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Abstract: We discuss the action of $O(d, d)$, and in particular T-duality, in the context of generalized geometry, focusing on the description of so-called non-geometric backgrounds. We derive local expressions for the pure spinors descibing the generalized geometry dual to an SU(3) structure background, and show that the equations for $N = 1$ vacua are invariant under T-duality. We also propose a local generalized geometrical definition of the charges f, H, Q and R appearing in effective four-dimensional theories, using the Courant bracket. We then address certain global aspects, in particular whether the local non-geometric charges can be gauged away in, for instance, backgrounds admitting a torus action, as well as the structure of generalized parallelizable backgrounds.

Keywords: Flux compactifications, String Duality

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1 Introduction

Recent developments in flux compactifications brought T-duality to the center stage [1–10]. Given a background with isometries, T-duality is a very effective tool for generating new backgrounds. Due to the mixing of the metric and NS two-form B, it can relate string backgrounds with drastically different properties.

A well known example is the action of a single T-duality along an isometry direction of a manifold with an H flux. At the local level, T-duality exchanges the off-diagonal components of the metric with those of the B field. Since the Killing vector generating the isometry must be globally defined, the manifold can be thought of as a circle fibration over a base M. Topologically the fibration can be characterized by the first Chern number. Another topological number is given by the integral over a two-cycle on the base of the two-form obtained by contracting the H-flux with the Killing vector. T-duality exchanges these numbers, leading to a change in topology [11]. Indeed, the dual manifold is again a circle fibration over the same base, but has in general a different first Chern number.

For backgrounds with d isometries, since the T-duality group $O(d,d)$ is much larger than the group of diffeomorphisms, it is not surprising that the action of T-duality can be more interesting and complicated [12].

At the level of string sigma model there is a well-defined procedure of performing the duality transformations (see e.g. [13]). Given a background with isometries, one gauges them and adds Lagrange multipliers. Integrating out the original directions of isometries, while leading to the same two-dimensional quantum field theory, yields a different target space. There are however global obstructions in performing the above procedure [14, 15]. In order to perform T-duality (in three or more directions) the component of the H flux fully lying in the directions to be dualized has to vanish, in other words, the B-field must respect the isometries. Also, the component of H with two legs along the duality directions must be trivial in cohomology, i.e. the corresponding component of the B-field is globally defined. A priori, when such global obstructions are present, T-duality is not possible.

However, it has been proposed in [14], that in some of the obstructed cases, T-duality can lead to consistent string backgrounds. While it is still possible to give local expressions for the metric and B-field, globally the resulting background will not have a conventional description as a good internal manifold, thus the terming "non-geometric compactification".

The interest in such "non-geometric" backgrounds is also motived by the analysis of four-dimensional effective theories. From the point of view of gauged supergravities in four dimensions, potentials governed by a large duality group seem to admit various minima, many of which cannot correspond to conventional string compactifications. It is interesting to identify those that can be lifted to full string solutions, and the backgrounds obtained via obstructed T-duality transformations are natural candidates. There is much recent work supporting this possibility [5, 6]. When trying to lift solutions of four-dimensional gauged supergravities to ten dimensions, the origin of the gauged symmetries as well as that of the structure constants in their algebra need to be given a string theory interpretation. For compactifications on d-dimensional homogeneous parallelizable manifolds (loosely called twisted tori), there are $2d$ symmetries corresponding to translations and to gauge transformations on B. The twisting of the frame bundle and the H-flux appear as structure constants of the "Kaloper-Myers algebra" [16]. This algebra is however not covariant under the duality group $O(d,d)$, since such an invariance would require twice as many charges. It has been argued that the missing half corresponds to "non-geometric fluxes", encoding, for example, monodromies in the T-duality group, mixing metric and B-field [2, 3].

The T-duality group $O(d,d)$ is also the structure group of the generalized tangent bundle, which combines the tangent and cotangent bundle of a d-dimensional manifold in Generalized Geometry [17, 18]. The generalized metric on the generalized bundle encodes the information about the metric and B-field of the manifold, which are exchanged by T-duality. Additionally, non-trivial patchings of the B-field are naturally incorporated in generalized geometry by defining an extension of the tangent bundle by the cotangent one. In this extension, the patching between two overlapping regions uses, besides the usual diffeomorphisms, an abelian subgroup of $O(d,d)$ involving the B-field. This suggests the use of generalized geometry to describe the action of T-duality [18, 19].

Generalized (complex) geometry is very well suited to describe $\mathcal{N}=1$ supersymmetric compactification with non trivial fluxes. When there are nowhere vanishing spinors on the manifold, one can construct bispinors by tensoring two $O(d)$ spinors with the same and with opposite chiralities. These $O(d,d)$ spinors also carry the information about the metric and B-field on the manifold. Each $O(d,d)$ spinor corresponds to an algebraic structure, namely, for six-dimensional manifolds, an $\mathrm{SU}(3,3) \subset O(6,6)$. The pair of $\mathrm{SU}(3,3)$ structures defines an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure on the generalized tangent bundle. In the context of flux compactifications this can can be understood as two independent $\mathrm{SU}(3)$ structures, one on the left and one on the right moving sector. The corresponding globally defined $O(d)$ spinors are the internal supersymmetry parameters. The conditions for $\mathcal{N}=1$ vacua reduce to a couple of first order differential equations for the pure spinors, implying the closure of one pure spinor and relating the failure of integrability of the other to the RR fluxes [20]. A manifold admitting a closed pure spinor is a Generalized Calabi-Yau.

Even though generalized geometry conventionally describes ordinary geometrical backgrounds, we will argue that it is still a suitable language to describe some aspects of non-geometric backgrounds. Specifically one is interested in the cases where the compactification remains locally a manifold, such as arises from obstructed T-duality of conventional geometries (some progress in this direction has been done in [8, 21]).

In this paper we discuss the action of $O(d,d)$ as well as the emergence of the extended Kaloper-Myers algebra, in the context of generalized geometry. We will first do this at the local level, and then discuss the global properties. It is at this point that the difference between geometry and non-geometry appears. At the local level, we find the T-duality action on the objects defining the generalized metric, namely the generalized vielbeins in the generic case, and the pure spinors in the case of reduced structure. We will mostly consider situations where the $O(d,d)$ transformations are along isometries of the background. We will assume that the B-field respects the isometries and yet is not globally well-defined (in particular the components to be fully T-dualized). In these cases we argue that T-duality yields perfectly good local expressions (mixing the metric and the B-field).

We also show that the charges in the extended Kaloper-Myers algebra have a simple interpretation as elements of a generalized spin-connection, or, equivalently, as structure constants (or rather functions, in the generic case) of the Courant algebra of generalized vielbeins. As such, the distention between geometric and non-geometric charges depends on the frame, and therefore loses physical content. However, when turning to global properties, we show that the transformation taking from one frame to another can be ill-defined if there are non-contractible loops. In this case the non-geometric charges cannot be globally gauged away. The distinction between geometric and non-geometric situations is nicely rephrased in terms of right and left mover sectors of string theory. Generalized geometry suggests the use of a different set of vielbeins for the left and right movers, which transform nicely under $O(d,d)$.[1] For geometric backgrounds it is always possible, after T-duality, to perform a well defined local $O(d)\times O(d)$ transformation to set the vielbeins for the right and

[1] Two sets of vielbeins on the target space were introduced in the context of T-duality in [22]. They also appear naturally in the doubled formalism [1].

left moving sectors to be the same everywhere. On the contrary, when such a transformation is only possible locally, the background is non-geometric. A similar situation arises in the doubled formalism, where for geometrical backgrounds one is again able to use $O(d) \times O(d)$ transformations to write the doubled vielbeins in a particular triangular form [10].

The two different sets of vielbeins on right and left movers have nice transformation properties under $O(d,d)$. Moreover they provide a way to determine the $O(d,d)$ transformation of an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure. In principle, given the two new SU(3) structures, one should be able to build the corresponding T-dual spinors. However, since the pure spinors are mixtures of left and right moving sectors, determining them explicitly can be quite challenging. Here we use a different approach and we study directly the action of $O(d,d)$ on the spinors. In particular we derive local expressions for the pure spinors dual, or mirror, to those corresponding to a single SU(3) structure (i.e. where the original structures on the left and on the right sector coincide). These correspond generically to $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structures.

As already mentioned, generalized geometry allows for an elegant classification of Type II flux background with $\mathcal{N} = 1$ supersymmetry. Given that supersymmetry equations are local, they could still be considered in the case of non-geometric backgrounds, when these admit a local description. In other words, one might wonder whether considering T-duality along isometries that commute with the supersymmetry generators, one might still have good local solutions, also when T-duality is obstructed. We show that the $\mathcal{N} = 1$ supersymmetry equations on pure spinors are invariant under T-duality.

A very simple example of a generalized geometrical background is one where there is a globally defined set of generalized vierbein, the analogue of a conventional parallelizable background. The Courant bracket on the preferred frame then provides a natural global definition of the generalized charges. Many of the simplest non-geometrical examples are of this "generalized parallelizable" type. We discuss some necessary conditions on the local geometry in this case and in particular show that the R charge always vanishes.

The paper is organized as follows. In section 2 we review the necessary ingredients of generalized geometry, and find the $O(d,d)$ transformations of the vielbeins. In section 3 we discuss how T-duality acts on the generalized structures, and find explicit expressions for the duals of an SU(3) structure. We also show that the equations for $\mathcal{N} = 1$ vacua are invariant under T-duality. In section 4 we introduce the generalized spin connection and we discuss how the charges of the extended Kaloper-Myers algebra arise locally from the Courant bracket. Finally in section 5 we discuss global issues and non-geometricity, as well as the structure of generalized parallelizable backgrounds.

2 Generalized geometry

This section starts with a review of generalized geometry, the generalized $O(d,d)$ spinors and the generalized metric $\mathcal{H}$, which encodes the ordinary metric g and the B-field. The generalized metric defines an $O(d) \times O(d)$ structure, and we also introduce a natural set of generalized vielbeins for $\mathcal{H}$. This latter has been previously analyzed by Hassan [22]. One new element here is the discussion of how the dilaton naturally enters the definition of $O(d,d)$ spinors.

We then turn, in the context of six-dimensional manifolds, to the various definitions of SU(3) $\times$ SU(3) structures relevant to supersymmetric backgrounds.

Of particular interest is how the $O(d,d)$ group acts on the generalized vielbeins and hence on the ordinary vielbein and B-field (also discussed in [22]). In addition we consider the action of $O(6,6)$ on an SU(3) $\times$ SU(3) structure. These will be useful in the following parts of the paper were we specialize to T-duality transformations in a number of specific cases. Here we consider only the action of $O(d,d)$ at a point in the manifold.

2.1 Generalized tangent bundle

The basic idea of generalized geometry [17, 18] is to combine vectors and one-forms into a single object. Formally, on a d-dimensional manifold M one introduces the generalized tangent bundle E which is a particular extension of T by T^*

$$0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0. \tag{2.1}$$

Sections of E are called generalized vectors. Locally they can be written as $X = x + \xi$ where $x \in TM$ and $\xi \in T^*M$. In going from one coordinate patch U_α to another U_β, we have to first make the usual patching of vectors and one-forms, and then give a further patching describing how T^*M is fibered over TM in E. This gives

$$x_{(\alpha)} + \xi_{(\alpha)} = a_{(\alpha\beta)} x_{(\beta)} + \left[a^{-T}_{(\alpha\beta)} \xi_{(\beta)} - i_{a_{(\alpha\beta)} x_{(\beta)}} \omega_{(\alpha\beta)} \right] , \tag{2.2}$$

where $a_{(\alpha\beta)} \in GL(d,\mathbb{R})$, $\omega_{(\alpha\beta)}$ is a two-form and $a^{-T} = (a^{-1})^T$. Using a two-component notation to distinguish the vector and form parts of X we can write

$$X_{(\alpha)} = \begin{pmatrix} x_{(\alpha)} \\ \xi_{(\alpha)} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ \omega_{(\alpha\beta)} & \mathbb{1} \end{pmatrix} \begin{pmatrix} a_{(\alpha\beta)} & 0 \\ 0 & a^{-T}_{(\alpha\beta)} \end{pmatrix} \begin{pmatrix} x_{(\beta)} \\ \xi_{(\beta)} \end{pmatrix} = p_{(\alpha\beta)} X_{(\beta)} \,. \tag{2.3}$$

In fact one makes the further restriction that $\omega_{(\alpha\beta)} = -\mathrm{d}\Lambda_{(\alpha\beta)}$, where $\Lambda_{(\alpha\beta)}$ are required to satisfy

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = g_{(\alpha\beta\gamma)} \mathrm{d} g_{(\alpha\beta\gamma)} \tag{2.4}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ and $g_{\alpha\beta\gamma} := \mathrm{e}^{\mathrm{i}\alpha}$ is a U(1) element. This is analogous to the patching of a U(1) bundle, except that the transition "functions" are one-forms, $\Lambda_{(\alpha\beta)}$. Formally it is called the "connective structure" of a gerbe. The point is that it is the geometrical structure one needs to introduce B, the two-form analogue of an ordinary one-form U(1) connection, with a correspondingly quantized field strength H.

Given the split into vectors and forms, there is a natural $O(d,d)$-invariant metric η on E, given, on each patch, by

$$\eta(X,X) = i_x \xi, \tag{2.5}$$

or, in two-component notation, $\eta(X,X) = X^T \eta X$ with

$$\eta = \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{2.6}$$

The metric is invariant under $O(d,d)$ transformations acting on the fibres of E. A general element $O \in O(d,d)$ can be written in terms of $d \times d$ matrices a, b, c, and d as

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.7}$$

under which a generic element $X \in E$ transforms by

$$X = \begin{pmatrix} x \\ \xi \end{pmatrix} \mapsto OX = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}. \tag{2.8}$$

The requirement that $\eta(OX,OX) = \eta(X,X)$ implies $a^T c + c^T a = 0$, $b^T d + d^T b = 0$ and $a^T d + c^T b = \mathbb{1}$. Note that the $GL(d)$ action on the fibres of TM and T^*M embeds as a subgroup of $O(d,d)$. Concretely it maps

$$X \mapsto X' = \begin{pmatrix} a & 0 \\ 0 & a^{-T} \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}, \tag{2.9}$$

where $a \in GL(d)$. Similarly the first factor in (2.3) is also an (Abelian) subgroup $G_B \subset O(d,d)$. Given a two-form ω, we write

$$\mathrm{e}^{\omega} = \begin{pmatrix} \mathbb{1} & 0 \\ \omega & \mathbb{1} \end{pmatrix} \qquad \text{such that} \qquad X = x + \xi \mapsto X' = x + (\xi - i_x \omega). \tag{2.10}$$

This is usually referred to as a B-transform. Given a bivector β one can similarly define another Abelian subgroup of β-transforms

$$\mathrm{e}^{\beta} = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix} \qquad \text{such that} \qquad X = x + \xi \mapsto X' = (x + \beta \cdot \xi) + \xi. \tag{2.11}$$

The patching (2.3) of E was by elements of $GL(d)$ and G_B. Together these form a subgroup which is a semi-direct product $G_{\text{geom}} = G_B \rtimes GL(d)$. A general element of G_{geom} can be written as

$$p = \mathrm{e}^{\omega} \begin{pmatrix} a & 0 \\ 0 & a^{-T} \end{pmatrix} = \begin{pmatrix} a & 0 \\ \omega a & a^{-T} \end{pmatrix}. \tag{2.12}$$

This patching means that the structure group of the generalized tangent space E actually reduces from $O(d,d)$ to G_{geom}. The embedding of $G_{\text{geom}} \subset O(d,d)$ is fixed by the projection $\pi : E \to TM$. It is the subgroup which leaves the image of the related embedding $T^*M \to E$ invariant.

There is also a natural bracket on generalized vectors known as the Courant bracket, which encodes the differentiable structure of E and will play an important role in what follows. It is defined as

$$[x + \xi, y + \eta] = [x,y]_{\text{Lie}} + \mathcal{L}_x \eta - \mathcal{L}_y \xi - \frac{1}{2} \mathrm{d}\left(i_x \eta - i_y \xi\right), \tag{2.13}$$

where $[x,y]_{\text{Lie}}$ is the usual Lie bracket between vectors and $\mathcal{L}_x$ is the Lie derivative. The Courant bracket is invariant under the action of elements of G_{geom}, (2.12), where the $GL(d)$ transformations a are generated by diffeomorphisms and the B-shifts ω are closed, $\mathrm{d}\omega = 0$.

2.2 Generalized metrics, generalized vielbeins and $O(d) \times O(d)$ structures

In the generalized geometry picture the metric g and the B-field combine into a single object which defines an $O(d)\times O(d)$ structure on E. To define an $O(d)\times O(d)$ structure we need the bundle E to split into two orthogonal d-dimensional sub-bundles $E = C_+ \oplus C_-$ such that the metric η decomposes into a positive-definite metric on C_+ and a negative-definite metric on C_-. The subgroup of $O(d,d)$ which preserves each metric separately is then $O(d) \times O(d)$. Since any element of E which is a pure vector or a pure one-form is null with respect to η, such elements cannot lie in C_+ or C_-. Hence we can write a generic element $X_+ \in C_+$ as $x + Mx$, where $x \in TM$ and, in components, the form part is given by $M_{mn}x^n$ for some general matrix M. (This actually describes an isomorphism between TM and C_+.) If we write $M_{mn} = B_{mn} + g_{mn}$, where g is symmetric and B antisymmetric, we see that the patching condition (2.3) implies that

$$g_{(\alpha)} = g_{(\beta)}, \qquad B_{(\alpha)} = B_{(\beta)} - \mathrm{d}\Lambda_{(\alpha\beta)}, \tag{2.14}$$

and hence is associated to the connective structure of a two-form B-field. Orthogonality between C_+ and C_- implies that a generic element of $X_- \in C_-$ can be written as $X_- = x + (B - g)x$.

Another way to define this structure is to introduce the $O(2d)$-invariant generalized metric[2]

$$\mathcal{H} = \eta|_{C_+} - \eta|_{C_-} . \tag{2.15}$$

Writing a general element $X = x + \xi \in E$ as $X = X_+ + X_-$ with $X_\pm = x_\pm + (B \pm g)x_\pm$ one finds that the generalized metric $\mathcal{H}$ takes the form

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} . \tag{2.16}$$

We can also introduce generalized vielbeins, where the local Lorentz symmetry is replaced by $O(d) \times O(d)$. They parametrise the coset $O(d,d)/O(d) \times O(d)$ and encode the metric g and the B-field. There are many different conventions one could use. Consider a basis of generalized one-forms $E_A \in E^*$ with $A = 1, \dots 2d$. (Note that η gives an isomorphism between E and E^* so we can equally well think of the E_A as generalized vectors.) One possibility is then to require that the metrics η and $\mathcal{H}$ take the form

$$\eta = E^T \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} E, \qquad \mathcal{H} = E^T \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} E. \tag{2.17}$$

Explicitly we have

$$E = \frac{1}{\sqrt{2}} \begin{pmatrix} e_+ - \hat{e}_+^T B & \hat{e}_+^T \\ -e_- - \hat{e}_-^T B & \hat{e}_-^T \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_+^T(g - B) & \hat{e}_+^T \\ -\hat{e}_-^T(g + B) & \hat{e}_-^T \end{pmatrix} , \tag{2.18}$$

[2] In [18], the $O(d,d)$ invariant generalized metric is defined via the product structure $G = -\mathcal{J}_1\mathcal{J}_2$, given two commuting generalized almost complex structures. This is related to our definition by $\mathcal{H} = \eta G$.

where we have introduced two sets of (ordinary) vielbeins $e^a_\pm$ and their inverse $\hat{e}_{\pm\,a}$ satisfying

$$\begin{aligned} g &= e^T_\pm e_\pm & &\text{or} & g_{mn} &= e^a_{\pm\,m} e^b_{\pm\,n}\delta_{ab}\,, \\ g^{-1} &= \hat{e}_\pm \hat{e}^T_\pm & &\text{or} & g^{mn} &= \hat{e}^m_{\pm\,a}\hat{e}^n_{\pm\,b}\delta^{ab}, \end{aligned} \tag{2.19}$$

and $e_\pm\hat{e}_\pm = \hat{e}_\pm e_\pm = \mathbb{1}$. With these conventions, the first d generalized vielbeins form a basis for C_+ and the second d form a basis for C_-. The local $O(d)\times O(d)$ action simply rotates each set of vielbeins. Concretely we can write

$$E \mapsto KE\,, \qquad K = \begin{pmatrix} O_+ & 0 \\ 0 & O_- \end{pmatrix} \quad \text{with} \quad O_\pm \in O(d)\,. \tag{2.20}$$

In type II string theory compactified on a six-dimensional manifold M, the subbundles $C_\pm$ have a natural interpretation in terms of the world-sheet theory: they are associated to the left and right mover sectors; $e_\pm$ are the corresponding vielbeins. The spinors transform under one or the other of the $O(d)$ groups. It is then usual to choose $e_+ = e_-$ so that the same spin-connections appear, for instance, in the derivatives of the two gravitini. However, this is, of course, not strictly necessary.

From the $O(d,d)$ action on the generalized metric and vielbein it is straightforward to recover the familiar $O(d,d)$ transformations on the metric, B-field and vielbein. The generalized metric (2.16) transforms under $O(d,d)$ as

$$\mathcal{H} \to \mathcal{H}' = O^T \mathcal{H} O\,, \tag{2.21}$$

with $\mathcal{H}$ and O given in (2.16) and (2.7), respectively. Given this transformation, we can derive the transformation of the bases $e_\pm$ under $O(d,d)$. The generalized basis forms E^A transform as $E \mapsto EO$, and hence the vielbeins transform as

$$\begin{aligned} \hat{e}_+ &\mapsto \left[d^T + b^T(B+g)\right]\hat{e}_+ \equiv \hat{\hat{e}}_+ \\ \hat{e}_- &\mapsto \left[d^T + b^T(B-g)\right]\hat{e}_- \equiv \hat{\hat{e}}_-\,. \end{aligned} \tag{2.22}$$

This agrees with the result given in [22]. Note that, if we initially set $e_+ = e_-$, generically this is no longer true after the $O(d,d)$ transformation, and one must make a compensating Lorentz transformation to restore the condition.

It is possible to use a different set of conventions where η and $\mathcal{H}$ take the form

$$\eta = \mathcal{E}^T \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \mathcal{E}, \qquad \mathcal{H} = \mathcal{E}^T \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \mathcal{E}. \tag{2.23}$$

In this basis the generalized vielbein can be written as

$$\mathcal{E} = \frac{1}{2}\begin{pmatrix} (e_+ + e_-) + (\hat{e}^T_+ - \hat{e}^T_-)B & (\hat{e}^T_+ - \hat{e}^T_-) \\ (e_+ - e_-) - (\hat{e}^T_+ + \hat{e}^T_-)B & (\hat{e}^T_+ + \hat{e}^T_-) \end{pmatrix}. \tag{2.24}$$

The $O(d)\times O(d)$ action is now of the form

$$\mathcal{E} \mapsto K\mathcal{E}\,, \qquad K = \frac{1}{2}\begin{pmatrix} O_+ + O_- & O_+ - O_- \\ O_+ - O_- & O_+ + O_- \end{pmatrix}. \tag{2.25}$$

As before, one can always make an $O(d) \times O(d)$ transformation to set $e_+ = e_- = e$ and put the generalized vielbein into the triangular form

$$\mathcal{E} = \begin{pmatrix} e & 0 \\ -\hat{e}^T B & \hat{e}^T \end{pmatrix} . \tag{2.26}$$

Note that in these conventions the vielbeins (2.24) are not a natural basis for $C_\pm$ since they do not diagonalise the $O(d,d)$ metric η. However they will be of particular interest in this paper because the latter form (2.26) is invariant under the G_{geom} subgroup of $O(d,d)$ transformations.

2.3 $O(d,d)$ spinors

Given the metric η, one can define $\mathrm{Spin}(d,d)$ spinors. These are Majorana-Weyl and we write the two helicity spin bundles as $S^\pm(E)$. Locally, the Clifford action of $X \in E$ on the spinors can be realized as an action on forms $\Phi \in \Lambda^{\text{even/odd}} T^*M\big|_{U_\alpha}$ given by

$$X \cdot \Phi := (x^m \check{\Gamma}_m + \xi_m \hat{\Gamma}^m)\Phi = i_x \Phi + \xi \wedge \Phi \,, \tag{2.27}$$

where $\check{\Gamma}$, $\hat{\Gamma}$ are the $O(d,d)$ gamma matrices. It is easy to see that

$$(XY + YX) \cdot \Phi = 2\eta(X,Y)\Phi \,, \tag{2.28}$$

as required. One also finds that, in going from one patch to another, the patching of E implies that

$$\Phi^\pm_{(\alpha)} = \mathrm{e}^{\mathrm{d}\Lambda_{(\alpha\beta)}} \Phi^\pm_{(\beta)} \,, \tag{2.29}$$

where the exponentiated action is by wedge product. Note that the usual action of the exterior derivative on the component forms is compatible with this patching and gives an action

$$\mathrm{d} : S^\pm(E) \to S^\mp(E) \,. \tag{2.30}$$

In terms of the $\mathrm{Spin}(d,d)$ group one can view this as a Dirac operator taking positive helicity spinors to negative helicity spinors and *vice versa*.

Let us now return to the $GL(d)$ action (2.9) on the tangent and cotangent bundles. If we take an infinitesimal transformation with $a = \mathbb{1} + \theta + \dots$, the induced action on the spinors is given by

$$\delta\Phi = \frac{1}{2}\theta^m{}_n \left(\check{\Gamma}_m \hat{\Gamma}^n - \hat{\Gamma}^n \check{\Gamma}_m \right) \Phi \,. \tag{2.31}$$

The degree of the component forms in Φ remains unchanged: in particular, each form transforms as

$$\delta\Phi_{m_1 \dots m_p} = -p\, \theta^n{}_{[m_1} \Phi_{|n|m_2 \dots m_p]} + \frac{1}{2}\theta^n{}_n \Phi_{m_1 \dots m_p} \,. \tag{2.32}$$

The first term correctly describes the transformation of an element of $\Lambda^p T^*M$ under $GL(d)$. The second term however corresponds to a rescaling of the form by a factor of $|\det a|^{1/2}$. This implies that we should locally identify [18]

$$\Phi \in |\Lambda^d T^*M|^{-1/2} \otimes \Lambda^{\text{even/odd}} T^*M\Big|_{U_\alpha} \,. \tag{2.33}$$

However, this presents a predicament: we cannot define the exterior derivative on such objects, because the extra $|\Lambda^d T^* M|^{-1/2}$ factor breaks the diffeomorphism symmetry. One solution is to identify

$$\Phi \in L \otimes \Lambda^{\text{even/odd}} T^* M\Big|_{U_\alpha} , \tag{2.34}$$

where we have introduced a new (trivial) real line bundle L with sections $\mathrm{e}^{-\phi} \in L$ that transform as

$$\mathrm{e}^{-\phi} \mapsto |\det a|^{1/2} \mathrm{e}^{-\phi} \tag{2.35}$$

under the $GL(d)$ action on TM, but which transform as scalars under diffeomorphisms. We have suggestively written the sections of L as $\mathrm{e}^{-\phi}$ since we will see in the next section that the ten-dimensional dilaton indeed transforms in this way.

Under the other two elements of $O(d,d)$ discussed in the previous section, eqs. (2.10) and (2.11), the spinor representation transforms

$$\Phi^\pm \mapsto e^{\omega+\beta} \Phi^\pm , \tag{2.36}$$

where ω acts by wedge product and β by contractions.

Using the generalized vielbeins (2.18) one can also introduce a basis for the $O(d,d)$ gamma matrices $\check{\Gamma}, \hat{\Gamma}$ adapted to the $O(d) \times O(d)$ structure. One defines

$$\begin{pmatrix} \Gamma^+ \\ \Gamma^- \end{pmatrix} = (E^{-1})^T \begin{pmatrix} \check{\Gamma} \\ \hat{\Gamma} \end{pmatrix} = \begin{pmatrix} \hat{e}_+^T \left(\check{\Gamma} + (g-B) \hat{\Gamma} \right) \\ \hat{e}_-^T \left(\check{\Gamma} - (g+B) \hat{\Gamma} \right) \end{pmatrix} , \tag{2.37}$$

which satisfy

$$\{\Gamma_a^+, \Gamma_b^-\} = 0, \quad \{\Gamma_a^+, \Gamma_b^+\} = 2\delta_{ab}, \quad \{\Gamma_a^-, \Gamma_b^-\} = -2\delta_{ab} . \tag{2.38}$$

One can then decompose the $\mathrm{Spin}(d,d)$ spinors into $\mathrm{Spin}(d) \times \mathrm{Spin}(d)$ objects. If d is even we can write

$$\Gamma_a^+ = \gamma_a \otimes \mathbb{1}, \qquad \Gamma_a^- = \mathrm{i}\tilde{\gamma} \otimes \gamma_a , \tag{2.39}$$

where γ_a are $\mathrm{Spin}(d)$ gamma matrices and $\tilde{\gamma} = \gamma_{(d)} = \gamma_1 \ldots \gamma_d$ if $d/2$ is even and $\tilde{\gamma} = -\mathrm{i}\gamma_{(d)}$ if $d/2$ is odd, so that $\tilde{\gamma}^2 = \mathbb{1}$. Similar expressions can be written when d is odd. The corresponding decompositions of the $\mathrm{Spin}(d,d)$ spinors are

$$\Phi^+ = \eta_+^1 \otimes \bar{\eta}_+^2 + \eta_-^1 \otimes \bar{\eta}_-^2 , \qquad \Phi^- = \eta_+^1 \otimes \bar{\eta}_-^2 + \eta_-^1 \otimes \bar{\eta}_+^2 , \tag{2.40}$$

where $\eta_\pm^1$ and $\eta_\pm^2$ are chiral $\mathrm{Spin}(d)$ spinors satisfying $\tilde{\gamma}\,\eta_\pm = \eta_\pm$.

The generalized metric allows us to relate the $O(d) \times O(d)$ decomposition of the $O(d,d)$ spinors to the $GL(d)$ decomposition (2.34). It is easiest to start by choosing the vielbeins such that $e_+ = e_-$. This identifies a common $O(d)$ subgroup of $O(d) \times O(d)$: η_+^1 and η_+^2 are now spinors of the same group so that, under this group, $\Phi^\pm$ is a spinor bilinear. However, any spinor bilinear can be expanded as a sum of forms using products of gamma matrices. In particular

$$\begin{aligned} \eta_+^1 \bar{\eta}_+^2 &= \frac{1}{n_d} \sum_{p \text{ even}} \frac{1}{p!} \left(\bar{\eta}_+^2 \gamma_{m_1 \ldots m_p} \eta_+^1 \right) \gamma^{m_p \ldots m_1} , \\ \eta_+^1 \bar{\eta}_-^2 &= \frac{1}{n_d} \sum_{p \text{ odd}} \frac{1}{p!} \left(\bar{\eta}_-^2 \gamma_{m_1 \ldots m_p} \eta_+^1 \right) \gamma^{m_p \ldots m_1} , \end{aligned} \tag{2.41}$$

where γ_m are $n_d \times n_d$ matrices, and we have used the metric g_{mn} to write the component forms in tangent space indices. Given an expansion of the form (2.41), the Clifford action on $\Phi^\pm$ is

$$X \cdot \Phi^\pm = \frac{1}{2}[x^m \gamma_m, \Phi^\pm]_\mp + \frac{1}{2}[\xi_m \gamma^m, \Phi^\pm]_\pm \,, \tag{2.42}$$

where $\Phi^\pm$ are defined in (2.44).

Note, however, that the forms (2.41) are neither twisted with $\mathrm{d}\Lambda_{\alpha\beta}$, as in (2.29), nor transform with the additional factor of $|\det a|^{1/2}$ under $GL(d)$. If we use the short-hand that $\eta^1_+ \bar{\eta}^2_\pm$ represent the corresponding sums of forms as in (2.41), naively we find that the decomposition of $\Phi^\pm$ under $GL(d)$ is related to the bispinor by

$$\Phi^+ = (\det g)^{-1/4}\, \mathrm{e}^{-B} \eta^1_+ \bar{\eta}^2_+ \,, \qquad \Phi^- = (\det g)^{-1/4}\, \mathrm{e}^{-B} \eta^1_+ \bar{\eta}^2_- \,. \tag{2.43}$$

However, this identifies $O(d,d)$ spinors as sections of (2.33), which precludes the use of the exterior derivative. Introducing the line bundle L we can take $\Phi^\pm$ to be sections of (2.34), and instead have

$$\Phi^+ = \mathrm{e}^{-\phi} \mathrm{e}^{-B} \eta^1_+ \bar{\eta}^2_+, \qquad \Phi^- = \mathrm{e}^{-\phi} \mathrm{e}^{-B} \eta^1_+ \bar{\eta}^2_- \,, \tag{2.44}$$

where $\mathrm{e}^{-\phi}$ is some section of L. By construction

$$\mathrm{e}^{2\phi}/\sqrt{\det g} \tag{2.45}$$

is invariant under $O(d,d)$. This is precisely the way the ten-dimensional dilaton transforms. Thus we see that the dilaton appears very naturally in generalized geometry: together with the generalized metric $\mathcal{H}$, encoding g and B, *the dilaton defines the isomorphism between* $S^\pm(E)$ *and* $\Lambda^{even/odd} T^* M$.

Finally we note that an $O(d,d)$ spinor is said to be pure if it is annihilated by half of the gamma matrices (or equivalently if its annihilator is a maximally isotropic subspace of E). Any pure spinor can be represented as a wedge product of an exponentiated complex two-form with a complex k-form. The degree k is called type of the pure spinor, and, when the latter is closed, it serves as a convenient way of characterizing the geometry.

A pure spinor defines an $\mathrm{SU}(d,d)$ structure on E. A further reduction of the structure group to $\mathrm{SU}(d) \times \mathrm{SU}(d)$ is given by the existence of a pair of compatible pure spinors. Two pure spinors are said to be compatible when they have $d/2$ common annihilators. By construction, the spinors (2.44) are pure and also compatible.

3 T-duality and SU(3) × SU(3)-structures

In this section we would like to address the question of how T-duality acts on backgrounds with $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure. Such geometries describe string compactifications leading to $\mathcal{N} = 2$ effective theories in four dimensions and can be defined by a pair of $O(6,6)$ spinors. We shall give explicit expressions for the new $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure in two natural cases. First we give the transformation of the structure when the original manifold is a T^3 fibration and we perform three T-dualities, that is the map to the mirror configuration. Then we consider the simpler case where the original manifold is a T^2 fibration and we perform a

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pair of T-dualities. In each case, we start with a given SU(3) structure with non-trivial H-flux. We shall see in particular that T-duality can change the type of structure.

Furthermore, in some cases we will find that naively the structure is ill-defined. We discuss this feature in detail for the T^2-fibrations, and argue that it arises precisely when the dual background is non-geometrical. The analysis is entirely consistent with the original discussions of non-geometry for such fibrations [3, 14]. Here we focus our attention on the transformation of the additional SU(3) $\times$ SU(3) structure.

Finally, we will also show that T-duality maps supersymmetric SU(3) $\times$ SU(3) backgrounds to supersymmetric SU(3) $\times$ SU(3) backgrounds. This requires that the Lie derivative along the T-duality direction of the pair of $O(6,6)$ spinors defining the geometry vanishes.

The section begins with a general discussion of T-duality in the context of generalized geometry. This leads to a simple expression for the action of T-duality on $O(d,d)$ spinors which are the defining objects for SU(3) $\times$ SU(3) structures. We then review the relation between SU(3) $\times$ SU(3) structures and supergravity backgrounds, before turning to considering T-duality on them. We conclude with the analysis of T-duality on supersymmetric backgrounds.

3.1 Generalized Lie derivative, generalized Killing vectors and T duality

In string theory T-duality is a non-local transformation. However, at the level of supergravity, there is a corresponding transformation, given by the Buscher rules [23], which can be viewed as a local transformation of the supergravity fields, taking solutions to solutions. In this section we discuss how this local T-duality acts on the generalized structure. We will see that formally it is simply a $O(d,d)$ gauge transformation on E.

Buscher rules apply when one has a supergravity background that admits a Killing vector field v satisfying[3]

$$\mathcal{L}_v g = \mathcal{L}_v H = 0\,. \tag{3.1}$$

The condition $\mathcal{L}_v H = 0$ implies that locally one can make a gauge transformation, $B' = B + \mathrm{d}\zeta'$, such that $\mathcal{L}_v B' = 0$ or, equivalently, $\mathcal{L}_v B - \mathrm{d}\zeta = 0$, where $\zeta = -i_v \mathrm{d}\zeta' + \mathrm{d}f$. Buscher rules are then applied to the gauge transformed background (g, B') and generate a new background $(\tilde{g}, \tilde{B})$. Thus, in a generic gauge we require

$$\begin{aligned} \mathcal{L}_v g &= 0\,, \\ \mathcal{L}_v B - \mathrm{d}\zeta &= 0\,, \end{aligned} \tag{3.2}$$

so that to define the T-duality action on the supergravity fields we really need a pair (v,ζ).

From the action (3.2) on B we see that (v,ζ) act as an infinitesimal diffeomorphism generated by v together with a gauge transformation. Writing $V = v + \zeta$, we can define the corresponding action on sections $X = x + \xi$ of E as a sort of "generalized Lie derivative"

$$\mathbb{L}_V X = [v, x]_{\text{Lie}} + (\mathcal{L}_v \xi - i_x \mathrm{d}\zeta)\,, \tag{3.3}$$

[3] If there is also a non-zero Ramond-Ramond flux, one must further require that the Lie derivative of the flux vanishes.

where $[v,x]_{\text{Lie}}$ is the Lie bracket and $\mathcal{L}_v$ is the ordinary Lie derivative. This combination of V and X is actually none other than the Dorfman bracket [18, 24] whose antisymmetrization gives the Courant bracket (2.13). It is the derived bracket for the exterior derivative d.

Note that this action is very natural given the bundle structure (2.1). We naturally identify as equivalent bundles E which are related by diffeomorphisms of the manifold M and gauge transformations which preserve the patching (2.3). Infinitesimally, together these are equivalent to an action of the generalized Lie derivative.

Given this definition of $\mathbb{L}_V$ on generalized vectors, it is then natural to define the generalized Lie derivative of $\mathcal{H}$ by

$$(\mathbb{L}_V\mathcal{H})(X,Y) = \mathbb{L}_V\left[\mathcal{H}(X,Y)\right] - \mathcal{H}(\mathbb{L}_V X, Y) - \mathcal{H}(X, \mathbb{L}_V Y)\,. \tag{3.4}$$

This is in analogy to the construction for a conventional Lie derivative and here, when acting on a scalar function such as $\mathcal{H}(X,Y)$, we define $\mathbb{L}_V f = \mathcal{L}_v f = i_v \mathrm{d}f$. It is then easy to see that

$$\mathbb{L}_V\mathcal{H} = \begin{pmatrix} \begin{matrix}\mathcal{L}_v g - (\mathcal{L}_v B - \mathrm{d}\zeta)g^{-1}B \\ \quad -B(\mathcal{L}_v g^{-1})B - Bg^{-1}(\mathcal{L}_v B - \mathrm{d}\zeta)\end{matrix} & (\mathcal{L}_v B - \mathrm{d}\zeta)g^{-1} + B(\mathcal{L}_v g^{-1}) \\ -g^{-1}(\mathcal{L}_v B - \mathrm{d}\zeta) - (\mathcal{L}_v g^{-1})B & \mathcal{L}_v g^{-1} \end{pmatrix}\,. \tag{3.5}$$

(Note that a similar calculation implies for the $O(d,d)$ metric (2.5) that $\mathbb{L}_V\eta = 0$.) The requirement (3.2) on (g,B) then simply translates into

$$\mathbb{L}_V\mathcal{H} = 0\,, \tag{3.6}$$

or, in other words, that V defines a "generalized Killing vector".

Given a generalized Killing vector V, we can then define the corresponding Buscher duality as follows. First recall that there was really an ambiguity in $V = v + \zeta$, since the generalized Lie derivative only depends on $\mathrm{d}\zeta$ so we can always shift ζ by $\mathrm{d}f$ for an arbitrary function f. Using this freedom we can always normalize V

$$\eta(V,V) = 1\,. \tag{3.7}$$

Concretely, for any vector field v we can introduce a coordinate t such that $v = \partial/\partial t$. In addition, from (3.1), we know we can write $\zeta = -i_v\mathrm{d}\zeta' + \mathrm{d}f$. Setting $f = t$ we have

$$V = \partial/\partial t + (\mathrm{d}t - i_{\partial/\partial t}\mathrm{d}\zeta')\,, \tag{3.8}$$

and hence $\eta(V,V) = 1$. We then construct the $O(d,d)$ element

$$T_V = \mathbb{1} - 2VV^T\eta\,. \tag{3.9}$$

The condition $\eta(V,V) = 1$ implies that $\eta(T_V X, T_V X) = \eta(X,X)$ so $T_V \in O(d,d)$ and, in addition, $T_V^2 = \mathbb{1}$. We can choose local bases on TM and T^*M such that, if $\hat{e}_1 = v = \partial/\partial t$ is the first basis element of TM and its dual one-form $e^1 = \mathrm{d}t$ is the first element for T^*M. Then taking $\zeta' = 0$, the T-duality matrix reads

$$T_{\tilde{e}_1 + e^1} = \begin{pmatrix} \mathbb{1} - m & m \\ m & \mathbb{1} - m \end{pmatrix}\,, \qquad m = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}\,. \tag{3.10}$$

The T-dual generalized metric $\tilde{\mathcal{H}}$ is simply given by

$$\tilde{\mathcal{H}}(X,X) = \mathcal{H}(T_V X, T_V X)\,, \tag{3.11}$$

or $\tilde{\mathcal{H}} = T_V^T \mathcal{H} T_V$. The action on $O(d,d)$ spinors is by an element of $Pin(d,d)$ equal simply to the Clifford action of V

$$\tilde{\Phi} = T_V\Phi = i_{\partial/\partial t}\Phi + \zeta \wedge \Phi\,, \tag{3.12}$$

where $\zeta = \mathrm{d}t - i_{\partial/\partial t}\zeta'$.

Note that T-duality is usually defined in the gauge where the NS two-form is given by $B' = B + \mathrm{d}\zeta'$ and hence satisfies $\mathcal{L}_v B' = 0$. Here we see that T_V can be written as

$$T_V = \mathrm{e}^{\mathrm{d}\zeta'} \cdot T_{V_0} \cdot \mathrm{e}^{-\mathrm{d}\zeta'}, \tag{3.13}$$

where $V_0 = \partial/\partial t + \mathrm{d}t$. Thus the action of T_V is to first make a gauge transformation on $\mathcal{H}$ to set the NS two form to B', and then act by conventional T-duality. Note also that, as always for Buscher duality, the choice of coordinate t used to write $v = \partial/\partial t$ is not unique. However, the effect after T-duality is simply an additional gauge transformation.

3.2 SU(3) × SU(3) structures and supergravity

In type II supergravities compactified on a six-manifold M, the two supersymmetry parameters decompose into two chiral Spin(6) spinors transforming under the Spin(6) groups associated with C_+ and C_-, respectively. When considering either a supersymmetric background, or a background leading to a low-energy supersymmetric effective action (such as a Calabi-Yau manifold with non-zero fluxes), the supersymmetry picks out a particular pair of globally defined, nowhere vanishing spinors (η^1_+, η^2_+).

Since Spin(6) $\simeq$ SU(4), a single spinor η_+ is invariant under an SU(3) subgroup of Spin(6). If η_+ is globally defined and nowhere vanishing, it defines an SU(3) structure. This is a topological restriction: the tangent bundle TM is patched using only SU(3) transformations. It is equivalent to the existence of a pair of globally defined, nowhere vanishing forms $J \in \Lambda^2 T^*M$ and $\Omega \in \Lambda^3 T^*_{\mathbb{C}}M$. Thus the pair of spinors (η^1_+, η^2_+) defines a pair of SU(3) structures. More precisely they are invariant under an SU(3) × SU(3) subgroup of $O(6,6)$, and we say they define an SU(3) × SU(3) structure. Note that the common subgroup preserving both η^1_+ and η^2_+ is generically SU(2), though at points where they are parallel it becomes SU(3); in this sense η^1_+ and η^2_+ define a "local" SU(2) structure.

Thus we see that the SU(3) × SU(3) structure can be defined in a number of equivalent ways:

(a) the generalized metric $\mathcal{H}$ (defining g and B) together with the pair (η^1_+, η^2_+);

(b) the two pairs of SU(3) structures (J^+, Ω^+) and (J^-, Ω^-) together with B;

(c) the (local) SU(2) structure, together with a complex scalar $\bar{\eta}^1_+\eta^2_+$ and B;

(d) a pair of complex generalized spinors $\Phi^\pm \in S^\pm_{\mathbb{C}}(E)$.

The relations between these various descriptions are as follows. First we fix the normalization of the spinors: $\bar{\eta}^1_+\eta^1_+ = \bar{\eta}^2_+\eta^2_+ = 1$. The two SU(3) structures are defined as

$$J^+_{mn} = -\, i\bar{\eta}^1_+\gamma_{mn}\eta^1_+, \qquad J^-_{mn} = -\, i\bar{\eta}^2_+\gamma_{mn}\eta^2_+\,, \tag{3.14}$$

$$\Omega^+_{mnp} = -\, i\bar{\eta}^1_-\gamma_{mnp}\eta^1_+, \qquad \Omega^-_{mnp} = -\, i\bar{\eta}^2_-\gamma_{mnp}\eta^2_+\,. \tag{3.15}$$

Here and in all the following definitions γ_m are Spin(6) gamma matrices and $\gamma_{(7)} = -i\gamma_1\ldots\gamma_6$. These two SU(3) structures are defined on C_+ and C_-, respectively. As such they can be always written in a standard form in terms of the vielbeins $e_\pm$

$$\begin{aligned} J_\pm &= e^1_\pm\wedge e^4_\pm + e^2_\pm\wedge e^5_\pm + e^3_\pm\wedge e^6_\pm\,,\\ \Omega_\pm &= (e^1_\pm + \mathrm{i}e^4_\pm)\wedge(e^2_\pm + \mathrm{i}e^5_\pm)\wedge(e^3_\pm + \mathrm{i}e^6_\pm)\,. \end{aligned} \tag{3.16}$$

Locally the two SU(3) structures define an SU(2) structure. The latter is defined by a complex one form $z = v + \mathrm{i}v'$, and a triplet of real two-forms (J_1, J_2, J_3), or, equivalently, a real two-form j and a complex two-form ω [25]. One can then always express the two SU(3)-structures in terms of the SU(2) objects, though the decomposition is not unique, since it depends on the different choices of j within the triplet (J_1, J_2, J_3). Here we will use a decomposition where j is naturally associated to (J_+, Ω_+) and $\eta^2_+ = k_\| \eta^1_+ + k_\perp(v + iv')_m\gamma^m\eta^1_-$. This gives

$$\begin{aligned} J^+ &= j - \frac{\mathrm{i}}{2}\, z\wedge\bar{z}\,, \qquad & J^- &= (|k_\||^2 - |k_\perp|^2)J^+ + \mathrm{Re}(\bar{k}_\| k_\perp\bar{\omega}) - 4\mathrm{i}|k_\perp|^2\, z\wedge\bar{z}\,,\\ \Omega^+ &= z\wedge\omega\,, & \Omega^- &= k_\|^2\Omega^+ - k_\perp^2\bar{\omega}\wedge z - 4k_\perp k_\| j\wedge z\,. \end{aligned} \tag{3.17}$$

To define the pure spinors we must decompose under the two Spin(6) subgroups of Spin(6, 6). We can realize the $O(6,6)$ gamma matrices as

$$\Gamma^+_m = \gamma_m\otimes\mathbb{1} \qquad \Gamma^-_m = \gamma_{(7)}\otimes\gamma_m\,. \tag{3.18}$$

Here we are implicitly assuming that $e_+ = e_-$. One can use this decomposition to write the $O(6)\times O(6)$ spinors as Spin(6) bispinors. For example, if η^1_+ and η^2_+ are chiral spinors of the first and second Spin(6) group, respectively, we can write

$$\Phi^+ = e^{-\phi-B}\eta^1_+\bar{\eta}^2_+ \in S^+(E), \qquad \Phi^- = e^{-\phi-B}\eta^1_+\bar{\eta}^2_- \in S^-(E). \tag{3.19}$$

Explicitly the two pure spinors read

$$\Phi^+ = e^{-\phi-B+\frac{1}{2}z\wedge\bar{z}}\,(\bar{k}_\| e^{-ij} - \mathrm{i}\bar{k}_\perp\omega)\,, \tag{3.20}$$

$$\Phi^- = e^{-\phi-B}\, z\,(k_\perp e^{-ij} + \mathrm{i}\, k_\|\omega)\,. \tag{3.21}$$

3.3 Examples

In this section we first determine the structure of the mirror of a generic manifold with a T^3 fibration, by doing T-dualities along the T^3 fiber. In particular we construct explicitly the resulting mirror local SU(2) structure. Mirror symmetry transformations of pure spinors

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for SU(3) and SU(2) structure were also studied in [21], by doing Fourier-Mukai transforms of the pure spinors.

As discussed in the previous section, the patching of the B-field, (2.14), induces the patching (2.29) on the spinors so that $e^{-B}\Phi$ is globally well defined on E. It is well known that, under a single T-duality, the components of the B-field with no legs along the T-dualized directions stay unchanged, while those with one leg are exchanged with the connection along the T-dualized fiber [11]. In that case, spinors are still globally well-defined. Under a second T-duality, however, if the original B-field has a component with both legs along the T-dualized directions, there is no connective structure allowing to define objects globally.

In the rest of this section we focus on the T-duality action for the latter case. We will illustrate this with two simple toroidal examples, where the B-field is purely along the two directions to be dualized.

All the calculations we perform in this section are local. We come back to global issues in section 5.

3.3.1 Mirror symmetry on an T^3-fibered manifold with H-flux

Consider the case of a manifold with a T^3-fibration and generic B field. We assume there is an SU(3) structure such that T-duality on the T^3-fibration corresponds to mirror symmetry, that is that the T^3 fibers are special-Lagrangian. We then act by T-duality and ask what is the structure of the mirror compactification. For the case of no B-field with two legs along the fiber, this computation was done in [26], where it was found that Φ^+ and Φ^- are exchanged under mirror symmetry. Here, with generic B, we find that the new structure is SU(3) $\times$ SU(3) rather than SU(3).

We use the same notation as in [26], except that we denote the coordinates of T^3 fibration by (y^1, y^2, y^3), and those of the base by (x^1, x^2, x^3). The metric and B-field are

$$
\begin{aligned}
ds^2 &= g_{ij} dx^i dx^j + h_{\alpha\beta}\eta^\alpha \eta^\beta \,, \\
B_2 &= \frac{1}{2} B^{(0)}_{ij} dx^i \wedge dx^j + \frac{1}{2} B^{(1)}_{i\alpha} dx^i \wedge (dy^\alpha + \eta^\alpha) + \frac{1}{2} B^{(2)}_{\alpha\beta} \eta^\alpha \wedge \eta^\beta \,,
\end{aligned}
\tag{3.22}
$$

where $\eta^\alpha \equiv dy^\alpha + \lambda^\alpha_i dx^i$ and the superindex on B indicates the number of legs along the fiber. The vielbein is $(e^{a'}_i dx^i, e^a_\alpha \eta^\alpha)$, where $a, a' = 1, 2, 3$ are fiber and base orthonormal indices, respectively

$$
\delta_{a'b'} e^{a'}_i e^{b'}_j = g_{ij} \,, \quad \delta_{ab} e^a_\alpha e^b_\beta = h_{\alpha\beta} \,. \tag{3.23}
$$

The holomorphic vielbeins are

$$
Z^a = e^a_\alpha \, \eta^\alpha + \mathrm{i}\, \delta^a_{a'} \, e^{a'}_i \, dx^i \,. \tag{3.24}
$$

The original SU(3) structure is given by the pure spinors[4]

$$
\begin{aligned}
\Phi^+ &= e^{-\phi - B - iJ} \,, & J &= \frac{\mathrm{i}}{2} Z^a \bar{Z}^{\bar{a}} \\
\Phi^- &= e^{-\phi - B} \Omega \,, & \Omega &= \frac{1}{6} \epsilon_{abc} Z^a \wedge Z^b \wedge Z^c \,.
\end{aligned}
\tag{3.25}
$$

[4]For a generic B-field, $B \wedge \Omega \neq 0$ so strictly speaking the original structure is not SU(3).

The three T-dualities on the fiber are generated by the generalized vectors $V_\alpha = \partial/\partial y^\alpha + dy^\alpha$. Writing $T = T_{V_1}T_{V_2}T_{V_3}$, we have $T\Phi^+ = \tilde{\Phi}^-$, $T\Phi^- = \tilde{\Phi}^+$, where $\tilde{\Phi}^-, \tilde{\Phi}^+$ can be written in the form given in (3.20), (3.21) with

$$
\begin{aligned}
\tilde{z} =& \frac{1}{|B^{(2)}|}\epsilon_{abc}B^{(2)}_{bc}\tilde{Z}^a\,, & e^{-\tilde{\phi}} =& e^{-\phi}\sqrt{|h+B^{(2)}|}\,,\\
\tilde{j} =& \frac{\mathrm{i}}{2}\tilde{Z}^a\bar{\tilde{Z}}^a - \frac{\mathrm{i}}{2}z\wedge\bar{z}\,, & k_\perp =& \mathrm{i}|B^{(2)}|\frac{\sqrt{h}}{\sqrt{|h+B^{(2)}|}}\,,\\
\tilde{\omega} =& -\frac{1}{|B^{(2)}|}B^{(2)}_{ab}\tilde{Z}^a\tilde{Z}^b\,, & k_\parallel =& \frac{\sqrt{h}}{\sqrt{|h+B^{(2)}|}}\,,
\end{aligned}
\tag{3.26}
$$

where $|B^{(2)}| = \sqrt{B^{(2)}_{ab}B^{(2)}_{ab}}$, and the dual holomorphic coordinates are

$$
\tilde{Z}^a = \tilde{e}^a_\alpha\tilde{\eta}^\alpha + \mathrm{i}\delta^a_{a'}\tilde{e}^{a'}_i\,dx^i\,, \qquad \tilde{\eta}^\alpha = dy^\alpha + \tilde{\lambda}^\alpha_i dx^i\,. \tag{3.27}
$$

The dual (plus) vielbeins for the dual metric and the connection are related to the original ones by

$$
\tilde{e}^{a'}_i = e^{a'}_i\,, \qquad \tilde{e}^a_\alpha = e^a_\beta\left((h+B^{(2)})^{-1}\right)^{\beta\alpha}\,, \qquad \tilde{\lambda}^\alpha_i = B^{(1)}_{i\alpha} \tag{3.28}
$$

The dual B-field is

$$
\tilde{B}^{(0)} = B^{(0)}\,, \tilde{B}^{(1)}_{i\alpha} = \lambda^\alpha_i\,, \tilde{B}^{(2)}_{\alpha\beta} = -\left((h+B^{(2)})^{-1}\right)^{\alpha\lambda}B_{\lambda\rho}\left((h-B^{(2)})^{-1}\right)^{\rho\beta}\,, \tag{3.29}
$$

as expected from Buscher rules. Note that in orthonormal indices $\tilde{B}^{(2)}_{ab} = -B^{(2)}_{ab}$.

In the limit of vanishing $B^{(2)}$, we recover the results of [26], namely Φ^+ and Φ^- get exchanged under T-duality (if we write them in terms of dual vielbeins), and define a good mirror SU(3) structure. For nonzero $B^{(2)}$, we get a mirror SU(3) $\times$ SU(3) structure that can be defined in patches, but does not appear to make sense globally. In the following we will focus on this issue in more detail in the slightly simpler case of a pair of T-dualities with B-field only on the T^2 fibres.

3.3.2 Two T-dualities on T^2-fibration with H-flux

We now consider the simpler example of a T^2-fibered manifold with SU(3) structure defined by

$$
\begin{aligned}
\Phi^+ =& e^{-\phi-B-\mathrm{i}J}\,, & J =& e^1\wedge e^4 + e^2\wedge e^5 + e^3\wedge e^6\\
\Phi^- =& e^{-\phi-B}\Omega\,, & \Omega =& (e^1+\mathrm{i}e^4)\wedge(e^2+\mathrm{i}e^5)\wedge(e^3+\mathrm{i}e^6)\,,
\end{aligned}
\tag{3.30}
$$

where e^i are a set of vielbeins. We will also assume there is a B-field along the fibre only. We will further assume that the fibration is trivial, implying we can introduce coordinates such that $e^i = r_i dx^i$ etc for the fibered directions. It would be straightforward to include a non-trivial fibration but it is well known that this is T-dual to a non-trivial B-field, so we instead consider the latter.

We will consider two distinct cases, where the T^2-fibration lies along e^1 and e^4 and e^2 and e^3, respectively. These two cases are inequivalent with respect to the SU(3) structure.

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Not type changing. We consider first the case where the T^2 fibration lies in two directions, e^1 and e^4, which are paired by the complex structure. The B-field on the T^2-fiber can be written as

$$B = \frac{b}{r_1 r_4} e^1 \wedge e^4 = b\, dx^1 \wedge dx^4 \, . \tag{3.31}$$

The factor b in the B-field can be a function of the base. For instance if the base was T^4, that is we compactify by identifying $x^i \sim x^i + 1$, we could take for example $b = hx^6$, corresponding to a flux $H_{146} = h$ (in coordinate indices).

We now perform two T-dualities along ∂_1, ∂_4, that is, in the notation of section 3.1, using the two generalized vectors $V_1 = \partial_1 + dx^1$ and $V_4 = \partial_4 + dx^4$. We obtain again an SU(3) structure of the form

$$\tilde{\Phi}^+ = e^{\mathrm{i}\theta_+} e^{-\tilde{\phi} - \tilde{B} - \mathrm{i}\tilde{J}} \, , \quad \tilde{J} = \frac{e^1 \wedge e^4}{b^2 + r_1^2 r_4^2} + e^2 \wedge e^5 + e^3 \wedge e^6 \, , \tag{3.32}$$

$$\tilde{\Omega} = \frac{b - \mathrm{i} r_1 r_4}{b^2 + r_1^2 r_4^2} \left(\frac{r_4}{r_1} e^1 + \mathrm{i}\frac{r_1}{r_4} e^4\right) \wedge (e^2 + \mathrm{i}e^5) \wedge (e^3 + \mathrm{i}e^6) \, , \tag{3.33}$$

$$\tilde{\Phi}^- = e^{\mathrm{i}\theta_- \tilde{\phi} - \tilde{B}} \tilde{\Omega} \, , \quad \tilde{B} = -\frac{b}{r_1 r_4 (b^2 + r_1^2 r_4^2)} \, e^1 \wedge e^4 \, , \tag{3.34}$$

$$e^{-\tilde{\phi}} = e^{-\phi} \sqrt{b^2 + r_1^2 r_4^2} \, , \tag{3.35}$$

$$\tan\theta_\pm = \mp \frac{r_1 r_4}{b} \, . \tag{3.36}$$

where $\tilde{\Phi}^\pm = T_{V_1} T_{V_4}(\Phi^\pm)$, are the dual pure spinors. This structure can be rewritten using either the set of $\tilde{e}^i_+$ vielbeins or $\tilde{e}^i_-$. These read[5]

$$\tilde{e}^1_\pm = \frac{\pm r_4^2 \, e^1 - b\, \frac{r_1}{r_4} e^4}{|h+B|_f} = r_1 \frac{\pm r_4^2 \, dx^1 - b\, dx^4}{|h+B|_f}$$

$$\tilde{e}^4_\pm = \frac{\pm r_1^2 \, e^4 + b\, \frac{r_4}{r_1} e^1}{|h+B|_f} = r_4 \frac{\pm r_1^2 \, dx^4 + b\, dx^1}{|h+B|_f} \, , \tag{3.37}$$

$$\tilde{e}^i_\pm = e^i \, , \ i \neq 1, 4 \, , \tag{3.38}$$

where $|h+B|_f = b^2 + r_1^2 r_4^2$ is the determinant of the matrix $h+B$ along the fiber directions e^1 and e^4. The structure after T-duality is still SU(3), since $\tilde{J}_+ = \tilde{J}_- = \tilde{J}$, and $\tilde{\Omega}_+ = \tilde{\Omega}_- = \tilde{\Omega}$.

Type changing. We now turn to the next to simplest example. Here we assume the T^2-fibration lies along the e^2 and e^3 directions, i.e. on two directions not paired by the

[5] The vielbeins are computed by inverting (2.22), where we read a, b from the $O(6,6)$ matrix generating the T-duality action of this example

$$T_{V_1} T_{V_4} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} , \qquad a = \begin{pmatrix} \mathbb{1}_3 - m & 0 \\ 0 & \mathbb{1}_3 - m \end{pmatrix} \qquad b = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} , \quad m \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

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complex structure. We again assume that B lies solely along the fibration so that[6]

$$B = \frac{b}{r_2 r_3} e^2 \wedge e^3 = b\, dx^2 \wedge dx^3 \ . \tag{3.39}$$

Performing two T-dualities generated by $V_2 = \partial_2 + dx^2$ and $V_3 = \partial_3 + dx^3$, we now get a local SU(2) structure on dual space. The structure is defined by the pure spinors $\tilde{\Phi}^+ = T_{V_2}T_{V_3}(\Phi^+)$, $\tilde{\Phi}^- = T_{V_2}T_{V_3}(\Phi^-)$, with $\tilde{\Phi}^+$, $\tilde{\Phi}^-$ given in (3.20), (3.21) and where the SU(2) structure can be written in terms of $\tilde{e}^i_+$

$$\begin{aligned}
\tilde{z} &= -\,\mathrm{i}(\tilde{e}^1_+ + \mathrm{i}\tilde{e}^4_+)\ , & & \\
\tilde{\jmath} &= \tilde{e}^2_+ \wedge \tilde{e}^5_+ + \tilde{e}^3_+ \wedge \tilde{e}^6_+\,, & k_\perp &= \mathrm{i}\frac{r_2 r_3}{\sqrt{b^2 + r_2^2 r_3^2}}\,, \\
\tilde{\omega} &= (\tilde{e}^2_+ + \mathrm{i}\tilde{e}^5_+) \wedge (\tilde{e}^3_+ + \mathrm{i}\tilde{e}^6_+)\,, & k_\parallel &= \frac{b}{\sqrt{b^2 + r_2^2 r_3^2}} \\
\tilde{B} &= -\frac{b}{r_2 r_3}\tilde{e}^2_+ \wedge \tilde{e}^3_+\,, & e^{-\tilde{\phi}} &= e^{-\phi}\sqrt{b^2 + r_2^2 r_3^2}
\end{aligned} \tag{3.40}$$

The T-dual vielbeins are[7]

$$\begin{aligned}
\tilde{e}^2_\pm &= \frac{\pm r_3^2\, e^2 - b\,\frac{r_2}{r_3} e^3}{|h+B|_f} = r_2 \frac{\pm r_3^2\, dx^2 - b\, dx^3}{|h+B|_f} \\
\tilde{e}^3_\pm &= \frac{\pm r_2^2\, e^3 + b\,\frac{r_3}{r_2} e^2}{|h+B|_f} = r_3 \frac{\pm r_2^2\, dx^3 + b\, dx^2}{|h+B|_f}\,,
\end{aligned} \tag{3.41}$$

$$\tilde{e}^i_\pm = e^i\,,\ \ i \neq 2,3\,, \tag{3.42}$$

and $|h+B|_f = b^2 + r_2^2 r_3^2$. The T-dual structure is an SU(2) since, unlike the case in the previous example, there are relative signs between $\tilde{J}_+$ and $\tilde{J}_-$:

$$\begin{aligned}
\tilde{J}_\pm &= \tilde{e}^1_\pm \wedge \tilde{e}^4_\pm + \tilde{e}^2_\pm \wedge \tilde{e}^5_\pm + \tilde{e}^3_\pm \wedge \tilde{e}^6_\pm \\
&= e^1 \wedge e^4 + \frac{1}{|h+B|_f}\left(\pm r_3^2\, e^2 \wedge e^5 \pm r_2^2\, e^3 \wedge e^6 + b\,\frac{r_3}{r_2}\, e^2 \wedge e^6 - b\,\frac{r_2}{r_3}\, e^3 \wedge e^5\right)\,,
\end{aligned} \tag{3.43}$$

and similarly for $\tilde{\Omega}_+$, $\tilde{\Omega}_-$. Because of these relative signs, the SU(3) $\times$ SU(3) structure defined on E reduces to a local SU(2) on TM. For $b = 0$, the T-dual structure would be a "static SU(2)" ($k_\parallel = 0$), in agreement with the examples studied in [27]. The effect of b in this case is to rotate this structure to a "dynamic SU(2)", with $k_\parallel, k_\perp \neq 0$.

---

[6] Strictly speaking the following is not an SU(3) structure, since $B \wedge \Omega \neq 0$. However, we can add for example an $e^5 \wedge e^6$ component to make B proportional to $\mathrm{Re}(z^2 \wedge \bar{z}^3)$. Since we will perform T-dualities in ∂_2, ∂_3, the additional component would play no role, and stay unaffected by the T-duality.

[7] The dual vielbeins are again computed from (2.22) where the $O(6,6)$ matrix generating the T-duality action for this case is

$$T_{V_2}T_{V_3} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\,, \qquad a = \begin{pmatrix} m & 0 \\ 0 & \mathbb{1}_3 \end{pmatrix} \qquad b = \begin{pmatrix} \mathbb{1}_3 - m & 0 \\ 0 & 0 \end{pmatrix}\,,$$

with m taking the same form as in footnote 5.

Relation to non-geometry. All the discussion thus far has really been local: we have essentially used $O(d,d)$ transformations on generalized spinors to map one local supergravity background into another. More generally one is interested in whether these local geometries can really be completed into sensible global string backgrounds. It is well known that performing T-dualities on compact backgrounds with flux can lead to non-geometrical dual backgrounds. Non-geometry is an essentially stringy phenomenon so we cannot expect to see it directly in the supergravity description. In our context this relates to the fact that T-duality does not act locally on the T^2 fibres. Nonetheless we see that our examples do reflect elements of the non-geometry when one simply takes into account that the base of the fibration is compact.

It is well known that a simple way of generating non-geometrical backgrounds is to take the T-dual of a T^2-fibration with non-trivial B-field on the fibre directions. This is precisely the case we have considered in the previous examples.

By construction $\Phi^\pm$ are independent of the fibre directions, as are $\tilde{\Phi}^\pm$. Thus effectively one may ignore the fibre and simply consider the dependence of the pure spinors on the base. If b depends non-trivially on the base, in general, the original pure spinors are only defined on the generalized tangent space E (2.1) twisted by the one-forms $\Lambda_{(\alpha\beta)}$ encoding the non-trivial patching of the B field. Put another way, globally, in the expressions (3.30), the spinors $\Phi^\pm$ are sections of $S^\pm(E)$ while $\Phi_0^+ = \mathrm{e}^{-\mathrm{i}J}$ and $\Phi_0^- = \Omega$ are sections of the untwisted spinor bundles $S^\pm(TM \oplus T^*M)$.

Now consider the T-dual pure spinors. In general we will see that they are not well defined. That is to say, they are not sections of $S^\pm(\tilde{E})$ for some generalized tangent bundle $\tilde{E}$ on the dual space. This is a reflection of the fact that the dual background is non-geometrical. To see explicitly that the spinors are not well defined, note that they can be written in terms of the original ones as a β-transform (2.36)

$$\tilde{\Phi}^\pm = \mp\mathrm{i}\, e^{-\beta}\Phi_0^\pm , \tag{3.44}$$

where the J and Ω defining $\Phi_0^\pm$ take the standard form (3.30) but are evaluated using the basis e^1/r_1^2, e^2, e^3, e^4/r_4^2, e_5, e_6 for the non-type changing example, and e^1, e^2/r_2^2, e^3/r_3^2, e^4, e_5, e_6 in the type-changing case. The bivector β is constructed from B by changing the form indices into vector indices, namely

$$\begin{aligned} &\beta = b\,\partial/\partial x^1 \wedge \partial/\partial x^4 && \text{for non-type-changing}\,,\\ &\beta = b\,\partial/\partial x^2 \wedge \partial/\partial x^3 && \text{for type-changing}\,. \end{aligned} \tag{3.45}$$

Note that this is a completely generic feature of T^2 fibrations. Splitting the TM and T^*M bundles into base and fibre components one can write a generic B-transformation as the matrix

$$e^B = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ B^{(0)} & B^{(1)} & \mathbb{1} & 0 \\ -B^{(1)} & B^{(2)} & 0 & \mathbb{1} \end{pmatrix} , \tag{3.46}$$

where the $B^{(0}$ is the component of B lying solely in the base, $B^{(1)}$ is the component with one leg in the base and one in the fibre and $B^{(2)}$ lies solely in the fibre. If T represents the action of T-duality on the T^2 fibre we have

$$e^B \;\mapsto\; Te^BT^{-1} = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ -B^{(1)} & \mathbb{1} & 0 & B^{(2)} \\ B^{(0)} & 0 & \mathbb{1} & B^{(1)} \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} , \qquad T = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \end{pmatrix} . \tag{3.47}$$

Note that $B^{(0)}$ stays in the same position, i.e. in the T-dual setup is still a B-transform, while $B^{(1)}$ and $B^{(2)}$ change positions. The former plays the role of a $GL(d)$ transformation connecting the base and the fiber, in agreement with (3.28), and the latter becomes a bivector just as in (3.45). We can now easily understand (3.44). In our T^2 examples we took $B^{(0)} = B^{(1)} = 0$. Thus our orginal pure spinors could be written as $e^{B^{(2)}}\Phi_0^\pm$ where $\Phi_0^+ = e^{-\mathrm{i}J}$ and $\Phi_0^- = \Omega$. Then

$$T(e^{-B^{(2)}}\Phi_0^\pm) = Te^{-B^{(2)}}T^{-1}T\Phi_0^\pm = e^{-\beta}\tilde{\Phi}_0^\pm \, . \tag{3.48}$$

in agreement with (3.44).

We now see the basic problem. If the original B-field on the fibres is non-trivial, the dual β-transform will be similarly non-trivial. Put another way, if $\Phi^\pm$ were sections of $S^\pm(E)$ where E is patched over the base by B-transformations along the fibre directions, then $\tilde{\Phi}^\pm$ are sections of some bundle were we must patch by β-transformations along the fibres. However, this is outside the domain of conventional generalized geometry, where, by definition E can only be twisted by B-transforms. Hence $\tilde{\Phi}^\pm$ appear to be not well defined.

Note that in the non-type changing case the problem is even more severe: not even the type of the pure spinors is well-defined since e^β changes it. The problem is not simply that the type depends on the location in the base, but rather that one cannot assign a unique type to the pure spinor at each point in the base. We also note that in both cases the metric defined by the pure spinors is similarly ill-defined, as pointed out for example in [3] from Buscher rules for T-duality [23]. Again, the T-dual structure makes sense locally, but there is no good global description.

One might have considered extending the notion of generalized tangent space to include β-transformations. The notion of such transforms was introduced in [18] and discussed in the physics literature in [28] in the context of supergravity duals of deformations of conformal gauge theory, while their connection to non-geometry was explored in [8, 21]. At first sight, they seem as nice as B-transforms. In order to patch the T-dual bundle, one could try and use the subgroup of $O(d,d)$ built out of $GL(d)$ and the β-transforms defined in (2.11). This would correspond to identification of $T \oplus T^*$ with an extension of T^* by T via β-transform. However, unlike the B-transform extension this can prove problematic. Specifically there are no consistent gluing conditions on the two-fold overlaps that would satisfy cocycle conditions. This can be associated with an obstruction given by the first cohomology of the base $H^1(B,\mathbb{Z})$. We come back to this point in section 5.

3.4 Supersymmetric vacua and T-duality

T-duality is a powerful solution-generating tool for string theory backgrounds. Provided the string background, that is the metric and the fluxes, has isometries, T-duality transformations map consistent string backgrounds into new consistent ones. At the level of supergravity, it maps solutions of the supergravity equations of motion into new solutions. In this section we will show that it also maps $\mathcal{N}=1$ supersymmetric backgrounds into $\mathcal{N}=1$ supersymmetric backgrounds.

The necessary conditions for preserving $\mathcal{N}=1$ supersymmetry can be expressed as the the closure of a pure spinor, and an integrability defect of its compatible partner which is determined by RR fluxes. Clearly, if T-duality connects two supersymmetric backgrounds, they must separately satisfy the pure spinor equations.

The supersymmetry equations for $\mathcal{N}=1$ Minkowski vacua given in terms of the pure spinors were found in [20] and read

$$\mathrm{d}(e^{2A}\Phi_1) = 0\,, \tag{3.49}$$

$$\mathrm{d}(e^{2A}\Phi_2) = e^{2A}\mathrm{d}A\wedge\bar{\Phi}_2 + \frac{i}{16}\left[c_- e^{A}G + \mathrm{i}c_+ e^{3A} *_E G\right]\ . \tag{3.50}$$

Let us explain the various ingredients in these equations. The pure spinors $\Phi_{1,2}$ are those of (2.44). The parity of Φ_1 is the same as that of the RR fluxes, while Φ_2 has the opposite parity. So $\Phi_{1(2)}=\Phi_{+(-)}$ for type IIA, and the opposite for type IIB. $c_\pm$ are real constants that give the relation between the norm of the 10D Killing spinors and the warp factor,[8] $G=e^{-B}F$ are the RR field strengths with Bianchi identity $\mathrm{d}G=0$ in the absence of sources. The Hodge star appearing in (3.49) is acting on sections of $S^\pm(E)$. It is related to the standard one acting on $\Lambda^\bullet T^*$ as $*_E = e^{-B} * \lambda\, e^{B}$ (where λ acts on p-forms by $\lambda G^{(p)} = (-1)^{[p/2]}G^{(p)}$) and it is the chirality operator on C_+, as we will show later.

Note that the $\mathcal{N}=1$ supersymmetry equations are written precisely on the objets that transform nicely under T-duality (the factor e^{2A} does not play any role here, since as it is part of the space-time metric, it does not transform). We want to claim that these equations are invariant under T-duality along a vector v that preserves the background, that is

$$\mathcal{L}_v\Phi_{1,2}=0, \qquad \mathcal{L}_vA=0, \qquad \mathcal{L}_vG=0\,. \tag{3.51}$$

Without loss of generality we can also take the B-field satisfying $\mathcal{L}_vB=0$, so that the generalized Killing vector is $V_0=\partial/\partial t + dt$, with $v=\partial/\partial t$. We will show that if $\Phi_{1,2}$ are a solution to the equations (3.49), (3.50), their T-duals, $\tilde{\Phi}_{1,2}$, solve equations of the same form, with T-dual RR field-strengths.

We first note that

$$\mathrm{d}\tilde{\Phi}_{1,2} = \mathrm{d}(\mathrm{d}t\wedge\Phi_{1,2}) + \mathrm{d}(i_{\partial/\partial t}\Phi_{1,2}) = -\mathrm{d}t\wedge\mathrm{d}\Phi_{1,2} - i_{\partial/\partial t}\mathrm{d}\Phi_{1,2} + \mathcal{L}_v\Phi_{1,2} = -T_V(\mathrm{d}\Phi_{1,2}) \tag{3.52}$$

where we have added and subtracted $i_{\partial/\partial t}\mathrm{d}\Phi_{1,2}$ to build the Lie derivative along v of $\Phi_{1,2}$. Next, one can show that

$$-T_V(\mathrm{d}A\wedge\Phi_{1,2}) = \mathrm{d}A\wedge\tilde{\Phi}_{1,2}\,, \quad -T_V(G)=\tilde{G}\,, \quad -T_V(*_E G) = \tilde{*}_E\tilde{G}\ . \tag{3.53}$$

[8] More precisely, $|\eta^1|^2 = c_+e^A + c_-e^{-A}$, $|\eta^2|^2 = c_+e^A - c_-e^{-A}$. Backgrounds with D-branes and/or orientifold planes require $|\eta^1|=|\eta^2|$ and therefore $c_-=0$.

The first result is straightforward using (3.51), which implies $i_v \mathrm{d}A = 0$. The second one is precisely the T-duality transformation of the RR fields. Indeed, the RR fields are $O(d,d)$ spinors, and therefore transform as Φ, (3.12). The third equality needs a little more thinking. Inserting $T_V^2 = 1$ we get

$$- T_V *_E T_V T_V G = (T_V *_E T_V)\tilde{G} = \tilde{*}_E \tilde{G}\,, \tag{3.54}$$

where in the last equality we have used that $*_E = e^{-B} * \lambda\, e^{B}$ transforms by conjugation. This can be understood by noting that this combination is the chirality operator $\Gamma^+_{(6)}$ on C_+. Indeed, Γ^+_a in (2.37) acts on $Spin(6,6)$ spinors as

$$\begin{aligned}\Gamma^+_a \cdot G &= i_{\hat{e}_{+a}} G + e_{+a} \wedge G - i_{\hat{e}_{+a}} B \wedge G \\ &= \mathrm{e}^{B} (i_{\hat{e}_{+a}} + e_{+a} \wedge\,)\, \mathrm{e}^{-B} G\,,\end{aligned} \tag{3.55}$$

and therefore the chirality operator is

$$\Gamma^+_{(6)} = \frac{1}{6!} \epsilon^{a_1 \ldots a_6} \Gamma^+_{a_1} \ldots \Gamma^+_{a_6} = \frac{1}{6!} \mathrm{e}^{B} \epsilon^{a_1 \ldots a_6} (i_{\hat{e}_{a_1}} + e_{a_1} \wedge\,) \ldots (i_{\hat{e}_{a_6}} + e_{a_6} \wedge\,) \mathrm{e}^{-B}\,, \tag{3.56}$$

where we have omitted the plus signs on e. Acting on a degree p form

$$\begin{aligned}\frac{1}{6!} \epsilon^{a_1 \ldots a_6} (i_{\hat{e}_{a_1}} + e_{a_1} \wedge\,) \ldots (i_{\hat{e}_{a_6}} + e_{a_6} \wedge\,)\, G^{(p)} &= \frac{p!(6-p)!}{6!} \epsilon^{a_1 \ldots a_6} e_{a_1} \wedge \cdots \wedge e_{a_{6-p}} i_{\hat{e}^{a_{6-p+1}}} \ldots i_{\hat{e}^{a_6}} G^{(p)} \\ &= (-)^{[p/2]} * G^{(p)} = *\lambda G^{(p)}\,,\end{aligned} \tag{3.57}$$

which implies $\Gamma^+_{(6)} = e^{B} * \lambda\, e^{-B}$. Changing the sign of B (which is conventional, and could have been taken opposite in (2.37)), this is just $*_E$. Since the chirality operator transforms under $O(d,d)$ by conjugation, we verify the last equality in (3.54).

We conclude that if $\Phi_{1,2}$ are pure spinors of an $\mathcal{N} = 1$ vacuum, their T-duals $\tilde{\Phi}_{1,2}$ are pure spinors of a vacuum with T-dual RR fields.

4 Generalized charges and the Courant bracket

One of the goals of this paper is to see how aspects of non-geometry might be encoded in the language of generalized geometry. In the previous section we saw examples of non-geometry appearing as a result of T-duality on backgrounds with $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structures. Specifically, assuming a torus fibration and restricting to the base of the manifold, we saw that the corresponding pure spinors $\Phi^\pm$ were no longer sections of $S^\pm(E)$. Instead, to be globally defined on the base, one had to patch by elements of $O(d,d)$, namely a β-transform, not contained in G_{geom}.

We now turn to a related problem. It has been argued that at the level of the effective theories non-geometric backgrounds are characterised by certain charges, or non-geometrical fluxes, Q and R. These are the analogues, and T-duals, of the "geometrical fluxes", the H-flux and the structure constants f of twisted torus compactifications. One way these fluxes appear is as structure constants in the $2n$-dimensional Lie algebra of the effective gauged supergravity theory [2, 3]. These can also be derived using world-sheet Hamiltonian methods [29, 30].

Alternatively, they appear to define a sort of generalized derivative operator on sums of forms [5],

$$\mathcal{D} = H\wedge\ + f\cdot\ + Q\cdot\ + R\llcorner\ . \tag{4.1}$$

Here $H \in \Lambda^3 T^*M$, $f \in TM\otimes\Lambda^2 T^*M$, $Q \in \Lambda^2 TM\otimes T^*M$ and $R\in\Lambda^3 TM$, and the action of f and Q on forms is by contraction on vector indices and antisymmetrization on the form indices. To date these charges have only been identified for very specific backgrounds.

In this section we propose a generalized geometrical definition of the generalized charges, as well as the operator $\mathcal{D}$ for generic backgrounds. That such a formulation exists is already suggested by the fact that $\mathcal{D}$ can be interpreted as an operator on generalized spinors, since these are sums of odd or even forms. We shall define the charges using the Courant bracket (2.13) and argue that they can be interpreted as components of a generalized spin connection [32]. A key point is that, as such, they will be gauge dependent, taking different values depending on the particular generalized vielbein one uses. We make the connection to various specific examples, and discuss the global issues in the following section.

4.1 The Lie bracket and the spin connection

In conventional differential geometry the Lie bracket is dual to the exterior derivative in the sense that one can always be defined in terms of the other. In particular, given a form α one has

$$i_{[x,y]}\alpha = \frac{1}{2}\mathrm{d}\left([i_x,i_y]\alpha\right) + i_x\mathrm{d}(i_y\alpha) - i_y\mathrm{d}(i_x\alpha) + \frac{1}{2}[i_x,i_y]\mathrm{d}\alpha\ . \tag{4.2}$$

This relation implies there are two equivalent ways of defining the spin-connection. Given any frame e^a and its inverse $\hat{e}_a$ we can define the objects $f^a{}_{bc}$ in two different ways

$$[\hat{e}_a,\hat{e}_b] = f^c{}_{ab}\hat{e}_c\,, \qquad \Leftrightarrow \qquad \mathrm{d}e^a = -\frac{1}{2}f^a{}_{bc}e^b\wedge e^c\ . \tag{4.3}$$

If e^a are vielbeins for some metric, then the requirement that the Levi-Civita connection is metric compatible and torsion-free implies that we can define the spin connection in terms of $f^a{}_{bc}$ as

$$\omega_{ab} = \frac{1}{2}\left(f_{cab} + f_{acb} - f_{bca}\right)e^c\,, \tag{4.4}$$

where we have raised and lowered frame indices with frame metric δ_{ab}.

4.2 Generalized charges, brackets and a generalized spin connection

The expression for $\omega^a{}_b$ in terms of the Lie bracket, suggests that, in generalized geometry, one can use the Courant bracket (2.13) to define a generalized spin connection Ω [32]. Suppose we have a basis given by the generalized vectors $\mathcal{E}_A$ with $A = 1,\ldots,2d$, and we use the conventions where η and $\mathcal{H}$ take the form (2.23), or equivalently

$$\begin{aligned}\eta &= \frac{1}{2}\left(\mathcal{E}_a\otimes\mathcal{E}^a + \mathcal{E}^a\otimes\mathcal{E}_a\right)\\ \mathcal{H} &= \frac{1}{2}\left(\delta^{ab}\mathcal{E}_a\otimes\mathcal{E}_b + \delta_{ab}\mathcal{E}^a\otimes\mathcal{E}^b\right).\end{aligned} \tag{4.5}$$

Here we have split $\mathcal{E}_A = (\mathcal{E}_a,\mathcal{E}^a)$ with $a = 1,\ldots,d$. In the language of ref. [32] this has given us a split of the generalized tangent space $E = C_0 + C_0^\perp$ spanned by $\mathcal{E}_a$ and $\mathcal{E}^a$

respectively. It requires that the resulting maps from C_0 and $C_0^\perp$ to TM and T^*M are non-degenerate.

In analogy to (4.3) one can then define

$$[\mathcal{E}_A, \mathcal{E}_B] = F^C{}_{AB}\mathcal{E}_C. \tag{4.6}$$

Our claim is that the components of $F^A{}_{BC}$ are the generalized fluxes f, H, Q and R. To see how this might work, let us first consider some special cases. If $e^+ = e^-$, the generalized vielbeins can be written as (2.26) so that

$$\mathcal{E}^a = e^a\,, \qquad \mathcal{E}_a = \hat{e}_a - i_{\hat{e}_a}B\ . \tag{4.7}$$

It is then easy to calculate

$$\begin{aligned}
[\mathcal{E}_a, \mathcal{E}_b] &= f^c{}_{ab}\mathcal{E}_c - H_{abc}\mathcal{E}^c\,, \\
\left[\mathcal{E}_a, \mathcal{E}^b\right] &= -f^b{}_{ac}\mathcal{E}^c\,, \\
\left[\mathcal{E}^a, \mathcal{E}^b\right] &= 0\,,
\end{aligned} \tag{4.8}$$

where $f^a{}_{bc}$ is defined as in (4.3) and

$$H_{abc} = -3(i_{\hat{e}_{[c}}\mathrm{d}B_{ab]} + f^d{}_{[ab}B_{c]d})\,. \tag{4.9}$$

One could also choose a basis based on the β-transform (2.11) where

$$\mathcal{E}^a = \tilde{e}^a + \beta\cdot\tilde{e}^a\,, \qquad \mathcal{E}_a = \hat{\tilde{e}}_a\,, \tag{4.10}$$

and, in order to reproduce the generalized metric (2.16), we have $\tilde{e}^a\tilde{e}^b\delta_{ab} = \tilde{g}$

$$\begin{aligned}
\tilde{g} &= g - Bg^{-1}B\,, \\
\beta &= -\tilde{g}^{-1}Bg^{-1}\ .
\end{aligned} \tag{4.11}$$

One then finds that

$$\begin{aligned}
[\mathcal{E}_a, \mathcal{E}_b] &= f^c{}_{ab}\mathcal{E}_c\,, \\
\left[\mathcal{E}_a, \mathcal{E}^b\right] &= -f^b{}_{ac}\mathcal{E}^c + Q^{bc}{}_a\mathcal{E}_c\,, \\
\left[\mathcal{E}^a, \mathcal{E}^b\right] &= Q^{ab}{}_c\mathcal{E}^c + R^{abc}\mathcal{E}_c\,,
\end{aligned} \tag{4.12}$$

where $f^a{}_{bc}$ is defined as in (4.3) but using $\tilde{e}^a$ and

$$Q^{ab}{}_c = i_{\hat{\tilde{e}}_c}\mathrm{d}\beta^{ab} + \beta^{ad}f^b{}_{cd} - \beta^{bd}f^a{}_{cd}\,, \tag{4.13}$$

and

$$R^{abc} = \beta^{ad}i_{\hat{\tilde{e}}_d}\mathrm{d}\beta^{bc} - \beta^{bd}i_{\hat{\tilde{e}}_d}\mathrm{d}\beta^{ac} + \beta^{ad}\beta^{be}f^c{}_{de}\,, \tag{4.14}$$

where $\beta = \frac{1}{2}\beta^{ab}\hat{\tilde{e}}_a\wedge\hat{\tilde{e}}_b$. Note that the new terms in the algebra only vanish if $\beta = 0$, showing that in contrast to a closed B-transform, a constant β-transform is not an automorphism of the Courant bracket. For special cases though, the contractions of β on f appearing in (4.13) and (4.14) vanish, as we will show in section 5.1.

The general case is as follows. First note that $\mathcal{E}^m_{(0)} = \mathrm{d}x^m$ and $\mathcal{E}_{(0)m} = \partial_m$ is a (local) frame for the $O(d,d)$ metric η, but not in general for $\mathcal{H}$. (Note that in this coordinate frame $F = 0$.) This implies that any given frame $\mathcal{E}$ can always be written as an $O(d,d)$ rotation of $\mathcal{E}_{(0)}$, that is $\mathcal{E} = \mathcal{E}_{(0)}O$ or in components

$$\mathcal{E}^a = \mathcal{A}^a{}_m \mathrm{d}x^m + \mathcal{B}^{am}\partial_m \,, \qquad \mathcal{E}_a = \mathcal{C}_{am}\mathrm{d}x^m + \mathcal{D}_a{}^m\partial_m \,. \tag{4.15}$$

The splitting condition implies that $\mathcal{A}^a{}_m$ and $\mathcal{D}_a{}^m$ are non-degenerate. This then leads to the general algebra

$$\begin{aligned} [\mathcal{E}_a, \mathcal{E}_b] &= f^c{}_{ab}\mathcal{E}_c - H_{abc}\mathcal{E}^c \,, \\ \left[\mathcal{E}_a, \mathcal{E}^b\right] &= -\tilde{f}^b{}_{ac}\mathcal{E}^c + Q^{bc}{}_a\mathcal{E}_c \,, \\ \left[\mathcal{E}^a, \mathcal{E}^b\right] &= \tilde{Q}^{ab}{}_c\mathcal{E}^c + R^{abc}\mathcal{E}_c \,. \end{aligned} \tag{4.16}$$

where the fluxes f, H, Q etc are given in terms of derivatives of $O^A{}_B$.

The commutators (4.8) and (4.12) agree in form with those appearing in the effective gauged supergravity theories. However, there the elements $\mathcal{E}_A$ are symmetry generators rather than generalized vectors. In addition, it is clear that our definition of $F^A{}_{BC}$ is *gauge dependent*. The frame $\mathcal{E}_A$ is not uniquely defined; instead equivalent frames are related by $O(d) \times O(d)$ transformations (2.25). Changing frame thus changes the charges f, H, Q and R. A very explicit example is provided by the two frames (4.7) and (4.10). Both define the same generalized metric, but lead to very different charges (4.8) and (4.12). In fact there is a stronger statement. Locally, one can *always* make an $O(d) \times O(d)$ transformation to go the basis (4.7). Thus it would appear that locally the Q and R charges can always be gauged away. As such it would seem hard, locally, to decide when a given set of charges implies we have a non-geometrical background and when not. We return to these points in the next section.

We want now to make the connection between the $F^A{}_{BC}$ and the generalized derivative $\mathcal{D}$ given in (4.1). Hitchin [31] has noted that, in analogy to the duality between the Lie bracket and the exterior derivative, the Courant bracket is dual to the action (2.30) of exterior derivative on $S^\pm(E)$. Explicitly, if $X \cdot \Phi$ is the Clifford action of $X \in E$ on an spinor Φ, then

$$[X,Y]\cdot\Phi = \frac{1}{2}\mathrm{d}\left[(X\cdot Y - Y\cdot X)\cdot\Phi\right] + X\cdot\mathrm{d}\left(Y\cdot\Phi\right) - Y\cdot\mathrm{d}\left(X\cdot\Phi\right) + \frac{1}{2}\left(X\cdot Y - Y\cdot X\right)\cdot\mathrm{d}\Phi \,. \tag{4.17}$$

This suggests that the charges $F^A{}_{BC}$ can be equally well defined using $O(d,d)$ spinors. To see how this works we need to consider what we mean by a generalized connection. Given a tensor bundle W, the ordinary Levi-Civita connection $\nabla = \partial + \omega$ is differential operator $\nabla : C^\infty(W) \to C^\infty(TM^* \otimes W)$. By analogy [32] a generalized connection is an operator[9]

$$D : C^\infty(W) \to C^\infty(E \otimes W) \,, \tag{4.18}$$

[9]Note that we can use the metric η to identify E and E^*

where W is some vector bundle which carries a representation of $O(d,d)$. Again we can think of D as $D = \partial + \Omega$, where the ordinary derivative ∂ simply gives a term in the T^*M part of E and nothing in the TM part. Thus one defines the derivative D, acting on a generalized vector $X = X^A \mathcal{E}_A$, as

$$DX = \left(\mathrm{d}X^A + \Omega^A{}_B X^B\right) \otimes \mathcal{E}_A \; . \tag{4.19}$$

Given a generalized connection one can then ask if it is compatible with η or with the generalized metric $\mathcal{H}$, that is $D\eta = 0$ or $D\mathcal{H} = 0$, and if it is torsion free in a generalized sense. For instance, in [31, 32] a natural η and $\mathcal{H}$ compatible connection is defined, which is not torsion free. If in particular one has a generalized connection that preserves the metric η, one can define a derivative of Spin(d,d) spinors by using the gamma matrices Γ^A associated to a particular frame $\mathcal{E}_A$, that is

$$D_A \Phi = \partial_A \Phi + \frac{1}{4} \Omega_A{}^{BC} \Gamma_{BC} \Phi \; . \tag{4.20}$$

If we now return to the exterior derivative we recall that it acts as a Dirac operator on the Spin(d,d) spinors, $\mathrm{d} : S^{\pm}(E) \to S^{\mp}(E)$. In the particular basis (2.27), where the generalized vielbein takes the form $(\mathcal{E}^m_{(0)}, \mathcal{E}_{(0)m}) = (\mathrm{d}x^m, \partial_m)$ we can write the exterior derivative in terms of a generalized η-compatible connection D

$$\not{D}\Phi = \Gamma^A D_A \Phi = \hat{\Gamma}^m \partial_m \Phi = \mathrm{d}\Phi \; . \tag{4.21}$$

In this basis the spin-connection Ω vanishes, consistent with the fact that $F^A{}_{BC} = 0$. As we commented above, a general frame, which is also a basis for $\mathcal{H}$, can be written as an $O(d,d)$ rotation of $\mathcal{E}_{(0)}$. In this basis Ω is non-zero and the Dirac operator can be written as

$$\not{D}\Phi = O\mathrm{d}\left(O^{-1}\Phi\right) = \mathrm{d}\Phi + \left(O\mathrm{d}O^{-1}\right)\Phi \tag{4.22}$$

where by construction, we are writing the spinor Φ in a frame associated to $\mathcal{E}_A$, that is, where it can be written as a tensor product of two spinors as in (2.40). Thus, for instance, if $\mathcal{E}_A$ takes the form (2.26), then we write elements of Φ in terms of the e^a basis[10]

$$\Phi = e^{-B} \sum_{n=0}^{d} \frac{1}{n!} \Phi_{a_1 \dots a_n} \, e^{a_1} \wedge \dots e^{e_n} \tag{4.23}$$

and in frame indices

$$(\not{D}\Phi)_{a_1 \dots a_n} = n\partial_{[a_1} \Phi_{a_2 \dots a_n]} + n f^b{}_{[a_1 a_2} \Phi_{|b| a_3 \dots a_n]} - \frac{n!}{3!(n-3)!} H_{[a_1 a_2 a_3} \Phi_{a_4 \dots a_n]} . \tag{4.24}$$

We see the appearance of the generalized fluxes f and H in the definition of $\not{D}$ just as in the definition of $\mathcal{D}$ given in (4.1). In a more general basis one would also generate Q and R terms. This is reflecting the duality between the exterior derivative and the Courant bracket. In summary, we see that the derivative $\mathcal{D}$ is simply the exterior derivative written in a frame adapted to the generalized vielbein $\mathcal{E}_A$.

[10] For simplicity we ignore the subtleties associated to the dilaton here.

5 Global properties, generalized charges and non-geometricity

In the previous section we proposed a generalized geometric expression for the charges f, H, Q and R, which arises from the Courant bracket between generalized vielbeins $\mathcal{E}_A$. This was a purely local notion. Crucially, it is also gauge dependent: changing the frame $\mathcal{E}_A$ changes the charges. A clear example was provided by the two bases (4.7) and (4.10). In fact, locally one can *always* choose the gauge where $\mathcal{E}_A$ takes the form (4.7) for which only the geometrical f and H charges appear. This implies that if Q and R are going to encode non-geometry, we can only see this *globally*: there must be some global obstructions to gauging them away.

In this section we try to address this issue in some particular cases. We will focus on backgrounds which admit a $\mathbb{T}^d$ action. We give first the general analysis and then focus on two known examples which can lead to non-geometry. The advantage of such backgrounds is that the local fibration structure picks out a preferred frame $\mathcal{E}_A$ with respect to which one can define the charges, which allows us to see how non-geometry can be characterized in terms of Courant brackets. Of course, for non-geometrical backgrounds, the $\mathbb{T}^d$ fibration will not patch to form a proper manifold. As such it cannot be described using supergravity. Nonetheless we will see that the twisting of the frame over the base of the fibration can be used to characterize the fact that the background is non-geometrical.

The existence of a preferred frame generically implies an additional structure beyond $O(d,d)$. The extreme case of this are "generalized parallelizable" backgrounds, where, in analogy with conventional parallelizable manifolds, there is a globally preferred frame $\mathcal{E}_A$. We end the section with a brief generic discussion of such backgrounds with additional structure.

5.1 Generalized charges and fibrations

In section 4.2 we showed how the two different choices of bases for the generalized vielbeins, (4.7) and (4.10), give rise to two different algebrae with charges f and H, and Q and R, respectively. Here we consider a particular realisation of these two bases. More precisely we consider a class of manifolds which are $\mathbb{T}^d$ fibrations. The structure of the metric and B-field is the same as in section 3.3.1, but now the dimension of the fibre is d rather than three.

If the metric admits a $\mathbb{T}^d$ action, the generalized vielbeins can be written as

$$\begin{pmatrix} \mathcal{E}^a \\ \mathcal{E}_a \end{pmatrix} = \begin{pmatrix} e^{a'}{}_i & 0 & 0 & 0 \\ \lambda^a{}_i & e^a{}_\alpha & 0 & 0 \\ B_{a'i} & B_{a'\alpha} & \hat{e}_{a'}{}^i & \hat{\lambda}_{a'}{}^\alpha \\ B_{ai} & B_{a\alpha} & 0 & \hat{e}_a{}^\alpha \end{pmatrix} \begin{pmatrix} dx^i \\ dy^\alpha \\ \partial_i \\ \partial_\alpha \end{pmatrix} . \tag{5.1}$$

In order not to clutter the expression above we defined the connections $\lambda^a{}_i = e^a{}_\alpha \lambda^\alpha{}_i$ and $\hat{\lambda}_{a'}{}^\alpha = -\hat{e}_{a'}{}^i \lambda_i{}^\alpha$. Similarly the components of the B-field are

$$B_{a'\alpha} = \hat{e}_{a'}{}^i B_{i\alpha} \qquad B_{a'i} = \hat{e}_{a'}{}^j(-B_{ij} + B_{j\alpha}\lambda^\alpha{}_i - \lambda_j{}^\alpha B_{\alpha i})\,, \tag{5.2}$$

$$B_{a\alpha} = \hat{e}_a{}^\beta B_{\beta\alpha} \qquad B_{ai} = -\hat{e}_a{}^\alpha(B_{\alpha\beta}\lambda^\beta{}_i + B_{\alpha i})\,. \tag{5.3}$$

As we see, the vectors on the base are shifted by the derivatives along the torus due to the nontrivial fibration. It is straightforward to check that the generalized vielbeins (5.1) satisfy the algebra (4.8), where, because of the isometries in the fibre directions, the only non trivial components of f are $f^{a'}{}_{b'c'} = i_{\hat{e}_{b'}} i_{\hat{e}_{c'}} \mathrm{d}e^{a'}$ and $f^{a}{}_{b'c'} = -\hat{e}_{[b'}{}^{i}\hat{e}_{c']}{}^{j}\partial_i \lambda^a_j$.

As we will see in the examples below, there are at least two different ways to obtain the β-transformed basis by $O(d,d)$ transformations. One possibility is to consider the torus fibration with a B-field with components in the fibre direction only, and to apply T-duality along the fibre. Since the torus directions are isometries, this is a perfectly lecit transformation. Alternatively we can set the B-field to zero and perform a β-deformation on the metric respecting the torus action. In both cases the resulting vielbein has the form

$$\begin{pmatrix} \tilde{\mathcal{E}}^a \\ \tilde{\mathcal{E}}_a \end{pmatrix} = \begin{pmatrix} e^{a'}_i & 0 & 0 & 0 \\ \lambda^a_i & 1 & 0 & \beta^{a\alpha} \\ 0 & 0 & \hat{e}_{a'}{}^i & \hat{\lambda}_{a'}{}^{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^i \\ dy^\alpha \\ \partial_i \\ \partial_\alpha \end{pmatrix}, \tag{5.4}$$

with $\beta^{a\alpha} = e^a{}_\beta \beta^{\beta\alpha}$. This generalized vielbein gives the algebra (4.12). Note that the derivatives along the fiber coordinates as well as the contractions of β and $f^a{}_{b'c'}$ vanish. Moreover the algebra (4.12) takes the canonical form, with $R^{abc} = 0$ and the only non-vanishing component of Q-charge being $Q^{ab}{}_{c'} = i_{\hat{\hat{e}}_{c'}} \mathrm{d}\beta^{ab} = \partial_{c'}\beta^{ab}$.

Note that a corollary of the above computation is that, on a manifold that admits a $\mathbb{T}^d$ action, a constant β-transform with components only along the $\mathbb{T}^d$ fibre is a symmetry of the Courant bracket.

On the other side, it is not hard to see that when β lies along the fibers, the R-charge is non-vanishing only if β depends on the torus coordinates. In our context, such a situation can arise when the B-field does not respect the isometries of the background.

As we already discussed, it is always possible to perform a local $O(d) \times O(d)$ transformation, (2.25), which preserves the form of the generalized metric $\mathcal{H}$, (2.16), and maps the β-transformed basis into the usual (B-transformed) basis on E

$$K\,\tilde{\mathcal{E}} = \frac{1}{2}\begin{pmatrix} O_+ + O_- & O_+ - O_- \\ O_+ - O_- & O_+ + O_- \end{pmatrix} \begin{pmatrix} e_{\mathrm{B}} & & & \\ & e_{\mathrm{F}} & & e_{\mathrm{F}}\beta \\ & & \hat{e}^T_{\mathrm{B}} & \\ & & & \hat{e}^T_{\mathrm{F}} \end{pmatrix} = \begin{pmatrix} e_{\mathrm{B}} & & & \\ & \tilde{e}_{\mathrm{F}} & & \\ & & \hat{e}^T_{\mathrm{B}} & \\ & -\hat{\tilde{e}}^T_{\mathrm{F}} B & & \hat{\tilde{e}}^T_{\mathrm{F}} \end{pmatrix}. \tag{5.5}$$

where the explicit expression for the matrices $O_\pm$ is

$$O_+ = \mathbb{I} \qquad\qquad O_- = \begin{pmatrix} 1 & \\ & (\hat{e}^T_{\mathrm{F}} + e_{\mathrm{F}}\beta)(\hat{e}^T_{\mathrm{F}} - e_{\mathrm{F}}\beta)^{-1} \end{pmatrix}. \tag{5.6}$$

Let us examine the global issues associated with such a transformation. As explained earlier (see (2.14)) the B-field is defined only locally. Moreover $B_{\alpha\beta}$ must not be a single-valued function in order for the H-flux to be non-trivial in cohomology. This in turn means that the matrix O and the resulting generalized vielbeins are not single-valued either. As a consequence the transformation in question, while not changing the generalized metric locally, cannot produce a well-defined metric. Put differently, $Q^{ab}{}_{c'}$ and

$H_{c'ab}$ can be deformed into each other by using local diffeomorphisms, provided they are exact. The difference in the vertical components (the position of the a, b indices) is not important here - the obstruction is given by the first cohomology of the base of the torus fibration: when the first cohomology of the base is trivial, there simply do not exist any B_{ab} which are not single-valued. This agrees with the T-duality obstruction derived form the world-sheet perspective [14, 15].

This is a general feature of the algebrae obtained from the generalized vielbeins: the Q charges can be gauged away and the algebra can be smoothly deformed into a conventional one with H and f (4.8), only if the first cohomology of the base is trivial.

5.2 Examples

In this section we illustrate with two basic and well known examples the general discussion above. The first one is probably the simplest and best known example of non-geometric background, namely the T-dual of the three-torus with a B-field along the T-duality directions. In this case the base of the fibration is not simply connected and we will see that the local transformation that should gauge the Q-charges away does not make sense globally.

The second example is the Lunin-Maldacena solution [33]. This is a good geometric background obtained via β-transformation along the directions of the T^2 fiber. In this case we will see that the Q-charges can indeed be gauged away by a good $O(d) \times O(d)$ transformation.

Three torus with H-flux. In this subsection we shall illustrate the construction on the prototypical example of a non-geometric background: the T-dual of the straight three-torus T^3 ($\mathrm{Vol}(T^3) = dx^1 \wedge dx^2 \wedge dx^3$) with a non-trivial NS three-form flux, $H = k dx^1 \wedge dx^2 \wedge dx^3$. As we will see, this is an example of a general parallelizable manifold.

Clearly there is a basis of well-defined vectors $\{\partial_1, \partial_2, \partial_3\}$ and a basis of one-forms dx^1, dx^2, dx^3 on the tangent and the cotangent bundle, respectively. We choose a gauge where $B = kx^1 \mathrm{d}x^2 \wedge \mathrm{d}x^3$. It is not hard to see that a global basis for the sections on E is given by

$$(\mathcal{E}_a, \mathcal{E}^a) = (\partial_1, \partial_2 - kx^1 dx^3, \partial_3 + kx^1 dx^2; dx^1, dx^2, dx^3)\,. \tag{5.7}$$

Note that this is of the standard triangular form (2.26). Calculating the Courant bracket yields the familiar algebra

$$\begin{aligned} [\mathcal{E}_a, \mathcal{E}_b] &= -H_{abc}\mathcal{E}^c\,, \\ \left[\mathcal{E}_a, \mathcal{E}^b\right] &= 0\,, \qquad\qquad \text{where} \quad H_{123} = k\,, \\ \left[\mathcal{E}^a, \mathcal{E}^b\right] &= 0\,. \end{aligned} \tag{5.8}$$

One can now act on the basis by an element of $O(3,3)$ to go to the T-dual configuration. T duality in the direction x^3 amounts to $\partial_3 \leftrightarrow dx^3$. In order for the new basis to be split (that is for the projections from C_0 and $C_0^{\perp}$ to TM and T^*M to be non-degenerate) we have to perform a local $O(d) \times O(d)$ transformation of the same form as the T-duality one. In this case this ends up in a relabeling of the vielbeins. We then arrive at the basis

$$(\tilde{\mathcal{E}}_a, \tilde{\mathcal{E}}^b) = (\partial_1, \partial_2 - kx^1\partial_3, \partial_3; dx^1, dx^2, dx^3 - kx^1 dx^2)\,, \tag{5.9}$$

where we have suppressed the tildes on the dual coordinates in order not to clutter the notation. Again this takes the standard form (2.26). It is well known that the dual background is a twisted torus with zero B-field. This is reflected in the fact that the new basis consists of well-defined sections of T and T^*. Computing the Courant bracket gives simply

$$\begin{aligned}
\left[\tilde{\mathcal{E}}_a, \tilde{\mathcal{E}}_b\right] &= f^c{}_{ab}\tilde{\mathcal{E}}_c\,, \\
\left[\tilde{\mathcal{E}}_a, \tilde{\mathcal{E}}^b\right] &= -f^b{}_{ac}\tilde{\mathcal{E}}^c\,, \qquad \text{where} \quad f^3{}_{12} = k\,, \\
\left[\tilde{\mathcal{E}}^a, \tilde{\mathcal{E}}^b\right] &= 0\,.
\end{aligned} \tag{5.10}$$

where we recognize the nilpotent Heisenberg algebra given by the structure constants $(0, 0, 12)$.

The second T-duality — now in direction x^2 — acts very much the same way and amounts to $\partial_2 \leftrightarrow dx^2$. The new basis (again after some relabeling) is

$$(\tilde{\tilde{\mathcal{E}}}_a, \tilde{\tilde{\mathcal{E}}}^a) = (\partial_1, \partial_2, \partial_3; dx^1, dx^2 + kx^1\partial_3, dx^3 - kx^1\partial_2) \tag{5.11}$$

and yields an algebra

$$\begin{aligned}
\left[\tilde{\tilde{\mathcal{E}}}_a, \tilde{\tilde{\mathcal{E}}}_b\right] &= 0\,, \\
\left[\tilde{\tilde{\mathcal{E}}}_a, \tilde{\tilde{\mathcal{E}}}^b\right] &= -Q^{bc}{}_a\tilde{\tilde{\mathcal{E}}}_c\,, \qquad \text{where} \quad Q^{23}{}_1 = k\,, \\
\left[\tilde{\tilde{\mathcal{E}}}^a, \tilde{\tilde{\mathcal{E}}}^b\right] &= Q^{ab}{}_c\tilde{\tilde{\mathcal{E}}}^c\,.
\end{aligned} \tag{5.12}$$

We now note that the new basis is *not* in the standard form (2.26). In fact, it is not a section of E for any choice of extension $T^*M \to E \to TM$. Rather, it is an extension of TM over T^*M. This is reflected in the fact that the generalized metric $\mathcal{H}$ built from $\tilde{\tilde{\mathcal{E}}}_A$ is not single valued as a function of x^1. We are used to this happening because B is not single valued, but here the monodromy in $\mathcal{H}$ is a β-transformation rather than a B-transformation.

One can, of course, find a local map $O(d) \times O(d)$ map to put the basis (5.11) into the standard form (2.26). Explicitly, in (2.25) one takes

$$O_+ = \mathbb{I} \qquad\qquad O_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Delta^{-2}(1 - k^2x^2) & -\Delta^{-2}2kx^1 \\ 0 & \Delta^{-2}2kx^1 & \Delta^{-2}(1 - k^2x^2) \end{pmatrix}\,, \tag{5.13}$$

where $\Delta = \sqrt{(kx^1)^2 + 1}$. The new basis is then

$$(K\tilde{\tilde{\mathcal{E}}}_a, K\tilde{\tilde{\mathcal{E}}}^a) = (\partial_1, \Delta\partial_2 - \Delta^{-1}kx^1\mathrm{d}x^3, \Delta\partial_3 + \Delta^{-1}kx^1\mathrm{d}x^2; dx^1, \Delta^{-1}dx^2, \Delta^{-1}dx^3)\,. \tag{5.14}$$

However O_- and hence K are clearly not single-valued, since x^1 is periodic. Thus although locally we can gauge the Q in (5.12) away (and replace it with f and H) we cannot do this globally.

This background is the simplest example of non-geometrical compactification, where the T^2 fibres, labeled by x^2 and x^3, are patched by a T-duality as one moves around

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the base S^1, labeled by x^1. As such, it is not a manifold since T-duality does not map points to points on the fibres. It is therefore hard to define in what sense the basis $\tilde{\tilde{\mathcal{E}}}_A$ is global. Nonetheless, the base S^1 is still a conventional manifold, and we can simply imagine restricting everything to this S^1 (or equivalently, ignoring the fact that the fibres are compact). The $\tilde{\tilde{\mathcal{E}}}_A$ are then a global basis for the restricted generalized tangent space over S^1. We also have the restrictions of TM and T^*M. The statement that Q cannot be gauged away then has a well defined meaning in terms of the restrictions, even if we cannot define the full compactification as a manifold.

We have seen that each of the three backgrounds, related by T-duality, are parallelizable in the sense that one can introduce a globally defined basis $\mathcal{E}_A$. The three algebras (5.8), (5.10) and (5.12) are actually equivalent: the only difference is the split of the basis into $\mathcal{E}_a$ and $\mathcal{E}^a$, which is related to how TM and T^*M embed in E. The non-geometry of (5.12) was encoded in the fact that the Q-charge could not be gauged away. Equivalently E, or rather its restriction to the base S^1, could not be viewed as an extension of TM by T^*M. Put another way, its structure group was not in the G_{geom} subgroup of $O(3,3)$.

The Lunin-Maldacena solution. The Lunin-Maldacena solution corresponds to a deformation of $AdS_5 \times S^5$ that was originally obtained by applying a T-duality, a rotation and a further T-duality on a T^2 inside S^5 [33]. $AdS_5 \times S^5$ can be written as a warped product of 4-dimensional Minkowski and the 6-dimensional flat metric. Defining the three complex coordinated on $\mathbb{R}^6$ as $z^i = \mu_i e^{i\phi}$,[11] the 6-dimensional metric can be written as a (trivial) T^3 fibration

$$\mathrm{d}s^2 = \sum_{i=1}^{3} (\mathrm{d}\mu_i)^2 + \mu_i^2 (\mathrm{d}\phi^i)^2 \,. \tag{5.16}$$

As shown in [28, 34], in this notation, the chain of transformations leading to the LM background is equivalent to a β-deformation. In particular one can act on the generalized vielbein with the β-transform (2.11), where

$$\beta = \gamma \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} , \tag{5.17}$$

where γ is the deformation parameter. Explicitely

$$\mathcal{E}' = O\mathcal{E} = \begin{pmatrix} \mathbb{1} & & & \\ & \mathbb{1} & & \beta \\ & & \mathbb{1} & \\ & & & \mathbb{1} \end{pmatrix} \begin{pmatrix} e_B & & & \\ & e_F & & \\ & & \hat{e}_B & \\ & & & \hat{e}_F \end{pmatrix} = \begin{pmatrix} e_B & & & \\ & e_F & & e_F\beta \\ & & \hat{e}_B & \\ & & & \hat{e}_F \end{pmatrix} , \tag{5.18}$$

[11] The coordinates μ_i are defined in terms of angles:

$$\mu_1 = \cos\alpha, \qquad \mu_2 = \sin\alpha\cos\theta, \qquad \mu_3 = \sin\alpha\sin\theta, \tag{5.15}$$

where $e_B = e^{a'} = \delta^{a'}{}_i \mathrm{d}\mu_i$ and $e_F = e^a = \delta^a{}_i \mu_i \mathrm{d}\phi^i$ are the vielbeins of the flat metric (5.16) and

$$e_F \beta = \gamma \begin{pmatrix} 0 & \mu_1 & -\mu_1 \\ -\mu_2 & 0 & \mu_2 \\ \mu_3 & -\mu_3 & 0 \end{pmatrix} . \tag{5.19}$$

From the generalized metric $\mathcal{H}$ it is easy to see that the new metric and B-field are indeed those of the LM solution

$$\begin{aligned} \mathrm{d}s^2 &= \sum_{i=1}^{3} (\mathrm{d}\mu_i + G \mathrm{d}\phi^i)^2 + \gamma^2 G (\mu_1\mu_2\mu_3)^2 (\sum_{i=1}^{3} \mathrm{d}\phi^i)^2 \,, && (5.20) \\ B &= \gamma G [(\mu_1\mu_2)^2 \mathrm{d}\phi^1 \wedge \mathrm{d}\phi^2 + (\mu_2\mu_3)^2 \mathrm{d}\phi^2 \wedge \mathrm{d}\phi^3 + (\mu_3\mu_1)^2 \mathrm{d}\phi^3 \wedge \mathrm{d}\phi^1] \,, && (5.21) \end{aligned}$$

with $G = [1 + \gamma^2((\mu_1\mu_2)^2 + (\mu_1\mu_3)^2 + (\mu_2\mu_3)^2)]^{-1}$.

As in the previous example we can find an $O(d) \times O(d)$ transformation bringing the generalized vielbein to the triangular form (2.26). In this case (5.6) takes the form

$$O_+ = \mathbb{I} \qquad\qquad O_- = \begin{pmatrix} 1 & \\ & G O_-^{\mathrm{F}} \end{pmatrix} \tag{5.22}$$

with

$$O_-^{\mathrm{F}} = \begin{pmatrix} 1 - \gamma^2(\mu_1^2\mu_2^2 - \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2) & 2\gamma\mu_1\mu_2(1+\gamma\mu_3) & 2\gamma\mu_1\mu_3(-1+\gamma\mu_2) \\ 2\gamma\mu_1\mu_2(-1+\gamma\mu_3) & 1 - \gamma^2(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 - \mu_1^2\mu_3^2) & 2\gamma\mu_2\mu_3(1+\gamma\mu_1) \\ 2\gamma\mu_1\mu_3(1+\gamma\mu_2) & 2\gamma\mu_2\mu_3(-1+\gamma\mu_1) & 1 - \gamma^2(\mu_2^2\mu_3^2 + \mu_1^2\mu_3^2 - \mu_1^2\mu_2^2) \end{pmatrix} \tag{5.23}$$

Differently from the previous example, the transformation O_- does not contain any non-single valued function of the base. This is also related to the fact that since we have a simply connected base it is not possible to choose a B-field with two legs along the fibre to be not single valued.

5.3 Generalized parallelizable backgrounds

The simplest way around the gauge-dependence of the charges F is to assume that there is some preferred frame $\mathcal{E}_A$, and to define F as the values in this frame. In the previous examples, such a class of frames was defined by the $\mathbb{T}^d$ fibration structure. In particular, for those based on the three-torus with H-flux, there was actually a fixed globally defined frame. This is an example of a "generalized parallelizable" background. In this section, we would like briefly to address some of the constraints on the generic form of the local geometry of such backgrounds, and in particular ask what charges F can appear. We will also see how T-duality acts on such backgrounds.

Recall that in conventional geometry on a parallelizable manifold there exists a globally defined frame e^a implying the tangent bundle TM is trivial. In addition one can further assume that the manifold admits a metric of the form $g = g_{ab} e^a \otimes e^b$ with g_{ab} constant. (In the mathematics literature this is known as "consistent absolute parallelism" [35, 36].)

Except for the special case of S^7, the manifold is then a Lie group and the functions $f^a{}_{bc}$ are the structure constants. In complete analogy one can define a "generalized parallelizable compactification" where there is now a globally defined frame $\mathcal{E}_A$ of E. We will also assume that the $O(d,d)$ metric takes the form (2.23), which we can also write as

$$\eta = \eta^{AB}\mathcal{E}_A \otimes \mathcal{E}_B\,, \tag{5.24}$$

but drop the requirement that $\mathcal{H}$ takes a particular form. Thus the $\mathcal{E}^A$ are defined up to global $O(d,d)$ transformations. Up to such rotations, there is then a unique set of charges defined by

$$[\mathcal{E}_A, \mathcal{E}_B] = F^C{}_{AB}\mathcal{E}_C\,, \tag{5.25}$$

which are taken to be constant. Again, the notion of "globally defined" becomes unclear when we talk about non-geometrical backgrounds. As it stands we will only assume such a local geometry and corresponding charges. The question of how these might complete into geometrical or non-geometrical backgrounds is not discussed. Note that such backgrounds are somewhat analogous to the general twisted double torus backgrounds discussed for instance in [10, 37]. The difference is that there the algebra is realized in terms of the Lie bracket of vector fields on a "doubled" $2d$-dimensional space. Here we are considering a more restricted example: we use the Courant bracket on generalized vectors on what is locally a conventional d-dimensional space.

The three-torus examples above are each generalized parallelizable manifolds. The algebras of the $\mathcal{E}_A$ are actually isomorphic in each case. It is the split of E into TM and T^*M (and hence of $\mathcal{E}_A$ into $(\mathcal{E}_a, \mathcal{E}^a)$) that gave the different interpretations of the structure constants as corresponding to H, f or Q charge.

Let us see what conditions the existence of the algebra (5.25) realized by the Courant bracket places on the local geometry of the background. The first conditions follow from the fact that we can define the $O(d,d)$ metric $\mathcal{H}$ as in (4.5). From Proposition 3.16 of [18] we see that, since the $F^A{}_{BC}$ are constant, the Courant bracket (5.25) on $\mathcal{E}_A$ satisfies the Jacobi identity and hence defines a Lie algebra $\mathfrak{h}$. Given Proposition 3.18 of [18], we also have

$$\eta_{CD}F^D{}_{AB} + \eta_{BD}F^D{}_{AC} = 0\,. \tag{5.26}$$

This implies that the adjoint representation of the algebra (4.16), where the generators are given by $(T_A)_B{}^C = F^C{}_{AB}$, acts as a sub-algebra $\mathfrak{h} \subset \mathfrak{o}(d,d)$.

Next recall that under the projection $\pi : E \to TM$ the Courant bracket reduces to the Lie bracket

$$\pi([X,Y]) = [\pi(X), \pi(Y)]_{\text{Lie}}\,. \tag{5.27}$$

Writing $v_A = \pi(\mathcal{E}_A)$ this simply states that $[v_A, v_B]_{\text{Lie}} = f^C{}_{AB}v_C$. Thus there is a realization of the algebra $\mathfrak{h}$ in terms of $2d$ vector fields on M, though of course this may be somewhat degenerate since some v_A may vanish identically. Since the $\mathcal{E}_A$ are a basis for E, the v_A must form a basis for TM, that is, there must be at least d non-vanishing v_A at each point p of M. Exponentiating the Lie algebra action into diffeomorphisms we see that M is locally a homogeneous space, with an action of a group H with Lie algebra $\mathfrak{h}$. Let us fix some point $p \in M$. If we identify $X = X^A \mathcal{E}_A$ with constant X^A as elements of the Lie algebra $\mathfrak{h}$ we define the set of vectors X with vanishing $\pi(X)$ at a given point $p \in M$

$$\mathfrak{k}_p = \{X \in \mathfrak{h} : \pi(X)|_p = 0\} \ . \tag{5.28}$$

This must be a d-dimensional subset of $\mathfrak{h}$. Since the Lie bracket of two vector fields that vanish at $p \in M$ must itself vanish at $p \in M$, we see that $\mathfrak{k}_p$ must form a closed subalgebra. Hence we see that locally M must be a coset space. We can write

$$M = H/K \,, \qquad K \subset H \subset O(d,d) \,, \tag{5.29}$$

where H is a $2d$-dimensional group with Lie algebra $\mathfrak{h}$ given by (5.25) and K is a d-dimensional subgroup with Lie algebra isomorphic to $\mathfrak{k}_p$. For a parallelizable manifold, M is (almost always) locally a group manifold. Thus we see, as one might expect, generalized parallelizable compactification appear to be more general.

Let us now turn to the fluxes F. At the point p, generalized vectors $X \in \mathfrak{k}_p$ lie solely in $T^*_p M$. Hence we can locally identify $\mathcal{E}^a$ as a basis for $\mathfrak{k}_p$ and, using the metric η, decompose $\mathfrak{h} = \mathfrak{k}_p \oplus \mathfrak{m}_p$, with $\mathcal{E}_a$ a basis for $\mathfrak{m}_p$. Hence, for any generalized parallelizable compactification, since $\mathfrak{k}_p$ is a closed subalgebra, we see that one cannot arrange all fluxes to be non-zero. In particular, one can always use a global $O(d,d)$ rotation to align the basis $\mathcal{E}_A$ such that $\mathcal{E}^a$ span $\mathfrak{k}_p$ and $\mathcal{E}_A$ span $\mathfrak{m}_p$ and

$$R^{abc} = 0 \ . \tag{5.30}$$

This is in agreement with the T^3 with flux examples discussed above.

In this discussion we have only considered some of the conditions on the generalized parallelizable background that follow from the Courant bracket structure. One would expect additional conditions, such as compatibility with a generalized metric of the form $\mathcal{H} = \mathcal{H}^{AB} \mathcal{E}_A \otimes \mathcal{E}_B$, and probably a curvature condition as in [10]. It would also be interesting to find specific examples where M is indeed locally a coset rather than a group manifold as in the T^3 with H-flux examples.

Let us end this section by discussing the generic action of T-duality on generalized parallelizable backgrounds. Suppose we have a generalized Killing vector V which preserves the parallelizable structure, that is

$$\mathbb{L}_V \mathcal{E}_A = 0 \qquad \forall A \ . \tag{5.31}$$

We can always normalize V such that $\eta(V,V) = 1$ and define the T-duality operator T_V as in (3.9). Using the general relations [18]

$$\begin{aligned} \mathbb{L}_X Y &= [X,Y] + \mathrm{d}\eta(X,Y) \,, \\ i_{\pi(X)} \mathrm{d}\eta(Y,Z) &= \eta(\mathbb{L}_X Y, Z) + \eta(Y, \mathbb{L}_X Z) \,, \\ [X, fY] &= f[X,Y] + (i_{\pi(X)} \mathrm{d}f) Y - \eta(X,Y) \mathrm{d}f \,, \end{aligned} \tag{5.32}$$

and the fact that $\eta(\mathcal{E}_A, \mathcal{E}_B) = \eta_{AB}$ and $\eta(V,V) = 1$ are constant, it is relatively straightforward to show that

$$\begin{aligned} [T_V \mathcal{E}_A, T_V \mathcal{E}_B] &= [\mathcal{E}_A, \mathcal{E}_B] - 2 \left(i_{\pi(\mathcal{E}_A)} \mathrm{d}\eta(V, \mathcal{E}_B) - i_{\pi(\mathcal{E}_B)} \mathrm{d}\eta(V, \mathcal{E}_A) \right) V \\ &= T_V \left[\mathcal{E}_A, \mathcal{E}_B \right] \ . \end{aligned} \tag{5.33}$$

Thus we see that T_V is an automorphism of the generalized parallelizable algebra. A particular example is the fact that the three algebras arising from T-duality of the T^3 with H-flux are all isomorphic. They are of course distinguished by the way one identifies vectors and forms in E.

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