define a cosmological solution in MGT or in GR like a FLRW, metric. The target metric $\mathbf{g} = {}^{\varepsilon}\mathbf{g}$ for an off-diagonal deformation of the metric structure, for a small parameter $0 \leqslant \varepsilon \ll 1$, is parameterized by N-adapted quadratic elements

$$\begin{aligned} ds^2 &= \overline{\eta}_i(x^k,t)\mathring{g}_i(x^k,t)(dx^i)^2 + \overline{\eta}_a(x^k,t)\mathring{g}_a(x^k,t)(\mathbf{e}^a)^2 \\ &= \check{a}^2(x^i,t)[\eta_i(x^k,t)(dx^i)^2 + \check{h}_3(x^k,t)(\mathbf{e}^3)^2 - (\check{\mathbf{e}}^4)^2], \\ \mathbf{e}^3 &= dy^3 + {}^n\eta_i(x^k,t)\mathring{n}_i(x^k,t)dx^i = dy^3 + \partial_k n\ dx^k, \\ \mathbf{e}^4 &= dt + {}^w\eta_i(x^k,t)\mathring{w}_i(x^k,t)dx^i = \check{\mathbf{e}}^4 = d\check{t} + \sqrt{|h_4|}(\partial_i t + w_i), \end{aligned} \tag{39}$$

with possible re-definitions of coordinates $\check{t} = \check{t}(x^k, t)$ for $\check{a}^2(x^i,t) \to \widehat{a}^2(x^i,t)$ and where, for instance, ${}^n\eta_i\mathring{n}_i dx^i = {}^n\eta_1\mathring{n}_1 dx^1 + {}^n\eta_2\mathring{n}_2 dx^2$. The polarization functions are ε-deformed following rules adapted to (17) and (37), when

$$\begin{aligned} \overline{\eta}_i &= \check{\eta}_i(x^k,t)[1+\varepsilon\overline{\chi}_i(x^k,t)], \overline{\eta}_a = 1+\varepsilon\overline{\chi}_a(x^k,t) \text{ and} \\ {}^n\eta_i &= 1+\varepsilon\ {}^n\chi_i(x^k,t),\ {}^w\eta_i = 1+\varepsilon\ {}^w\chi_i(x^k,t), \\ \eta_i &\simeq 1+\varepsilon\chi_i(x^k,\widehat{t}), \partial_k n \simeq \varepsilon\widehat{n}_i(x^k), \sqrt{|h_4|}\ w_i \simeq \varepsilon\widehat{w}_i(x^k,\widehat{t}). \end{aligned} \tag{40}$$

[18] We note that in general, $\mathring{\mathbf{g}}$ (38) may not be a solution of gravitational field equations but it will be nonholonomically deformed into such solutions.

Such 'double' N-adapted deformations are convenient for generating new classes of solutions and further physical interpretation of such solutions with limits of quasi-FLRW metrics to some homogenous diagonal cosmological metrics.

The target generic off-diagonal cosmological metrics

$$\mathbf{g} = {}^{\varepsilon}\mathbf{g} = ({}^{\varepsilon}g_i,\ {}^{\varepsilon}h_a,\ {}^{\varepsilon}N_b^j) = (g_\alpha = \eta_\alpha \mathring{g}_\alpha,\ {}^{n}\eta_i n_i,\ {}^{w}\eta_i \mathring{w}_i)\ (39) \to \mathring{\mathbf{g}}\ (38) \text{ for } \varepsilon \to 0,$$

define, for instance, cosmological QC configurations with parametric ε-dependence determined by a class of solutions (25) (or any variant of solutions (31), (34)–(37)). The effective ε-polarizations of constants (see (32)) are written

$$\check{m} = m_0 \check{\eta}(x^i,t) \simeq m_0(1+\varepsilon\chi(x^i,t)) \text{ and } \check{M}_{Pl} = M^0_{Pl}\sqrt{f(\phi)} = M^0_{Pl}\eta_{Pl}(x^i,t) = M^0_{Pl}(1+\varepsilon\chi_{Pl}(x^i,t)),$$

see formulas (7), (8) and (22), where ${}^{A}\rho = \overline{\rho}$ (19) in locally anisotropic and inhomogeneous first and second Friedmann equation (24).

4.2. ε-deformations to off-diagonal cosmological metrics

The deformations of h-components of the cosmological d-metrics are

$${}^{\varepsilon}g_i(x^k) = \mathring{g}_i(x^k,t)\check{\eta}_i(x^k,t)[1+\varepsilon\chi_i(x^k,t)] = \mathrm{e}^{\psi(x^k)}$$

defined by a solution of the 2d Poisson equation. Considering $\psi = {}^{0}\psi(x^k) + \varepsilon\ {}^{1}\psi(x^k)$ and

$${}_{h}\mathbf{\Upsilon}(x^k) = {}^{0}_{h}\Upsilon(x^k) + \varepsilon\, {}^{1}_{h}\Upsilon(x^k) = {}^{F}_{h}\mathbf{\Upsilon}(x^i) + {}^{m}_{h}\mathbf{\Upsilon}(x^i) + {}_{h}\overline{\mathbf{\Upsilon}}(x^i)$$

(in particular, we can take ${}^{1}_{h}\Upsilon$), see (13), we compute the deformation polarization functions

$$\overline{\chi}_i = \mathrm{e}^{\ {}^{0}_{\ni}\psi}\ {}^{1}\psi/\mathring{g}_i\check{\eta}_i\ \ {}^{1}_{h}\Upsilon. \tag{41}$$

Let us compute ε-deformations of v-components using formulas for a source $\mathbf{\Upsilon} = {}^{F}\mathbf{\Upsilon} + {}^{m}\mathbf{\Upsilon} + \overline{\mathbf{\Upsilon}}$. We consider

$${}^{\varepsilon}h_3 = h_3^{[0]}(x^k) - \frac{1}{4}\int \mathrm{d}t \frac{(\Psi^2)^*}{\Upsilon} = (1+\varepsilon\overline{\chi}_3)\mathring{g}_3;\ {}^{\varepsilon}h_4 = -\frac{1}{4}\frac{(\Psi^*)^2}{(\Upsilon)^2}\left(h_4^{[0]} - \frac{1}{4}\int \mathrm{d}t \frac{(\Psi^2)^*}{\Upsilon}\right)^{-1} = (1+\varepsilon\,\overline{\chi}_4)\mathring{g}_4, \tag{42}$$

when the generation function can also be ε-deformed,

$$\Psi = {}^{\varepsilon}\Psi = \mathring{\Psi}(x^k,t)[1+\varepsilon\overline{\chi}(x^k,t)]. \tag{43}$$

Introducing ${}^{\varepsilon}\Psi$ in (42), we compute

$$\overline{\chi}_3 = -\frac{1}{4\mathring{g}_3}\int \mathrm{d}t \frac{(\mathring{\Psi}^2\overline{\chi})^*}{\Upsilon} \text{ and } \int \mathrm{d}t \frac{(\mathring{\Psi}^2)^*}{\Upsilon} = 4(h_3^{[0]} - \mathring{g}_3). \tag{44}$$

We conclude that $\overline{\chi}_3$ can be computed for any deformation $\overline{\chi}$ in (43) adapted to a time like oriented family of 2-hypersurfaces $t = t(x^k)$. This family given in non-explicit form by $\mathring{\Psi} = \mathring{\Psi}(x^k,t)$ when the integration function $h_3^{[0]}(x^k)$, $\mathring{g}_3(x^k)$ and $(\mathring{\Psi}^2)^*/\ \Upsilon$ satisfy the conditions (44).

Using (43) and (42), we get

$$\overline{\chi}_4 = 2(\overline{\chi} + \frac{\mathring{\Psi}}{\mathring{\Psi}^*}\overline{\chi}^*) - \overline{\chi}_3 = 2(\overline{\chi} + \frac{\mathring{\Psi}}{\mathring{\Psi}^*}\overline{\chi}^*) + \frac{1}{4\mathring{g}_3}\int \mathrm{d}t \frac{(\mathring{\Psi}^2\overline{\chi})^*}{\Upsilon}.$$

As a result, we can compute $\overline{\chi}_3$ for any data $\left(\mathring{\Psi}, \mathring{g}_3, \overline{\chi}\right)$ and a compatible source $\Upsilon = \pm \mathring{\Psi}^*/2\sqrt{|\mathring{g}_4 h_3^{[0]}|}$. Such conditions and (44) define a time oriented family of 2d hypersurfaces, parameterized by $t = t(x^k)$ defined in non-explicit form from

$$\int \mathrm{d}t \mathring{\Psi} = \pm (h_3^{[0]} - \mathring{g}_3)/\sqrt{|\mathring{g}_4 h_3^{[0]}|}. \tag{45}$$

The final step consists of ε-deformations N-connection coefficients $w_i = \partial_i \Psi/\Psi^*$ for nontrivial $\mathring{w}_i = \partial_i\ \mathring{\Psi}/\ \mathring{\Psi}^*$, which are computed following formulas (43) and (40), ${}^{w}\chi_i = \frac{\partial_i(\overline{\chi}\ \mathring{\Psi})}{\partial_i\ \mathring{\Psi}} - \frac{(\overline{\chi}\ \mathring{\Psi})^\diamond}{\mathring{\Psi}^\diamond}$. We omit similar computations of ε-deformations of n-coefficients (we omit such details which are not important if we restrict our research only to LC-configurations).

Summarizing (41)–(45), we obtain the following formulas for ε-deformations of a prime cosmological metric (38) into a target cosmological metric:

$$
\begin{aligned}
{}^{\varepsilon}g_i &= [1 + \varepsilon \overline{\chi}_i(x^k, t)]\mathring{g}_i \check{\eta}_i = [1 + \varepsilon \mathrm{e}^{\ {}^0\psi}\ {}^1\psi/\mathring{g}_i\check{\eta}_i\ \ {}^0_h\Upsilon]\mathring{g}_i \text{ solution of 2d Poisson equations});\\
{}^{\varepsilon}h_3 &= [1 + \varepsilon\ \overline{\chi}_3]\mathring{g}_3 = \left[1 - \varepsilon \frac{1}{4\mathring{g}_3}\int \mathrm{d}t \frac{(\mathring{\Psi}^2 \overline{\chi})^*}{\Upsilon}\right]\mathring{g}_3;\\
{}^{\varepsilon}h_4 &= [1 + \varepsilon\ \overline{\chi}_4]\mathring{g}_4 = \left[1 + \varepsilon \left(2(\overline{\chi} + \frac{\mathring{\Psi}}{\mathring{\Psi}^*}\overline{\chi}^*) + \frac{1}{4\mathring{g}_3}\int \mathrm{d}t \frac{(\mathring{\Psi}^2\overline{\chi})^*}{\Upsilon}\right)\right]\mathring{g}_4;\\
{}^{\varepsilon}n_i &= [1 + \varepsilon\ {}^{n}\chi_i]\mathring{n}_i = \left[1 + \varepsilon\ \widetilde{n}_i \int \mathrm{d}t\ \frac{1}{\Upsilon^2}\left(\overline{\chi} + \frac{\mathring{\Psi}}{\mathring{\Psi}^*}\overline{\chi}^* + \frac{5}{8}\frac{1}{\mathring{g}_3}\frac{(\mathring{\Psi}^2\overline{\chi})^*}{\Upsilon}\right)\right]\mathring{n}_i;\\
{}^{\varepsilon}w_i &= [1 + \varepsilon\ {}^{w}\chi_i]\mathring{w}_i = \left[1 + \varepsilon(\frac{\partial_i(\overline{\chi}\ \mathring{\Psi})}{\partial_i\ \mathring{\Psi}} - \frac{(\overline{\chi}\ \mathring{\Psi})^*}{\mathring{\Psi}^*})\right]\mathring{w}_i.
\end{aligned} \tag{46}
$$

The factor $\widetilde{n}_i(x^k)$ is a redefined integration function.

The quadratic element for such inhomogeneous and locally anisotropic cosmological spaces with coefficients (46) can be written in N-adapted form

$$
\begin{aligned}
\mathrm{d}s^2 = {}^{\varepsilon}g_{\alpha\beta}(x^k, t)\mathrm{d}u^\alpha \mathrm{d}u^\beta &= {}^{\varepsilon}g_i\left(x^k\right)[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2]\\
&+ {}^{\varepsilon}h_3(x^k, t)\ [\mathrm{d}y^3 + {}^{\varepsilon}n_i \mathrm{d}x^i]^2 + {}^{\varepsilon}h_4(x^k, t)[\mathrm{d}t + {}^{\varepsilon}w_k\ (x^k, t)\mathrm{d}x^k]^2.
\end{aligned} \tag{47}
$$

Further assumptions on generating and integration functions and source can be considered in order to find solutions of type ${}^{\varepsilon}g_{\alpha\beta}(x^k, t) \simeq {}^{\varepsilon}g_{\alpha\beta}(t)$.

4.3. Cosmological ε-deformations with anamorphic QCs and LQG

We apply the procedure of ε-deformations described in the previous section in order to generate solutions of type (37). We prescribe $\overline{\chi}(x^k, t)$ and ${}^0_h\Upsilon(x^k)$ for any compatible $\left(\mathring{\Psi}, \mathring{g}_3\right)$ and source

$$\mathring{\Psi}^* = \pm 2\sqrt{|\mathring{g}_4 h_3^{[0]}|}\ (\ {}^{qc}\mathbf{\Upsilon} +\ {}^{qm}\Upsilon - \overline{\rho}^2[\ {}^{q}\phi]/3\overline{\rho}_c).$$

The generated d-metric with coefficients (46) is of type (47) for $\Upsilon = {}^{qc}\mathbf{\Upsilon} + {}^{qm}\Upsilon - \overline{\rho}^2[\ {}^{q}\phi]/3\overline{\rho}_c$,

$$\mathrm{d}s^2 = [1+\varepsilon \mathrm{e}^{\ ^0\psi}\ {}^1\psi/\mathring{g}_i \check{\eta}_i\ \ {}^0_h\Upsilon]\mathring{g}_i[(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2]$$
$$+[1-\varepsilon\frac{1}{4\mathring{g}_3}\int \mathrm{d}t\frac{(\mathring{\Psi}^2\overline{\chi})^*}{\Upsilon}]\mathring{g}_3\left[\mathrm{d}y^3+\ [1+\varepsilon\,\tilde{n}_i\int \mathrm{d}t\,\frac{1}{\Upsilon^2}\left(\overline{\chi}+\frac{\mathring{\Psi}}{\mathring{\Psi}^*}\overline{\chi}^*+\frac{5}{8}\frac{1}{\mathring{g}_3}\frac{(\mathring{\Psi}^2\overline{\chi})^*}{\Upsilon}\right)]\mathring{n}_i\mathrm{d}x^i\right]^2$$
$$+[1+\varepsilon\,(2(\overline{\chi}+\frac{\mathring{\Psi}}{\mathring{\Psi}^*}\overline{\chi}^*)+\frac{1}{4\mathring{g}_3}\int \mathrm{d}t\frac{(\mathring{\Psi}^2\overline{\chi})^*}{\Upsilon})]\mathring{g}_4\left[\mathrm{d}t+\ [1+\varepsilon(\frac{\partial_i(\overline{\chi}\,\mathring{\Psi})}{\partial_i\,\mathring{\Psi}}-\frac{(\overline{\chi}\,\mathring{\Psi})^*}{\mathring{\Psi}^*})]\mathring{w}_k\mathrm{d}x^k\right]^2.$$

Hierarchies of coupled three QC cosmological structures are generated by a functional $\overline{\chi}=\overline{\chi}[\ ^{qc}\phi,\ ^m\phi,\ ^q\phi]$ subjected to conditions of type

$$\frac{\partial\overline{\chi}}{\partial t}=\ ^b\widehat{\Delta}\left[\frac{\delta\mathcal{F}}{\delta\Psi}\right]=-\ ^b\widehat{\Delta}(\Theta\overline{\chi}+Q\overline{\chi}^2-\overline{\chi}^3),$$

with functionals for effective free energy $\mathcal{F}[\overline{\chi}]=\int\left[-\frac{1}{2}\overline{\chi}\Theta\overline{\chi}-\frac{Q}{3}\overline{\chi}^3+\frac{1}{4}\overline{\chi}^4\right]\sqrt{b}\mathrm{d}x^1\mathrm{d}x^2\delta y^3$, written in conventional integro-functional forms. The value

$$^\varepsilon h_4=-\ ^\varepsilon\widehat{a}^2(x^i,t)=[1+\varepsilon\,(2(\overline{\chi}+\frac{\mathring{\Psi}}{\mathring{\Psi}^*}\overline{\chi}^*)+\frac{1}{4\mathring{g}_3}\int \mathrm{d}t\frac{(\mathring{\Psi}^2\overline{\chi})^*}{\Upsilon})]\mathring{g}_4, \quad (48)$$

with $\mathring{g}_4=a(t)$, allows us to compute the Weyl type invariants characterizing anamporphic phases in QC cosmological spacetimes,

$$^m_\varepsilon\Theta\,M^0_{Pl}(1+\varepsilon\chi_{PL}):=\widehat{H}+\overline{H}+(1+\varepsilon\chi)^*=(\ln|\ ^\varepsilon\widehat{a}(1+\varepsilon\chi)|)^*\ \text{and}$$
$$^{Pl}_{\ \varepsilon}\Theta\,M^0_{Pl}(1+\varepsilon\chi_{PL}):=\widehat{H}+\overline{H}+(1+\varepsilon\chi)^*=(\ln|\ ^\varepsilon\widehat{a}(1+\varepsilon\chi_{PL})|)^*,$$

where the ε-polarized Hubble functions, $^\varepsilon\widehat{H}$ (16) and $^\varepsilon\overline{H}$ (20) are respectively computed for $^\varepsilon h_4$

$$^\varepsilon\widehat{H}=(\ln\ ^\varepsilon\widehat{a})^*=\frac{1}{2}\left(\ln\left|[1+\varepsilon\,(2(\overline{\chi}+\frac{\mathring{\Psi}}{\mathring{\Psi}^*}\overline{\chi}^*)+\frac{1}{4\mathring{g}_3}\int \mathrm{d}t\frac{(\mathring{\Psi}^2\overline{\chi})^*}{\Upsilon})]\mathring{g}_4\right|\right)^*\ \text{and}\ ^\varepsilon\overline{H}=\sqrt{\left|\frac{\overline{\rho}}{3}(1-\frac{\overline{\rho}}{\overline{\rho}_c})\right|}.$$

The possibility to induce and preserve certain anamorphic cosmological phases following the conditions (23). For mixed 3 type QC structures, the Weyl type anamorphic ε-deformed characteristics are determined also by the data for the integration function $h_4^{[0]}(x^k)$; effective sources $^F\boldsymbol{\Upsilon}$, $^m\boldsymbol{\Upsilon}$, $\overline{\boldsymbol{\Upsilon}}$ and $\overline{\rho}$ contained in the sum $^\varepsilon\widehat{H}+\ ^\varepsilon\overline{H}$. We compute

	anamorphosis	inflation	ekpyrosis
$M^0_{Pl}\ {}^m_\varepsilon\Theta=\frac{(\ln|\sqrt{|\ ^\varepsilon h_4|}(1+\varepsilon\chi)|)^*}{(1+\varepsilon\chi_{PL})}$	<0 (contracts)	>0 (expands)	<0 (contracts)
$M^0_{Pl}\ {}^{Pl}_{\ \varepsilon}\Theta=\frac{(\ln|\sqrt{|\ ^\varepsilon h_4|}(1+\varepsilon\chi_{PL})|)^*}{(1+\varepsilon\chi_{PL})}$	>0 (grow)	>0 (grow)	>0 (decay).

In such criteria, we use the value $^\varepsilon h_4$ (48) conditions imposing additional nonholonomic constraints on generating functions and all types of sources and integration functions and constants which induce QC structures. In a similar form, we can generate ε-analogs of (34)–(36), (17) and analyze if respective conditions for anamorphic phases can be satisfied.

In [70, 71], detailed studies using analytical methods and numerical computations for holonomy corrections were performed in order to demonstrate respectively how such terms may prevent the Big Rip singularity in LQC and how to avoid singularities in certain MGTs.

It was concluded that the dynamics with holonomy corrections is very different from that for original $R + \alpha R^2$ models (for such theories, 'the universe is singular at early times and never bounces'). The research in [55, 56, 69–71, 78, 79] was performed for diagonal ansatz constraining from the very beginning the possibility to generate quasi-periodic, pattern forming, and/or 3d soliton cosmological configurations depending, in principle, on all spacetime coordinates. The AFDM (see a survey in appendix, and references therein) allows us to construct very general classes of locally anisotropic and inhomogeneous cosmological solutions in varios types of MGTs [18–20, 49–52]. Such generic off-diagonal cosmological models may describe anamorphic phases with quasi-periodic structures, singular or nonsigular configurations etc. It was not clear if the results and conclusions on holonomy corrections, for instance, those obtained in [70, 71] hold true for generic off-diagonal configurations. The results of this section prove that at least for off-diagonal ε-deformations the holonomy corrections may also prevent/avoid singularities (as in the cases isotropic and homogeneous configurations). Finally we note that the holonomy corrections $\overline{H}(x^i,t) = \sin(\sqrt{2\sqrt{3}\gamma}\overline{\beta}(x^i,t))/\sqrt{2\sqrt{3}\gamma}$ considered section 2.5 can be computed for arbitrary classes of off-diagonal solutions. The conclusion on avoiding or generating singularities depends on the type of nonlinear cosmological configurations we define by generating functions and/or effective sources. This should be analyzed in explicit form for a chosen example of cosmological metric, for instance, determined by a cosmological solitonic configuration, or a pattern forming structure.

5. Concluding remarks

The Planck temperature anistoropy maps were used to probe the large-scale spacetime structure [16]. The observational data were completed with respective calculus for the Baysesian likelihood with simulations for specific topological models (in universes with locally flat, hyperbolic and spherical geometries). All such work found no evidence for a multiply-connected spacetime topology (when the assumption on the fundamental domain is considered within the last scattering surface). No matching circles, which would result from the intersection of fundamental topological domains with the surface of last scattering, were found. It is supposed that future Planck measurements of CMB polarization may provide more definitive conclusions on anisotropic geometries and non-trivial topologies. At present, the Planck data provides certain phenomenological evidence for a Bianchi VII_h component when parameters are decoupled from standard cosmology. There is no a well defined set of cosmological parameters which can produce existing patterns and observed anisotropies on other scales.

Following new results of Planck2015 [13–17] (with the ratio of tensor perturbation amplitude $r < 0.1$) authors of [1–6] concluded that such observational data seem to 'virtually eliminate all the simplest textbook inflationary models'. In order to solve this problem and update cosmological scenarios, theorists elaborate [7–10] on three classes of cosmological theories:

- There are alternative plateau-like and multi-parameter models adjusted in such ways that necessary r is reproduced. This results in new challenges like 'unlikeness' and multiverse-un predictiability problems with more tuning and of parameters and initial conditions.
- The classic inflationary paradigm is changed into a 'postmodern' one and a MGT that allow certain flexibility to fit any combination of observations. Even a series of conceptual problem of initial conditions and multiverse is known and unresolved for decades, many theorists still advocate this direction.

- There are developed 'bouncing' cosmologies, for instance, certain versions of ekpyrotic (cyclic) cosmology and, also, anamorphic cosmology. In such models, the large scale structure of the universe is set via a period of slow contraction when the big bang is replaced by a big bounce. The anamorphic approach is also considered as a different scenario with a smoothing and flattening of the universe via a contracting phase. This way, a nearly scale-ivariant spectrum of perturbations is generated.

The ekpyrotic cosmology [4] fits quite well the Planck2015 data even in the simplest version with the least numbers of parameters and the least amount of tuning. It provides a mechanism for getting a smooth and flat cosmological background via a period of ultra slow contraction before the big bang. For such a model, there are not required improbable initial conditions and the multiverse problem is avoided. Realistic ekpyrotic theories [4, 80–82] involve two scalar fields when only one has a negative potential in such a form that a non-canonical kinetic coupling acts as an additional friction term for a scalar field freezing the second one. A standard stability analysis proves that diagonal cosmological solutions for such a model are scale-invariant and stable.

We note that the anamorphic cosmology [1–6] was developed as an attempt to describe the early-universe in a form combining the advantages of the 'old and modern' inflationary and ekpyrotic models. The main assumption is that the effective Plank mass, $M_{Pl}(t)$, has a different time dependence on t, compared to the mass of a massive particle $m(t)$ in any Weyl frame during the primordial genesis phase. Such cosmological models with similar, or different, variations of fundamental constants and masses of particles can be developed in the framework of various MGTs, see discussions in [1, 3]. In our works [49–52, 77], we proved that it is possible to construct exact solutions with effective polarization of constants (in general, depending on all spacetime coordinates, $M_{Pl}(x^i, y^3, t)$ and $m(x^i, y^3, t)$) in GR mimicking time-like dependencies in MGTs if generic off-diagonal metrics and nonholonomically deformations of connections are considered for constructing new classes of cosmological solutions. Such exact/ parametric solutions can be constructed in general form using the anholonomic frame deformation method, AFDM, see review of results in [75, 76] and references therein. Following this geometric method, we perform such nonholonomic deformations of the coefficients of frames, generic off-diagonal metrics and (generalized) connections when the (generalized) Einstein equations can be decoupled in general forms and integrated for various classes of metrics $g_{\alpha\beta}(x^i, y^3, t)$.

Finally, we note that noholonomic anamorphic scenarios allow us to preserve the paradigm of Einstein's GR theory and to produce cosmological (expanding for certain phases and contracting in other cases) inflation and acceleration, if generic off-diagonal gravitational interactions model equivalently modifications of diagonal configurations in MGTs. This is possible if more general classes of cosmological solutions encoding QC structures and LQG corrections are considered.

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Appendix. Off-diagonal cosmological solutions in MGTs

We present a brief review of the anholonomic frame deformation method, AFDM, for generating off-diagonal solutions in MGTs and GR, see details in [50–52, 75] and references therein.

The N-adapted coefficients of the canonical d-connection $\widehat{\mathbf{D}} = \{\widehat{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})\}$ (5) are computed following formulas

$$\widehat{L}^i_{jk} = \frac{1}{2} g^{ir}\left(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}\right), \widehat{C}^a_{bc} = \frac{1}{2} g^{ad}\left(e_c g_{bd} + e_b g_{cd} - e_d g_{bc}\right),$$
$$\widehat{C}^i_{jc} = \frac{1}{2} g^{ik} e_c g_{jk},\ \widehat{L}^a_{bk} = e_b(N^a_k) + \frac{1}{2} g^{ac}\left(\mathbf{e}_k g_{bc} - g_{dc}\, e_b N^d_k - g_{db}\, e_c N^d_k\right). \tag{A.1}$$

The torsion, $\widehat{\mathcal{T}}$, and the curvature, $\widehat{\mathcal{R}}$, tensors of $\widehat{\mathbf{D}} = (h\widehat{D}, v\widehat{D})$ are defined in standard froms for any distinguished vectors, d-vectors, $\mathbf{X}$ and $\mathbf{Y}$,

$$\widehat{\mathcal{T}}(\mathbf{X},\mathbf{Y}) := \widehat{\mathbf{D}}_{\mathbf{X}}\mathbf{Y} - \widehat{\mathbf{D}}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}] \text{ and } \mathcal{R}(\mathbf{X},\mathbf{Y}) := \widehat{\mathbf{D}}_{\mathbf{X}}\widehat{\mathbf{D}}_{\mathbf{Y}} - \widehat{\mathbf{D}}_{\mathbf{Y}}\widehat{\mathbf{D}}_{\mathbf{X}} - \widehat{\mathbf{D}}_{[\mathbf{X},\mathbf{Y}]}.$$

Such formulas (including definitions of the Ricci d-tensor and related scalar curvature) can be written and computed in N-adapted form as in footnote 10. For (A.1), we express the nontrivial d-torsion coefficients $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$ in the form $\widehat{T}^i_{\ jk} = \widehat{L}^i_{jk} - \widehat{L}^i_{kj}$, $\widehat{T}^i_{\ ja} = \widehat{C}^i_{jb}$, $\widehat{T}^a_{\ ji} = -\Omega^a_{\ ji}$, $\widehat{T}^c_{aj} = \widehat{L}^c_{aj} - e_a(N^c_j)$, $\widehat{T}^a_{\ bc} = \widehat{C}^a_{bc} - \widehat{C}^a_{cb}$. These d-torsion coefficients vanish if there are satisfied the conditions

$$\widehat{L}^c_{aj} = e_a(N^c_j),\ \widehat{C}^i_{jb} = 0, \Omega^a_{\ ji} = 0. \tag{A.2}$$

Using above formulas for (A.1) and any d-metric (1), we can compute the coefficients of the Riemann d-tensor, $\widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\delta}$, the Ricci d-tensor, $\widehat{\mathbf{R}}_{\alpha\beta}$, and the Einstein d-tensor $\widehat{\mathbf{E}}_{\alpha\beta} := \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\, \widehat{R}$. Such values can be similarly computed for a LC-connection $\nabla = \{\Gamma^{\gamma}_{\ \alpha\beta}\}$.

To generate locally anisotropic and nonhomogeneous cosmological solutions, we consider a d-metric $\mathbf{g}$ (1) which via frame and coordinate transforms can be parameterized in the form

$$g_i = \mathrm{e}^{\psi(x^k)},\quad g_a = \omega(x^k, y^b) h_a(x^k, t),\ N^3_i = n_i(x^k, t),\quad N^4_i = w_i(x^k, t). \tag{A.3}$$

For $\omega = 1$, such off-diagonal cosmological metrics posses a space like killing symmetry on $\partial_3 = \partial q/\partial\varphi$. We can use brief notations of partial derivatives $\partial_\alpha q = \partial q/\partial u^\alpha$ when, for any function $q(x^k, y^a)$, one compute $\partial_1 q = q^\bullet = \partial q/\partial x^1, \partial_2 q = q' = \partial q/\partial x^2, \partial_3 q = \partial q/\partial y^3 = \partial q/\partial\varphi = q^\diamond, \partial_4 q = \partial q/\partial t = \partial_t q = q^*, \partial^2_{33} = \partial^2 q/\partial\varphi^2 = \partial^2_{\varphi\varphi} q = q^{\diamond\diamond}, \partial^2_{44} = \partial^2 q/\partial t^2 = \partial^2_{tt} q = q^{**}$. The sources (10) for (effective) matter field configurations can be parameterized via frame transforms in respective N-adapted forms, $\mathbf{\Upsilon}^{\mu}_{\ \nu} = \mathbf{e}^{\mu}_{\ \mu'}\mathbf{e}_{\nu}^{\ \nu'}\ \mathbf{\Upsilon}^{\mu'}_{\ \nu'}] = [\ {}_h\Upsilon(x^i)\delta^i_j, \Upsilon(x^i, t)\delta^a_b]$ (13). The values ${}_h\Upsilon(x^i)$ and $\Upsilon(x^i, t)]$ can be taken as generating functions for (effective) matter sources. They impose nonholonomic frame constraints on cosmological dynamics of (effective) matter fields. For simplicity, we consider generation of generic off-diagonal cosmological solutions with $\partial_b h_a \neq 0$ and $[\ {}_h\overline{\Upsilon}(x^i), \overline{\Upsilon}(x^i, t)] \neq 0$ (such conditions can be satisfied for certain frame/coordianate systems)[19].

Let us prove that above introduced parameterizations of cosmological d-metrics and (effective) sources allows us to integrate in explicit form the MGT field equations (9). Here

[19] It is possible to construct important (non) vacuum solutions if such conditions are not satisfied with respect to certain systems of references. This requests more special methods.

we note that constructing off-diagonal solutions for some nonholonomic cosmological configurations we can impose additional nonholonomic constraints and obtain configurations with $g_{\alpha\beta}(x^i,t) \approx g_{\alpha\beta}(t)$ which can be related to Bianchi type, or FLRW, like cosmological metrics.

Considering d-metrics with data (A.3) for $\omega = 1$, we compute $\widehat{\mathbf{D}} = \{\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}\}$ (A.1) and $\widehat{\mathbf{R}}_{\alpha\beta}$. The modified Einstein equations (9) transform into a system of nonlinear PDEs,

$$\begin{aligned}
\widehat{R}^1_1 = \widehat{R}^2_2 &= \frac{1}{2g_1g_2}[\frac{g_1^\bullet g_2^\bullet}{2g_1} + \frac{(g_2^\bullet)^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1'g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1''] = -\ {}_h\Upsilon,\\
\widehat{R}^3_3 = \widehat{R}^4_4 &= \frac{1}{2h_3h_4}[\frac{(h_3^*)^2}{2h_3} + \frac{h_3^*h_4^*}{2h_4} - h_3^{**}] = -\Upsilon\\
\widehat{R}_{3k} &= \frac{h_3}{2h_4}n_k^{**} + (\frac{h_3}{h_4}h_4^* - \frac{3}{2}h_3^*)\frac{n_k^*}{2h_4} = 0,\\
\widehat{R}_{4k} &= -\frac{w_k}{2h_3}[\frac{(h_3^*)^2}{2h_3} + \frac{h_3^*h_4^*}{2h_4} - h_3^{**}] + \frac{h_3^*}{4h_3}(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}) - \frac{\partial_k h_3^*}{2h_3} = 0,
\end{aligned} \tag{A.4}$$

for partial derivatives $\partial_t q = \partial_4 q = q^*$ and $\partial_i q = (\partial_1 q = q^\bullet, \partial_2 q = q')$. The zero torsion conditions request impose for data (A.3) additional LC-conditions (A.2) which can be written in the form

$$\partial_t w_i = (\partial_i - w_i\partial_t)\ln\sqrt{|h_4|}, (\partial_i - w_i\partial_t)\ln\sqrt{|h_3|} = 0, \partial_k w_i = \partial_i w_k, \partial_t n_i = 0, \partial_i n_k = \partial_k n_i. \tag{A.5}$$

The system (A.4) can be written in the form

$$\begin{aligned}
\psi^{\bullet\bullet} + \psi'' &= 2\ {}_h\Upsilon\\
\varpi^* h_3^* &= 2h_3h_4\Upsilon\\
n_i^{**} + \gamma n_i^* &= 0,\\
\beta w_i - \alpha_i &= 0,
\end{aligned} \tag{A.6}$$

where the system of reference is chosen for $\partial_t h_a \neq 0$ and $\partial_t\varpi \neq 0$ and the coefficients are computed,

$$\alpha_i = (\partial_t h_3)\ (\partial_i\varpi),\ \beta = (\partial_t h_3)\ (\partial_t\varpi),\ \gamma = \partial_t\left(\ln|h_3|^{3/2}/|h_4|\right),\ \text{where } \varpi = \ln|\partial_t h_3/\sqrt{|h_3h_4|}|. \tag{A.7}$$

The system of nonlinear PDEs (A.6) reflects a decoupling property of equations for functions ψ, h_a, n_i^* and can be integrated in general form for any generating function $\Psi(x^i,t) := \mathrm{e}^{\varpi}$ and sources ${}_h\Upsilon(x^i)$ and $\Upsilon(x^k,t)$.

The system (A.6) can be integrating in general form 'step by step'. In result, we generate exact solutions of the modified Einstein equation (9) parameterized by such coefficients of a

$$\begin{aligned}
&\text{d}-\text{metric}: \ g_i = \mathrm{e}^{\,\psi(x^k)} \text{ as a solution of 2d Poisson equations. } \psi^{\bullet\bullet} + \psi'' = 2\ {}_h\Upsilon;\\
&g_3 = h_3(x^i,t) = -(\Psi^2)^*/4\Upsilon^2 h_4 = -(\Psi^2)^*/4\Upsilon^2(h_4^{[0]}(x^k) - \int \mathrm{d}t(\Psi^2)^*/4\Upsilon)\\
&g_4 = h_4(x^i,t) = h_4^{[0]}(x^k) - \int \mathrm{d}t(\Psi^2)^*/4\Upsilon;\\
&N-\text{connection}: N_k^3 = n_k(x^i,t) = {}_1n_k(x^i) + {}_2n_k(x^i)\int \mathrm{d}t(\Psi^*)^2/\Upsilon^2|h_4^{[0]}(x^i) - \int \mathrm{d}t(\Psi^2)^*/4\Upsilon|^{5/2};\\
&N_i^4 = w_i(x^i,t) = \partial_i\Psi/\Psi^* = \partial_i\Psi^2/\ (\Psi^2)^*.
\end{aligned} \tag{A.8}$$

In these formulas, the values $h_4^{[0]}(x^k)$, ${}_1n_k(x^i)$, and ${}_2n_k(x^i)$ are integration functions. The coefficients (A.8) define generic off-diagonal cosmological solutions if some anholonomy coefficients $C^{\gamma}_{\alpha\beta}(x^i,t)$ (3) are not zero. Such locally cosmological solutions can be with nontrivial nonholonomically induced d-torsion or for LC-configurations if the conditions (A.5) are satisfied. We generate as particular cases some well-known cosmological FLRW, or Bianchi, type metrics, for cerain data of type $(\Psi(t),\Upsilon(t))$ with integration functions which allow frame/coordinate transforms to respective (off-) diagonal configurations $g_{\alpha\beta}(t)$.

Introducing the coefficients (A.8) into (A.3) $\mathbf{g}$ (1), we construct construct a class of linear quadratic elements for locally anisotropic cosmological solutions,

$$
\begin{aligned}
\mathrm{d}s^2 = & \; \mathrm{e}^{\,\psi(x^k)}[(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2] + (h_4^{[0]} - \int \mathrm{d}t \frac{(\overline{\Psi}^2)^*}{4\overline{\Upsilon}})[\mathrm{d}t + \frac{\partial_i\,\overline{\Psi}}{\overline{\Psi}^*}\mathrm{d}x^i] \\
& - \frac{(\overline{\Psi}^2)^*}{4\Upsilon^2\left(h_4^{[0]} - \int \mathrm{d}t \frac{(\overline{\Psi}^2)^*}{4\overline{\Upsilon}}\right)}[\mathrm{d}y^3 + ({}_1n_k + \;{}_2n_k \int \mathrm{d}t \frac{(\overline{\Psi}^*)^2}{\overline{\Upsilon}^2 |h_4^{[0]} - \int \mathrm{d}t \frac{(\overline{\Psi}^2)^*}{4\overline{\Upsilon}}|^{5/2}})\mathrm{d}x^k].
\end{aligned} \tag{A.9}
$$

Such solutions posses a Killing symmetry on ∂_3 and can be re-written in terms of η-polarization function functions for target locally anisotropic cosmological metrics $\widehat{\mathbf{g}} = [g_\alpha = \eta_\alpha \mathring{g}_\alpha,\ \eta_i^a \mathring{N}_i^a]$ encoding primary cosmological data $[\mathring{g}_\alpha, \mathring{N}_i^a]$.

We can extract cosmological spacetimes in GR (with zero torsion) if the conditions (A.5) are imposed and solved for a special class of generating functions and sources. For instance, taking a $\Psi = \check{\Psi}(x^i,t)$ subjected to the conditions $(\partial_i\check{\Psi})^* = \partial_i(\check{\Psi}^*)$ and $\Upsilon(x^i,t) = \Upsilon[\check{\Psi}] = \check{\Upsilon}$, or $\Upsilon = \mathrm{const}$, we generate LC-configurations for some functions $\check{A}(x^i,t)$ and $n(x^i)$ when the N-connection coefficients are computed $\overline{n}_k = \check{n}_k = \partial_k n(x^i)$ and $w_i = \partial_i\check{A} = \partial_i\check{\Psi}/\check{\Psi}^*$. Such off-diagonal locally anisotropic cosmological solutions in GR are defined as subclasses of solutions (A.9) with zero torsion,

$$
\begin{aligned}
\mathrm{d}s^2 = & \; \mathrm{e}^{\,\psi(x^k)}[(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2] - \frac{(\check{\Psi}^2)^*}{4\check{\Upsilon}^2(h_4^{[0]} - \int \mathrm{d}t \frac{(\check{\Psi}^2)^*}{4\check{\Upsilon}})}[\mathrm{d}y^3 + (\partial_k n)\mathrm{d}x^k] \\
& + (h_4^{[0]} - \int \mathrm{d}t \frac{(\check{\Psi}^2)^*}{4\check{\Upsilon}})[\mathrm{d}t + (\partial_i\check{A})\mathrm{d}x^i].
\end{aligned} \tag{A.10}
$$

Quadratic linear elements for exact off-diagonal solutions (A.9) or (A.10) constructed above can be parameterized in the form (39). Such parameterizations are in terms of polarization functions $\eta_\alpha = (\eta_i, \eta_a)$ and η_i^a defining nonholonomic deformations of a prime d-metric, $\mathring{\mathbf{g}}$, into a target d-metric, $\widehat{\mathbf{g}} = [g_\alpha = \eta_\alpha \mathring{g}_\alpha,\ \eta_i^a \mathring{N}_i^a] \rightarrow \mathring{\mathbf{g}}$. Such parameterizations with ε-deformations are useful for analyzing possible physical implications of general off-diagonal deformations of some physically important solutions when, for instance, $\mathring{\mathbf{g}}$ is taken for a standard cosmological, solution in GR, or in a MGT.

Finally, we note that the AFDM allows to construct off-diagonal cosmological solutions in general form depending in arbitrary classes of generating functions and effective sources. Such functions may remove existing singularities of a prime metric, or inversely, to transform nonsingular configurations into singular locally anisotropic cosmological ones because of singular nonholonomic deformations. In general, it is not clear what physical implications may have some classes off-diagonal solutions. In section 3, we demonstrate that well-defined physical interpretations can be provided for anamorphic cosmological quasi-periodic, pattern-forming and/or solitonic structures generated in in general off-diagonal form. For

ϵ-deformations studied in sections 2.3 and 4, such locally anisotropic cosmological structures can be considered for modeling dark matter and dark energy effects determined by 'small' off-diagonal deformations of FLRW, or Bianchi, type metrics in standard cosmology.

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