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First-order flow equations for extremal and non-extremal black holes

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Abstract: We derive a general form of first-order flow equations for extremal and non-extremal, static, spherically symmetric black holes in theories with massless scalars and vectors coupled to gravity. By rewriting the action as a sum of squares à la Bogomol'nyi, we identify the function governing the first-order gradient flow, the 'generalised superpotential', which reduces to the 'fake superpotential' for non-supersymmetric extremal black holes and to the central charge for supersymmetric black holes. For theories whose scalar manifold is a symmetric space after a timelike dimensional reduction, we present the condition for the existence of a generalised superpotential. We provide examples to illustrate the formalism in four and five spacetime dimensions.

Keywords: Black Holes in String Theory, Black Holes, Integrable Equations in Physics

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1 Introduction

A distinctive feature of supersymmetric extremal black holes with regular event horizons in theories of gravity coupled to neutral scalar fields and Abelian vector fields is the attractor mechanism [1–4]. Its name derives from the fact that the radial evolution of the scalars

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follows a set of first-order equations, such that near the event horizon the scalars are driven to values determined by the electric and magnetic charges carried by the black hole. For supersymmetric black holes these equations are implied by supersymmetry and constitute a gradient flow on the scalar manifold, governed by the central charge. The attractor mechanism also applies to some extremal non-supersymmetric black holes. This suggests the possibility of non-supersymmetric gradient flows. Indeed, fake superpotentials have been found for some non-supersymmetric extremal black holes [5–7].

Conversely, it has been shown that the existence of first-order flow equations is not necessarily tied to the attractor mechanism, since there exist non-extremal solutions described by such flow equations [8] and non-extremal black holes cannot be attractive (see e.g. [9]). This raises the question of whether one can find a general form of the flow equations which is valid for both extremal and non-extremal black holes, and under what conditions these equations constitute a gradient flow. Having a first-order description at hand for non-extremal solutions might shed light on some open problems concerning the relation between the scalar charges and the entropy of non-extremal black holes [10]. Furthermore, a fake superpotential is the natural candidate for a c-function for non-BPS solutions [6, 11].

In this paper we present the general form of the gradient flow equations valid for extremal and non-extremal, static and spherically symmetric solutions, extending the formalism developed in [5] for extremal solutions. We name the function that determines the gradient flow the 'generalised superpotential' in analogy with the fake supergravity formalism for domain walls [12–18]. In addition, for theories whose moduli space is a symmetric space after a timelike dimensional reduction, we derive the condition for a generalised superpotential to exist. In these cases the black hole equations of motion are explicitly integrable [19–22]. In fact, in the case of extremal, but not necessarily supersymmetric black holes with regular horizons, the procedure proposed in [6] should be sufficient to construct a fake superpotential for symmetric moduli spaces in $\mathcal{N}$-extended supergravities, as long as the fake superpotential can be expressed in terms of duality invariants. However, for general extremal and non-extremal solutions (including those without regular horizons) not much is known. The goal of our work is to fill this gap, providing a uniform description of extremal and non-extremal black holes.

We begin our discussion by recalling the necessary background material (section 2). In particular, we briefly recall what is known about the construction of a black hole effective action and first-order flow equations from the existing literature. We show in section 3 how one can obtain such a one-dimensional effective action with a black hole potential, in an arbitrary number of spacetime dimensions, and how to find the most general first order flow equations from a 'generalised superpotential', assuming that it exists. This is illustrated by an example, the dilatonic black hole, in section 4. Section 5 discusses the question of the existence of a superpotential in arbitrary dimensions. In section 6, we explain how to obtain a free-geodesic form of the effective action by timelike dimensional reduction, for systems whose scalar manifold is a symmetric space after the reduction, and derive from it first-order equations. In sections 7 and 8 we then study this condition for a single-scalar and a multi-scalar example. We end with a discussion of our results and a comparison with the literature on domain walls in section 9.

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2 Prerequisites

2.1 Two forms of the black hole effective action

We will consider static, spherically symmetric black hole solutions in gravity coupled to a number of neutral scalars ϕ^a and vector fields A^I in $D+1$ dimensions,

$$S = \int \mathrm{d}^{D+1}x\sqrt{|g|}\Big(\mathcal{R}_{D+1} - \tfrac{1}{2}G_{ab}\partial_\mu\phi^a\partial^\mu\phi^b - \tfrac{1}{2}\mu_{IJ}F^I_{\mu\nu}F^{J\,\mu\nu}\Big), \tag{2.1}$$

where G_{ab} and μ_{IJ} are functions that depend on the scalar fields ϕ^a, and $F^I_{\mu\nu}$ are Abelian field strengths. Greek indices are raised and lowered with the spacetime metric $g_{\mu\nu}$ and $g = \det g_{\mu\nu}$. For now we leave the dimension unspecified, but note that in the special case $D+1 = 4$ there can be another term in the action of the form, $-\frac{1}{2}\nu_{IJ}F^I_{\mu\nu}(\star F^J)^{\mu\nu}$, where ν_{IJ} also depends on the scalar fields. To keep our discussion as general as possible we shall make no further assumptions about this theory, but the reader should notice that it is of the appropriate form to describe the bosonic sector of ungauged supergravity.

There are two techniques to construct the effective action for such systems. Both are based on the fact that static, spherically symmetric solutions depend only on the radial parameter, so that effectively the problem is one-dimensional. The first technique [4, 23] expresses the Maxwell field strengths in terms of the magnetic and electric charges (the fluxes of F and $\star F$ at spatial infinity) via the respective equations of motion (and Bianchi identities). Consider for example the metric ansatz for a black hole in $D+1 = 4$ dimensions

$$\mathrm{d}s^2 = -\mathrm{e}^{2U(\tau)}\mathrm{d}t^2 + \mathrm{e}^{-2U(\tau)}\gamma_{mn}\mathrm{d}x^m\mathrm{d}x^n\,, \tag{2.2}$$

where $U(\tau)$ is often referred to as the black hole warp factor and depends only on the radial coordinate τ on the spherically symmetric spatial slice with the metric γ_{mn}. The one-dimensional effective action obtained as explained above turns out to be that of a particle subject to an external force field given by the effective black hole potential V:

$$S = \int \mathrm{d}\tau\,\Big(2\dot{U}^2 + \tfrac{1}{2}G_{ab}(\phi)\dot{\phi}^a\dot{\phi}^b + \mathrm{e}^{2U}V(\phi)\Big), \tag{2.3}$$

where a dot means differentiation with respect to the radial parameter τ. The configuration space of this 'fiducial' particle is a direct product $\mathcal{M}\times\mathbb{R}$ where $\mathcal{M}$ is the scalar target space, with metric G_{ab}, and $\mathbb{R}$ represents the warp factor. The 'mass parameters' in the black hole potential V are given by the electric and magnetic charges. Solutions to this action have to obey a constraint, stemming from part of the information in the $D+1$-dimensional Einstein equations that cannot be derived from the effective action.[1] In section 3 we will explain how to use this first method in arbitrary $D+1$ dimensions.

The second technique for constructing a one-dimensional effective action, first described in the $D+1 = 4$ case in [19], is based on the observation that a static solution in $D+1$ dimensions can be dimensionally reduced over time to a Euclidean D-dimensional instanton

[1] This constraint can be found from the effective action if one introduces an 'einbein' corresponding to the reparametrisations of the radial coordinate. This einbein then acts as a Lagrange multiplier that enforces the constraint [24].

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solution. Because of the assumed spherical symmetry, the resulting instanton solutions are carried only by the metric and the scalars in D dimensions. Moreover, since the reduction is performed over a Killing direction, the D-dimensional solutions fully specify the solutions in $D+1$ dimensions. As explained in [19, 22] the equations for the D-dimensional metric decouple and are easily solved. The scalar field equations of motion are found from the following effective one-dimensional action

$$S = \int \mathrm{d}\tau \, \tilde{G}_{ij} \dot{\tilde{\phi}}^i \dot{\tilde{\phi}}^j \,, \tag{2.4}$$

which describes the free geodesic motion of a fiducial particle in an enlarged target space of scalar fields $\tilde{\phi}^i$ that contain the scalar fields ϕ^a of the $(D+1)$-dimensional theory plus axion-type scalar fields arising from the reduced vector potentials. In the remainder of this paper, we will always use the notation $\tilde{G}$ for the moduli space metric in the reduced (Euclidean) gravity theory. Note that in this procedure the vectors (or equivalently, the axions) are not eliminated by their equations of motion. It is these axionic scalars that have the opposite sign for their kinetic term, which causes the metric $\tilde{G}$ to have an indefinite signature. If we were to start in four dimensions and would then integrate out those axions, we would find the other black hole effective action (2.3). This second technique will be explored in section 6. There we will discuss systems for which the moduli space after reduction to D dimensions is a symmetric space and show how to extract the $D+1$ dimensional first order equations.

2.2 Flow equations

The question of which technique (or effective action) is best suited for the given task depends on the theory one considers and on which aspects of black hole solutions one wishes to investigate. For instance, if the scalar target space in the effective action of the second type (2.4) is a symmetric space then the geodesic equations are manifestly integrable and can be used to construct explicit solutions, see for instance [22] for more details. When one is interested in studying supersymmetry and the black hole attractor mechanism, the first approach is more commonly used. For a supersymmetric (BPS) black hole ansatz the first-order Killing spinor equations in $D+1$ dimensions provide an integrated form of the second order equations of motion derived from (2.3), and are of the type

$$\dot{\phi}^a = \pm G^{ab} \partial_b |Z| \,, \tag{2.5}$$

where the function Z has the property that[2]

$$|Z|^2 + 2\tfrac{(D-1)}{(D-2)} G^{ab} \partial_a |Z| \partial_b |Z| = V \,. \tag{2.6}$$

and (when evaluated at infinity) is the (complex) central charge. The set of equations (2.5) is called BPS or *gradient flow equations*, and describes an *attractor flow* if there is an attractive fixed point (that is, when the black hole potential has a minimum). For non-supersymmetric black holes the first-order equations are no longer guaranteed to exist.

[2] Note that in our conventions the gravitational coupling constant κ^2 in the Einstein-Hilbert term $\frac{1}{2\kappa^2}\sqrt{|g|}\mathcal{R}$ is set to $\frac{1}{2}$. This influences the coefficients in formula (2.6).

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Nonetheless, as non-supersymmetric *extremal* black holes can still exhibit attractor behaviour [25–28], it would seem plausible that they admit a first-order description. Indeed, Ceresole and Dall'Agata [5] have shown that it is possible for extremal non-supersymmetric black holes to mimic the BPS equations (2.5) of their supersymmetric counterparts, with the central charge replaced by a suitable 'superpotential' function $W \neq |Z|$ (which is not necessarily a remnant of supersymmetry in one dimension higher [7]). Subsequent work has provided further examples of the hidden structure in non-supersymmetric extremal solutions, see e.g. [6, 10, 29–31] and references therein. Of most direct relevance for this work is [6], where a fake superpotential was presented for $\mathcal{N} > 2$ theories in $D + 1 = 4$, which all possess a symmetric moduli space after timelike reduction.

While non-extremal black holes are of considerable interest, little is known about their possible interpretation as solutions of first-order equations. As non-extremal solutions cannot be attractors even when a regular horizon exists (see [9]), it is perhaps already surprising that some non-extremal solutions can be found from first-order equations derived from a superpotential [32]. Miller et al. [8] later provided the simplest possible example — the non-extremal Reissner-Nordström black hole — by making use of Bogomol'nyi's trick from non-gravitational field theories, namely completing the squares.[3] In these theories one is able to rewrite the energy functional as a strict sum of squares. The energy-minimising solutions are found by solving the first-order Bogomol'nyi equations that result from setting each of the squares to zero, and correspond to the BPS solutions in the supersymmetric completion of the original field theory. It was pointed out in [8] that the coupling to gravity introduces at least one term with a relative minus sign, which would appear to ruin this scheme. However, one can show that the extremal, static, BPS solutions can be found by solving the equivalent set of first-order equations that arise in rewriting the total action in terms of squared expressions. It transpires that the relative minus sign makes the rewriting of the action as a sum of squares non-unique and allows one to introduce a one-parameter deformation. This leads to the non-extremal version of first-order equations, with the deformation parameter measuring the deviation from extremality.

The Bogomol'nyi approach has been generalised to include the non-extremal dilatonic black hole and p-brane solutions, as well as time-dependent (cosmological) solutions in arbitrary dimensions [24] and non-extremal black holes in gauged supergravity [32, 37]. The first-order formalism for time-dependent solutions is of interest as it provides further evidence for hidden structures in cosmologies, as first suggested by the domain wall/cosmology correspondence [14]. The explicit structure of non-extremal flow equations in theories with more complicated scalar matter coupling is not known, although some suggestions were made in [6].

[3] The Bogomol'nyi trick was first applied to self-gravitating solutions in the case of cosmic strings [33]; see also [34, 35] for recent discussions. The same procedure can be applied to time-dependent gravitating solutions [36].

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3 General flow equations in $D+1$ dimensions

Consider the metric describing static, spherically symmetric black hole solutions of the theory described by the action (2.1). The most general form of the spacetime metric consistent with these symmetries is

$$\mathrm{d}s^2_{D+1} = -\mathrm{e}^{2\beta\varphi(\tau)}\mathrm{d}t^2 + \mathrm{e}^{2\alpha\varphi(\tau)}\Big(\mathrm{e}^{2(D-1)A(\tau)}\mathrm{d}\tau^2 + \mathrm{e}^{2A(\tau)}\mathrm{d}\Omega^2_{D-1}\Big), \tag{3.1}$$

where

$$\alpha = -1/\sqrt{2(D-1)(D-2)}\,, \qquad \beta = -(D-2)\alpha\,, \tag{3.2}$$

and the scalars depend solely on the radial coordinate: $\phi^a = \phi^a(\tau)$. In four and five dimensions, a common notation for the black hole warp factor is $U = -\alpha\varphi$, i.e. in the four-dimensional case ($D+1=4$) $U = \varphi/2$, while in five dimensions ($D+1=5$) $U = \varphi/\sqrt{3}$. Following the procedure of [4, 23], described beneath (2.2), we eliminate the vector fields in terms of the charges through their equations of motion and obtain a one-dimensional action of the form

$$S = \int \mathrm{d}\tau\, \Big(\frac{\dot{A}^2 + \mathrm{e}^{2(D-2)A}}{2\alpha^2} - \tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}G_{ab}\dot{\phi}^a\dot{\phi}^b - \mathrm{e}^{2\beta\varphi}V(\phi^a)\Big), \tag{3.3}$$

where a dot denotes a derivative with respect to τ. We use small Latin indices from the beginning of the alphabet $a, b, \ldots$ to label the scalars of the $(D+1)$-dimensional theory and G denotes the moduli space metric in the same theory. This action is supplemented with a Hamiltonian constraint, which states that the radial evolution of the fields happens on a slice of constant total energy

$$(2\alpha^2)^{-1}(\dot{A}^2 - \mathrm{e}^{2(D-2)A}) = \tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}G_{ab}\dot{\phi}^a\dot{\phi}^b - \mathrm{e}^{2\beta\varphi}V(\phi) \equiv E\,. \tag{3.4}$$

The constraint is the remnant of the original $D+1$-dimensional Einstein equations that is not reproduced by the effective action (3.3). The total gravitational energy E contains a charge contribution, such that extremal black holes have vanishing energy ($E=0$) and non-extremal black holes have positive energy ($E>0$).

Generalised superpotential. Let us now assume that there exists a function $Y(\varphi, \phi^a)$, which we call the 'generalised superpotential', such that

$$e^{2\beta\varphi}V(\phi^a) = \tfrac{1}{2}\partial_\varphi Y\partial_\varphi Y + \tfrac{1}{2}\partial_a Y\partial^a Y + \Delta\,, \tag{3.5}$$

where Δ is a constant to be determined later (see eq. (3.10)). The effective action (3.3) can then be written in the following form[4]

$$S = \frac{1}{2}\int \mathrm{d}\tau\, \Big[\frac{1}{\alpha^2}\Big(\dot{A} + \sqrt{\mathrm{e}^{2(D-2)A} + \gamma^2}\Big)^2 - (\dot{\varphi} + \partial_\varphi Y)^2 - (\dot{\phi}^a + \partial^a Y)^2\Big], \tag{3.6}$$

plus a total derivative; γ is a constant.

[4] In fact a minus sign is also possible within the squares, but this choice amounts to a redefinition of τ and Y, so without loss of generality we may choose plus.

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The first-order form of the equations of motion is then obtained by putting the terms within brackets in (3.6) to zero, giving a stationary point of the action. We first note that the solution to the first-order equation for A is independent of the details of the model under consideration:

$$\mathrm{e}^{-(D-2)A} = \gamma^{-1} \sinh[(D-2)\gamma\tau + \delta]\,, \tag{3.7}$$

where δ is an integration constant. The constant γ^2 appearing under the square root must be non-negative to ensure the absence of naked singularities. We then call the remaining equations *generalised flow equations*

$$\dot{\varphi} + \partial_\varphi Y = 0\,, \tag{3.8}$$

$$\dot{\phi}^a + G^{ab}\partial_b Y = 0\,. \tag{3.9}$$

The Hamiltonian constraint (3.4) fixes the constant Δ appearing in (3.5) to be

$$(D-1)(D-2)\gamma^2 = -\Delta = E\,. \tag{3.10}$$

Extremal case. When $\Delta = 0$ (extremality) equation (3.5) implies that $Y(\varphi, \phi^a)$ must factor as

$$Y(\varphi, \phi^a) = \mathrm{e}^{\beta\varphi} W(\phi^a)\,, \tag{3.11}$$

such that the formula for the black hole potential assumes the familiar form

$$V = \tfrac{1}{2}\beta^2 W^2 + \tfrac{1}{2}\partial_a W \partial^a W\,, \tag{3.12}$$

and the flow equations become the known expressions for extremal black hole solutions

$$\dot{\varphi} + \beta \mathrm{e}^{\beta\varphi} W = 0\,, \tag{3.13}$$

$$\dot{\phi}^a + \mathrm{e}^{\beta\varphi}\partial^a W = 0\,. \tag{3.14}$$

This means that the main difference between the flow equations describing extremal and non-extremal solutions is the factorisation property (3.11) of the generalised superpotential $Y(\varphi, \phi^a)$.

The form of the flow equations for non-extremal solutions presented here differs somewhat from the conjecture made in [6], which proposes to preserve the form of the flow equations from the extremal case (3.13), (3.14), but allow W to explicitly depend on τ: $W(\phi, \tau)$. Noting that explicit τ-dependence can locally be rephrased as φ-dependence, with τ then considered as a function of φ, one sees that this is in a similar vein as our proposal. The two are not equivalent, however, as in [6] the dependence of W on τ is of a specific kind, $\partial_\tau W \sim -\gamma^2 \mathrm{e}^{-\varphi/2}$. With this form of $\partial_\tau W$ the two sets of equations (3.8), (3.9) and (3.13), (3.14) can hold simultaneously only in the extremal case. In section 4 we give an explicit example where all non-extremal solutions obey equations (3.8), (3.9), but not (3.13), (3.14).

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4 An illustration: dilatonic black hole

The simplest theory involving scalar fields that admits charged black hole solutions is given by the Einstein-dilaton-Maxwell action

$$S = \int \mathrm{d}^4x\sqrt{|g|}\Big(\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}\mathrm{e}^{a\phi}F^2\Big)\,, \tag{4.1}$$

where the dilaton coupling a is a non-zero constant. In reference [24] the 'fake' BPS equations for the purely electric, extremal and non-extremal solutions of this theory were given by writing the action as a sum and difference of squares, generalising the results on the pure Reissner-Nordström black hole in [8]. In the following we reconsider these results in the language of section 3 and extend to the full dyonic solution. For dyonic solutions, however, we will notice that only in the $a = 1$ case can we easily find the fake superpotential. In section 7 we return to this subject and discuss the $a = \sqrt{3}$ example. We refer the reader to [38] for the original treatment of dilatonic black hole solutions.

Following the language of section 3, we will now consider the first-order equations and the construction of a generalized superpotential for the dilaton ϕ and the 'warp factor' φ appearing in the metric (3.1). Note that for $D+1 = 4$ we have $\beta = -\alpha = 1/2$. As explained above, the equations of motion for φ and ϕ can be derived from a one-dimensional action of the form (3.3), where now $G_{ab}\dot{\phi}^a\dot{\phi}^b = \dot{\phi}^2$ and the black hole effective potential is given by

$$V(\phi) = \tfrac{1}{2}Q_{\mathrm{e}}^2\mathrm{e}^{-a\phi} + \tfrac{1}{2}Q_{\mathrm{m}}^2\mathrm{e}^{+a\phi}\,, \tag{4.2}$$

where Q_{e} is the electric charge and Q_{m} is the magnetic charge (which, in what follows, we assume to be non-negative).

4.1 Purely electric or magnetic solutions

The first-order equations found in [24] for purely electric solutions are[5]

$$f^{\varphi}(\varphi,\phi) \equiv \dot{\varphi} = -\tfrac{2}{1+a^2}\sqrt{\tfrac{1+a^2}{4}Q_{\mathrm{e}}^2\mathrm{e}^{\varphi-a\phi} + \beta_2^2} - \tfrac{2a}{1+a^2}\beta_3\,, \tag{4.3}$$

$$f^{\phi}(\varphi,\phi) \equiv \dot{\phi} = +\tfrac{2a}{1+a^2}\sqrt{\tfrac{1+a^2}{4}Q_{\mathrm{e}}^2\mathrm{e}^{\varphi-a\phi} + \beta_2^2} - \tfrac{2}{1+a^2}\beta_3\,, \tag{4.4}$$

where we introduced a set of integration constants $(\gamma, \beta_2, \beta_3)$ that obey the Hamiltonian constraint

$$(1+a^2)\gamma^2 = \beta_2^2 + \beta_3^2\,. \tag{4.5}$$

In order for a generalised superpotential Y to exist, the above two-dimensional flow must be a gradient flow. This is locally the case, as one immediately verifies that the curl, $\partial_{[\phi}f_{\varphi]}$, vanishes.[6] It is not difficult to construct the generalised superpotential explicitly,

$$Y(\varphi,\phi) = -\frac{2}{1+a^2}\Big(2\sqrt{s_{\mathrm{e}}} - 2\beta_2\log(\beta_2+\sqrt{s_{\mathrm{e}}}) + \beta_2(\varphi - a\phi) + \beta_3(a\varphi+\phi)\Big)\,, \tag{4.6}$$

[5] We changed the sign of β_3 and divided it by 2, compared to the definition in [24].

[6] In this example no distinction needs to be made between lower and upper indices, but we maintain it for consistency with section 5.

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where $\sqrt{s_{\rm e}}$ is shorthand for $\sqrt{\frac{1+a^2}{4}Q_{\rm e}^2{\rm e}^{\varphi-a\phi}+\beta_2^2}$. As expected, extremality ($\gamma=\beta_2=\beta_3=0$) implies that the superpotential Y factorises according to (3.11)

$$Y(\varphi,\phi)={\rm e}^{\varphi/2}\big(-\tfrac{2}{\sqrt{1+a^2}}Q_{\rm e}{\rm e}^{-a\phi/2}\big)\equiv{\rm e}^{\varphi/2}W(\phi)\,. \tag{4.7}$$

We now come back to the remark made at the end of the previous section. If we compare with the flow equations of [6], by locally inverting $\varphi(\tau)$, we do not find the form of $\partial_\tau W(\phi,\varphi(\tau))$ suggested in [6], unless $\gamma=\beta_2=\beta_3=0$. However, from the expression (4.6) for the generalised superpotential, we see that we can explicitly construct Y for all possible values of the nonextremality parameters β_2,β_3.

In the case of purely magnetic charge, the above equations hold when the following electromagnetic duality rule is imposed:

$$Q_{\rm e}\to Q_{\rm m}\,,\qquad \phi\to-\phi\,,\qquad \beta_2\leftrightarrow\beta_3\,. \tag{4.8}$$

4.2 Dyonic solutions

The dyonic case with arbitrary dilaton coupling a is more involved. The theory with $a=1$ is the simplest and, in the *extremal case*, it is not difficult to see that the correct vector field f can be found by summing the electric and magnetic ones

$$f^\varphi(\varphi,\phi)=-\tfrac{1}{\sqrt{2}}Q_{\rm e}{\rm e}^{(\varphi-\phi)/2}-\tfrac{1}{\sqrt{2}}Q_{\rm m}{\rm e}^{(\varphi+\phi)/2}\,, \tag{4.9}$$

$$f^\phi(\varphi,\phi)=+\tfrac{1}{\sqrt{2}}Q_{\rm e}{\rm e}^{(\varphi-\phi)/2}-\tfrac{1}{\sqrt{2}}Q_{\rm m}{\rm e}^{(\varphi+\phi)/2}\,. \tag{4.10}$$

The corresponding superpotential Y is

$$Y(\varphi,\phi)=-{\rm e}^{\varphi/2}\sqrt{2}\left(Q_{\rm e}{\rm e}^{-\phi/2}+Q_{\rm m}{\rm e}^{+\phi/2}\right)\equiv{\rm e}^{\varphi/2}W(\phi), \tag{4.11}$$

and is the sum of the pure electric and magnetic superpotentials. An extremum of the superpotential $W(\phi)$, and consequently of the black hole potential $V(\phi)$, only exists in the dyonic case, corresponding to the fact that an attractive AdS_2 horizon exists only in the extremal dyonic case.

Let us now extend to *non-extremal solutions* using the technique of [8], as explained in the previous section. This gives

$$f^\varphi(\varphi,\phi)=-\sqrt{\tfrac{1}{2}Q_{\rm e}^2{\rm e}^{\varphi-\phi}+\beta_2^2}-\sqrt{\tfrac{1}{2}Q_{\rm m}^2{\rm e}^{\varphi+\phi}+\beta_3^2}\,, \tag{4.12}$$

$$f^\phi(\varphi,\phi)=+\sqrt{\tfrac{1}{2}Q_{\rm e}^2{\rm e}^{\varphi-\phi}+\beta_2^2}-\sqrt{\tfrac{1}{2}Q_{\rm m}^2{\rm e}^{\varphi+\phi}+\beta_3^2}\,. \tag{4.13}$$

The corresponding generalised superpotential Y reads

$$\begin{aligned}Y(\varphi,\phi)=\;&-2\sqrt{s_{\rm e}}+2\beta_2\log(\beta_2+\sqrt{s_{\rm e}})-\beta_2(\varphi-\phi)\\&-2\sqrt{s_{\rm m}}+2\beta_3\log(\beta_3+\sqrt{s_{\rm m}})-\beta_3(\varphi+\phi)\,,\end{aligned} \tag{4.14}$$

where $\sqrt{s_{\rm e}}$ is defined as in the electric case and $\sqrt{s_{\rm m}}$ is shorthand for $\sqrt{\frac{1}{2}Q_{\rm m}^2{\rm e}^{\varphi+\phi}+\beta_3^2}$.

We have not been able to integrate the second-order equations for φ and ϕ when $a\neq1$. However, we demonstrate in section 7 that the case $a=\sqrt{3}$ can also be solved explicitly

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with the aid of the group-theoretical methods of section 6. For general dilaton coupling a we are not aware of whether the solution for the dyonic case is explicitly known or not, but in the next section we argue that the *extremal* solution (if it exists) obeys first-order flow equations.

5 Existence of a generalised superpotential

In this section we comment on the question of whether a generalised superpotential exists or not. First, we consider black holes with a single scalar field and explain that, at least in the extremal case, a generalised superpotential always exists, as argued in a different way already in [5]. We then investigate the multiscalar case and see that a generalised superpotential exists when the velocity field on the enlargement of the scalar manifold in $D+1$ dimensions with the warp factor φ is irrotational (curl-free), generalising the condition of [5] for extremal black holes to non-extremal ones. In the following section we will study this velocity field in detail for theories with a symmetric moduli space after a timelike dimensional reduction.

5.1 Black holes with a single scalar field

The argument for the existence of a fake superpotential for extremal black hole solutions involving one scalar field is taken from the fake supergravity formalism for single scalar domain walls [14] and proceeds as follows. Assume that the extremal dyonic solution exists, then equation (3.13) can be used to give W in terms of the radial parameter τ, i.e. this defines the function $W(\tau)$. Since the black hole is supported by a single scalar ϕ, we have that W depends only on ϕ. Locally we can always invert the function $\phi(\tau)$ to $\tau(\phi)$ and this defines $W(\phi)$.

Having constructed the fake superpotential $W(\phi)$ for the extremal solution, we could then attempt the deformation technique of [8] to obtain the function $Y(\varphi,\phi)$ in the non-extremal case. This approach, however, requires the Lagrangian to satisfy certain conditions (see [8] for details). For the dyonic example of the previous section it turns out that only the Lagrangian with $a=1$ obeys these constraints. Therefore, when $a\neq 1$, even though flow equations might exist, the procedure cannot be applied. For $a=\sqrt{3}$ the hidden symmetries of the theory will allow us to demonstrate the existence of generalised flow equations also in the non-extremal case (section 7).

5.2 Black holes with multiple scalar fields

When a black hole solution is carried by multiple scalars, the above argument for the existence of *extremal* flow equations does not apply.[7] Furthermore, for domain walls an example has been found, where a solution does not admit a first-order flow that can be derived from a fake superpotential [18].

We shall now reconsider the question of the existence of a gradient flow for black holes. Remember that, in the formalism of section 3, the gradient of the generalised superpotential

[7] Unless some complicated conditions are satisfied, as explained in the case of domain walls in [13, 18].

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determines the first order derivatives of both the 'warp factor' φ appearing in the $D+1$-dimensional metric and the $D+1$-dimensional scalars ϕ^a, through equations (3.8) and (3.9). Therefore we consider all these scalars on the same footing and will combine them in a vector ϕ^A:

$$\phi^A = \{\varphi, \phi^a\}\,. \tag{5.1}$$

In the following section we will investigate a class of theories that have a symmetric moduli space when reduced over one dimension, as their equations of motion are known to be integrable. Using the integrability of the effective action we can explicitly write down the velocity vector field f on the enlarged scalar manifold in D dimensions

$$\dot{\phi}^A \equiv f^A(\phi, \chi)\,, \tag{5.2}$$

$$\dot{\chi^\alpha} \equiv f^\alpha(\phi, \chi)\,, \tag{5.3}$$

where the χ^α are the scalars descending from the vector potentials upon dimensional reduction. One can demonstrate that there are enough 'integrals of motion' to fully eliminate the χ^α in terms of the ϕ^A, such that one can write down a velocity field on the original target space in $D+1$ dimensions:

$$\dot{\phi}^A = f^A(\phi, \chi(\phi))\,. \tag{5.4}$$

Having obtained the velocity field (5.4) on the moduli space in $D+1$ dimensions, it suffices to show that the velocity one-form f_A is locally exact

$$f_A(\phi, \chi(\phi)) \equiv \tilde{G}_{AB}(\phi) f^B(\phi, \chi(\phi)) = \partial_A Y(\phi)\,, \tag{5.5}$$

where $\tilde{G}$ is the metric on the scalar manifold in the D-dimensional theory. A necessary and sufficient condition for this to hold locally is, by Poincaré's lemma, that the one-form is closed

$$\partial_{[A} f_{B]} = 0\,. \tag{5.6}$$

Whether or not the field $Y(\phi)$ is defined over the whole target space is of less relevance to us and depends on the cohomology of the target space.

For specific non-supersymmetric solutions it might be very difficult in practice to find the superpotential Y. In spite of this, by verifying the vanishing curl condition (5.6) one can demonstrate the existence of a gradient flow.[8] For this reason we restrict ourselves to those theories that have a symmetric moduli space after timelike reduction, where we know that f exists. It will therefore be convenient to now briefly review the relationship between black holes and geodesics on symmetric spaces.

6 Black holes and geodesics

Now we would like to examine the condition discussed in section 5.2 for the generalised superpotential to exist. We will consider a timelike reduction of the $D+1$-dimensional

[8] In some cases a direct integration turns out to be possible for an *extremal* ansatz, as in [39, 40]. One can readily check that the velocity field is irrotational in these examples.

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theory. We showed that it suffices for the curl (5.6) of a velocity field on the scalar manifold of D dimensions to vanish. In this section we concentrate on theories for which the D-dimensional scalar manifold is a symmetric space. For these theories we construct the velocity field needed to investigate the curl-condition (5.6). We begin with explaining the timelike dimensional reduction to D dimensions and will then give the necessary background on symmetric spaces to arrive at an expression for the velocity field $\dot{\phi}^A = \{\dot{\varphi}, \dot{\phi}^a\}$.

6.1 Timelike dimensional reduction

The ansatz for stationary black holes can always be interpreted as the ansatz for the dimensional reduction over time

$$\mathrm{d}s^2_{D+1} = -\mathrm{e}^{2\beta\varphi}(\mathrm{d}t - B^0)^2 + \mathrm{e}^{2\alpha\varphi}\mathrm{d}s^2_D\,, \tag{6.1}$$

$$A^I = \chi^I(\mathrm{d}t - B^0) + B^I_m \mathrm{d}x^m\,, \tag{6.2}$$

where B^0 and φ are the Kaluza-Klein (KK) vector and (KK) dilaton, respectively. Normalisations are chosen in such a way that the $(D,0)$-dimensional theory is in the Einstein frame and that φ is canonically normalised.

We will restrict to spherically symmetric solutions and truncate the KK vector ($B^0 = 0$), since its presence would violate staticity.[9] In $D+1=4$ we make an exception: in $D=3$ the Taub-NUT vector B^0 can be dualised to a scalar, $\tilde{\chi}^0$, which is part of the scalar manifold. We will make use of the group structure associated to this manifold and truncate the Taub-NUT scalar at the end of the calculation. In fact, when $D=3$ also B^I can be dualised to axionic scalars $\tilde{\chi}^I$. One can then verify that the kinetic terms of the axions χ^I and $\tilde{\chi}^I$ appear with the opposite sign [19].

From (3.1) we make following ansatz for the dimensionally reduced black hole (instanton)

$$\mathrm{d}s^2_D = \mathrm{e}^{2(D-1)A(\tau)}\mathrm{d}\tau^2 + \mathrm{e}^{2A(\tau)}\mathrm{d}\Omega^2_{D-1}\,, \qquad \tilde{\phi}^i = \tilde{\phi}^i(\tau)\,, \tag{6.3}$$

with $\tilde{\phi}^i$ denoting the scalars in the D-dimensional Euclidean theory

$$\tilde{\phi}^i = \{\phi^A, \chi^\alpha\}\,. \tag{6.4}$$

The fields ϕ^A contain both the scalars of the $(D,1)$- dimensional theory (ϕ^a) and the KK dilaton φ, whereas the χ^α are the axions χ^I (and $\tilde{\chi}^I$, $\tilde{\chi}^0$ when $D=3$). The effective field equations, which arise by substituting ansatz (6.3) into the equations of motion, can be found by varying the following effective action (see e.g. [24])

$$S_{\text{eff}} = \int \mathrm{d}\tau \Big((2\alpha^2)^{-1}(\dot{A}^2 + \mathrm{e}^{2(D-2)A}) - \tfrac{1}{2}\tilde{G}_{ij}\dot{\tilde{\phi}}^i\dot{\tilde{\phi}}^j \Big), \tag{6.5}$$

where dots denote derivatives with respect to τ. Note that we reserve the symbol $\tilde{G}_{ij}$ for the moduli space metric in D dimensions. This action has to be complemented by the Hamiltonian constraint [24]

$$\alpha^{-2}(\dot{A}^2 - \mathrm{e}^{2(D-2)A}) = \tilde{G}_{ij}\dot{\tilde{\phi}}^i\dot{\tilde{\phi}}^j \equiv 2E\,, \tag{6.6}$$

[9] 'Static' means that the spacetime admits a global, nowhere zero, timelike hypersurface orthogonal Killing vector field. A generalization are the 'stationary' spacetimes, which admit a global, nowhere zero timelike Killing vector field. In particular, stationary, spherically symmetric spacetimes are static.

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with E a constant. Those D-dimensional solutions that lift to extremal black holes in $(D,1)$ dimensions have flat D-dimensional geometries, or equivalently $E=0$, which implies that the geodesic is null

$$\tilde{G}_{ij}\dot{\tilde{\phi}}^i\dot{\tilde{\phi}}^j = 0\,. \tag{6.7}$$

In $D>3$ one can eliminate the χ^α from the action, since the moduli space metric has the following properties

$$\tilde{G}_{\alpha A}=0\,,\qquad \partial_\alpha \tilde{G}_{ij}=0\,,\qquad \tilde{G}_{ab}=G_{ab}\,,\qquad \tilde{G}_{\varphi\varphi}=1\,,\qquad \tilde{G}_{\varphi a}=0\,. \tag{6.8}$$

The first two identities can be derived from the fact that the shift symmetries of the scalars ϕ^α commute. In $D=3$ this is not the case as the shift symmetries associated with electric and magnetic charges q_I, p^I no longer commute. With a slight abuse of notation, we have

$$[p^I, q_J] = \Omega^I{}_J\, Q_{\rm T}\,, \tag{6.9}$$

where $Q_{\rm T}$ is the NUT charge and $\Omega^I{}_J$ is a symplectic invariant matrix. If the NUT charge is zero, the properties of the moduli space metric (6.8) are also valid in $D=3$ upon truncation of the vectorial direction that corresponds to the NUT charge. Henceforth we always restrict to solutions with a vanishing NUT charge (thus, spherically symmetric in $D+1$ dimensions).

6.2 Geodesics on symmetric spaces

The assumption we shall make is that the target space in D dimensions is a symmetric coset space G/H, where G is a Lie group and H some subgroup subject to certain conditions that we shall state below. In the theories we consider, such as various supergravities in arbitrary dimensions, the Lie algebra of G is always semi-simple. The condition that the target space is a symmetric space is always valid for supergravity theories with more than eight supercharges and is sometimes valid for theories with less supersymmetry. Nevertheless, our analysis here is independent of any supersymmetry considerations.

We will take L, an element of G, to be a coset representative. We first define the group multiplication from the left, $L\to gL$, $\forall g\in G$, and we let the local symmetry act from the right $L\to Lh$, $\forall h\in H$. The definition of a coset element of G/H then implies that we identify L and Lh. The Lie algebras associated to G and H are denoted by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The defining property of a symmetric space G/H is that there exists a Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{f}\,, \tag{6.10}$$

with respect to the Cartan automorphic involution θ, such that $\theta(\mathfrak{f})=-\mathfrak{f}$ and $\theta(\mathfrak{h})=+\mathfrak{h}$. From the Cartan involution we can construct the symmetric coset matrix $M=LL^\sharp$, where $\sharp$ is the generalised transpose, defined as

$$L^\sharp = \exp[-\theta(\log L)]\,. \tag{6.11}$$

The matrix M is invariant under H-transformations that act from the right on L. Under G-transformations from the left, M transforms as follows

$$M\to gMg^\sharp\,. \tag{6.12}$$

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With the aid of the matrix M the line element on the space G/H with coordinates $\tilde{\phi}^i$ can be written as

$$\mathrm{d}s^2 = \tilde{G}_{ij}\mathrm{d}\tilde{\phi}^i\mathrm{d}\tilde{\phi}^j = -\tfrac{1}{2}\,\mathrm{Tr}\big(\mathrm{d}M\mathrm{d}M^{-1}\big)\,. \tag{6.13}$$

We expand the matrix valued one-form $M^{-1}\mathrm{d}M$ in the generators T_Λ of Lie algebra $\mathfrak{g}$ and in the basis $\mathrm{d}\tilde{\phi}^i$ as follows

$$M^{-1}\mathrm{d}M = (M^{-1}\partial_i M)^\Lambda \mathrm{d}\tilde{\phi}^i T_\Lambda \equiv e_i^\Lambda \mathrm{d}\tilde{\phi}^i T_\Lambda\,, \tag{6.14}$$

where we introduced the symbol e_i^Λ. This plays a role similar to the vielbein on G/H, but one should note that e_i^Λ is not a square matrix, since $\Lambda = 1,\ldots,\dim G$ and $i = 1,\ldots,\dim G/H$. In this language, the metric can also be written as

$$\tilde{G}_{ij} = e_i^\Lambda\,\eta_{\Lambda\Sigma}\,e_j^\Sigma\,, \qquad \eta_{\Lambda\Sigma} = \tfrac{1}{2}\,\mathrm{Tr}\,(T_\Lambda T_\Sigma)\,. \tag{6.15}$$

In the above η is proportional to the Cartan-Killing metric of $\mathfrak{g}$ and is non-degenerate, since $\mathfrak{g}$ is semi-simple in the theories we consider.

The metric is invariant under a *local* action of H on L from the right and under a *global* action of G on L from the left. The latter implies that G is the isometry group of G/H, as expected. The action for the geodesic curves on G/H is then given by

$$S = \int \mathrm{d}\tau\ \tilde{G}_{ij}\dot{\tilde{\phi}}^i\dot{\tilde{\phi}}^j = -\tfrac{1}{2}\int \mathrm{d}\tau\ \mathrm{Tr}\left(\tfrac{\mathrm{d}}{\mathrm{d}\tau}M\tfrac{\mathrm{d}}{\mathrm{d}\tau}(M^{-1})\right), \tag{6.16}$$

where τ is an affine coordinate parametrising the geodesic curves and the resulting equations of motion are

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}(M^{-1}\tfrac{\mathrm{d}}{\mathrm{d}\tau}M) = 0 \qquad \Rightarrow \qquad M^{-1}\tfrac{\mathrm{d}}{\mathrm{d}\tau}M = Q\,, \tag{6.17}$$

with the matrix of Noether charges Q being a constant matrix in some representation of $\mathfrak{g}$. We now see that the geodesic equations are indeed integrable and their general solution is

$$M(\tau) = M(0)\mathrm{e}^{Q\tau}\,. \tag{6.18}$$

The affine velocity squared of the geodesic curve is (the dot stands for ordinary matrix multiplication)

$$\tilde{G}_{ij}\dot{\tilde{\phi}}^i\dot{\tilde{\phi}}^j = \tfrac{1}{2}\,\mathrm{Tr}(Q\cdot Q)\,, \tag{6.19}$$

and coincides with the Hamiltonian constraint (6.6).

An integrable geodesic motion on an n-dimensional space is characterised by $2n$ constants: the initial position and velocity of the geodesic curve. In our case the geodesic motion on G/H is thus specified by $2(\dim G - \dim H)$ integration constants. In equation (6.18) $M(0)$ contains $(\dim G - \dim H)$ arbitrary constants that correspond to the initial position and Q corresponds to the initial velocity, so we also expect $(\dim G - \dim H)$ arbitrary constants there. This can be understood from the constraint

$$M^\sharp(\tau) = M(\tau) \qquad \Rightarrow \qquad \theta(Q) = -M(0)^{-1}QM(0)\,, \tag{6.20}$$

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which indeed reduces the number of arbitrary constants in Q from $\dim G$ to $(\dim G - \dim H)$.

The first-order equation (6.17) can be written compactly as $e_i^\Lambda \dot{\tilde{\phi}}^i = Q^\Lambda$ or equivalently,

$$\dot{\tilde{\phi}}^i = \tilde{G}^{ij} e_j^\Sigma \eta_{\Sigma\Lambda} Q^\Lambda \,. \tag{6.21}$$

These are only $(\dim G - \dim H)$ equations. After substituting (6.21) into eq. (6.17), the remaining $\dim H$ components become non-differential equations. This shows the power of (6.17): we split the $\dim G$ differential equations in $M^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}M = Q$ into $(\dim G - \dim H)$ first-order equations and $\dim H$ equations without any derivatives. In the context of section 5 these non-differential equations are precisely what is needed to eliminate the additional scalars resulting from dimensional reduction, so that we obtain first-order equations in terms of the scalars in $D+1$ dimensions, as in eq. (5.4).

7 The dilatonic black hole revisited

When the Einstein-dilaton-Maxwell action (4.1) has the specific dilaton coupling $a = \pm\sqrt{3}$, a symmetry enhancement takes place upon dimensional reduction over a (timelike or spacelike) circle. This can be explained by the fact that (4.1) is the action obtained from reducing five-dimensional gravity over a spacelike circle and subsequent reduction should display at least the $\mathrm{GL}(2,\mathbb{R})$ symmetry of the internal torus. Furthermore, if the 3d vectors are dualised to scalars the $\mathrm{GL}(2,\mathbb{R})$-symmetry turns out to be part of a larger $\mathrm{SL}(3,\mathbb{R})$-symmetry. The Euclidean 3d action then describes gravity minimally coupled to the $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(2,1)$ sigma model with coordinates $\tilde{\phi}^i = \{\phi^1, \phi^2, \chi^0, \chi^1, \chi^2\}$:

$$\begin{aligned}\tilde{G}_{ij}\mathrm{d}\tilde{\phi}^i\mathrm{d}\tilde{\phi}^j = {} & (\mathrm{d}\phi^1)^2 + (\mathrm{d}\phi^2)^2 - \mathrm{e}^{-\sqrt{3}\phi^1+\phi^2}(\mathrm{d}\chi^0)^2 + \mathrm{e}^{2\phi^2}(\mathrm{d}\chi^2)^2 \\ & - \left[\mathrm{e}^{+\sqrt{3}\phi^1+\phi^2} - \mathrm{e}^{2\phi^2}(\chi^0)^2\right](\mathrm{d}\chi^1)^2 + 2\chi^0\mathrm{e}^{2\phi^2}\mathrm{d}\chi^1\mathrm{d}\chi^2 \,.\end{aligned} \tag{7.1}$$

The details of the Kaluza-Klein reduction can be found in appendix A.1; for details on the $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(2,1)$ sigma model, see appendix B. The scalars ϕ^1 and ϕ^2 are both a linear combination of the black hole warp factor φ and the dilaton ϕ. The scalars χ^0 and χ^1 are the electric and magnetic potentials. χ^2 comes from the dualisation of the KK vector in the reduction from four to three dimensions and is hence related to the NUT charge Q_T via

$$Q_\mathrm{T} \sim \dot{\chi}^2 + \chi^0\dot{\chi}^1 \,. \tag{7.2}$$

As explained before, a vanishing NUT charge leads to a truncated target space, where χ^0 and χ^1 have a shift symmetry

$$\mathrm{d}s^2 = (\mathrm{d}\phi^1)^2 + (\mathrm{d}\phi^2)^2 - \mathrm{e}^{-\sqrt{3}\phi^1+\phi^2}(\mathrm{d}\chi^0)^2 - \mathrm{e}^{+\sqrt{3}\phi^1+\phi^2}(\mathrm{d}\chi^1)^2 \,. \tag{7.3}$$

Upon eliminating $\mathrm{d}\chi^1$ and $\mathrm{d}\chi^2$ by their equations of motion, one obtains the black hole potential for ϕ^1 and ϕ^2.

Let us discuss the geodesic equations of motion for the full sigma model (7.1). The charge matrix $Q \in \mathfrak{sl}(3)$ that specifies a geodesic solution contains eight arbitrary parameters $Q = Q^\Lambda T_\Lambda$, where the eight generators of $\mathfrak{sl}(3)$, denoted T_Λ, are given in the appendix,

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eq. (B.4). We will reduce the number of integration constants to four. Firstly, we demand that the geodesic curve goes through the origin which, using (6.20), gives an involution condition on Q

$$Q = -\theta(Q)\,, \tag{7.4}$$

and requires identifying

$$Q^3 = -Q^6\,, \qquad Q^4 = -Q^7\,, \qquad Q^5 = Q^8\,. \tag{7.5}$$

This condition is gauge-equivalent to the most general expression for Q. It amounts to fixing the U(1)-gauge transformation and the boundary conditions for the black hole warp factor and dilaton at spatial infinity. This specification is without loss of generality.[10] Secondly, we restrict ourselves to solutions with a vanishing NUT charge, which amounts to $Q^5 = Q^8 = 0$, so that we are left with

$$Q = Q^\Lambda T_\Lambda = \begin{pmatrix} -\frac{1}{\sqrt{3}}Q^1 - Q^2 & -Q^6 & 0 \\ Q^6 & \frac{2}{\sqrt{3}}Q^1 & -Q^7 \\ 0 & Q^7 & -\frac{1}{\sqrt{3}}Q^1 + Q^2 \end{pmatrix}. \tag{7.6}$$

Thus we count four independent integration constants to describe the dilatonic black hole solutions: the mass, the electric and magnetic charge and the scalar charge. Upon demanding a regular horizon one can write the scalar charge in terms of the three others [38], but we do not make that restriction here for the sake of generality.

We now have sufficient information to construct the velocity vector field $f^i(\phi^j) = \dot{\phi}^i$ for the charge configuration (7.6)

$$f^{\phi^1} = Q^1 + \tfrac{\sqrt{3}}{2}(Q^7\chi^0 - Q^6\chi^1)\,, \tag{7.7}$$

$$f^{\phi^2} = Q^2 - \tfrac{1}{2}(Q^7\chi^0 + Q^6\chi^1)\,, \tag{7.8}$$

$$f^{\chi^0} = Q^7 \mathrm{e}^{\sqrt{3}\phi^1 - \phi^2}\,, \tag{7.9}$$

$$f^{\chi^1} = Q^6 \mathrm{e}^{-\sqrt{3}\phi^1 - \phi^2}\,, \tag{7.10}$$

$$f^{\chi^2} = -\chi^0 f^{\chi^1}\,. \tag{7.11}$$

Note that we have already used the component of the velocity field for the Taub-NUT scalar ($\dot{\chi}^2 = f^{\chi^2}$) to eliminate $\dot{\chi}^2$ from the other components of the velocity field via eq. (7.2) with $Q_{\mathrm{T}} = 0$. From the asymptotic behaviour of the velocity field we can then identify the charges Q^Λ: Q^1 is proportional to the ADM mass, Q^2 is proportional to the dilaton charge, while Q^6 and Q^7 are equal to the magnetic and electric charge respectively.

Aside from the explicit expression for the velocity field, there is more information in the *eight* first-order equations $M^{-1}\dot{M} = Q$. The velocity field uses five out of these eight.

[10] We put $\phi(r \to \infty) = \varphi(r \to \infty) = 0$. This condition on the warp factor φ can always be achieved by a coordinate transformation. The condition for the dilaton cannot be changed, but any other boundary value is equivalent upon a shift of the dilaton and accordingly a compensating rescaling of magnetic and electric charge.

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The remaining three equations are non-differential and we call them *constraint equations*:

$$0 = Q^6 + e^{\sqrt{3}\phi^1+\phi^2}\left[(\sqrt{3}Q^1 + Q^2)\chi^1 - Q^6\left(1 + (\chi^1)^2\right) - Q^7\chi^2\right], \tag{7.12}$$

$$0 = Q^7 + e^{-\sqrt{3}\phi^1+\phi^2}\left[(-\sqrt{3}Q^1 + Q^2)\chi^0 - Q^7\left(1 + (\chi^0)^2\right) + Q^6(\chi^2 + \chi^0\chi^1)\right], \tag{7.13}$$

$$\begin{aligned} 0 = {} & 2Q^2(2\chi^2 + \chi^0\chi^1) - Q^6\left[(1 + e^{-\sqrt{3}\phi^1-\phi^2})\chi^0 + \chi^1\chi^2\right] \\ & + Q^7\left[(1 + e^{\sqrt{3}\phi^1-\phi^2})\chi^1 - \chi^0(\chi^2 + \chi^0\chi^1)\right]. \end{aligned} \tag{7.14}$$

Note that we already used the constraint equations to simplify the first order derivatives of χ^0 and χ^1: (7.9) and (7.10). The constraint equations (at least theoretically) enable one to extract the functional dependence of the χ^α on the ϕ^1 and ϕ^2, such that we can write $f^{\phi^1}(\phi,\chi)$ and $f^{\phi^2}(\phi,\chi)$ purely in terms of ϕ^1 and ϕ^2:

$$f^{\phi^1}(\phi) \equiv f^{\phi^1}[\phi,\chi(\phi)], \qquad f^{\phi^2}(\phi) \equiv f^{\phi^2}[\phi,\chi(\phi)]. \tag{7.15}$$

The condition for the existence of a first-order gradient flow then becomes

$$\partial_{[\phi^1} f_{\phi^2]} = -\tfrac{1}{4}Q^7(\sqrt{3}\partial_{\phi^2} + \partial_{\phi^1})\chi^0(\phi^1,\phi^2) + \tfrac{1}{4}Q^6(\sqrt{3}\partial_{\phi^2} - \partial_{\phi^1})\chi^1(\phi^1,\phi^2) = 0, \tag{7.16}$$

and we can evaluate under which conditions on the charges Q^Λ the expression (7.16) holds.

In principle, we have three constraint equations at our disposal to eliminate the three axions $\chi^\alpha(\phi^1,\phi^2), \alpha = 0,1,2$, but in practice this would require solving relatively complicated non-linear simultaneous equations, which is not straightforward. Fortunately, the curl (7.16) requires only a knowledge of the derivatives of the axions with respect to the dilatons, i.e. the Jacobian matrix $[J^\alpha_A](\phi) \equiv \partial_{\phi^A}\chi^\alpha$, $\alpha = 0,1$. It turns out that the inverse Jacobian matrix $[J^A_\alpha](\chi) \equiv \partial_{\chi^\alpha}\phi^A$ is easily computable using the constraint equations. If we then use

$$[J^\alpha_A](\phi(\chi)) = [J^A_\alpha]^{-1}(\chi), \tag{7.17}$$

where the inverse is with respect to the whole matrix, we can evaluate the curl in terms of the fields χ^0,χ^1. An explicit calculation shows that the curl vanishes. We thus conclude that *when $a = \sqrt{3}$, all the dilatonic black holes with arbitrary mass, electric, magnetic and scalar charge possess a generalised superpotential.*

We have not attempted the construction of the generalised superpotential for arbitrary solutions with $a = \sqrt{3}$, but rather only for extremal cases. Then one can use the factorisation property (3.11) of the superpotential to deduce W as a function of τ from the flow equation (3.13) with the help of the known explicit solution [38] and invert $\phi(\tau)$ to obtain $W(\phi)$. Even in this simplified setting the result is very long compared to the $a = 1$ case and not illuminating, we therefore refrain from quoting it here.

8 Kaluza-Klein black hole in five dimensions

Let us now consider black holes carried by multiple scalars and vectors. In $D+1 = 5$ there is an example for which we can use the same hidden symmetry as for the KK dilatonic black hole, namely $\mathrm{SL}(3,\mathbb{R})$. This theory is obtained by reducing 7d gravity on a two-torus.

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This gives a 5d theory with two vectors, and three scalars: an axion-dilaton system and an extra dilaton $\tilde{\varphi}$

$$S = \int \mathrm{d}^5x\sqrt{|g|}\Big(\mathcal{R} - \tfrac{1}{2}(\partial\tilde{\varphi})^2 + \tfrac{1}{4}\,\mathrm{Tr}(\partial K\partial K^{-1}) - \tfrac{1}{4}\mathrm{e}^{\sqrt{\frac{5}{3}}\tilde{\varphi}}K_{mn}F^mF^n\Big), \tag{8.1}$$

where the matrix K defines the $\mathrm{SL}(2,\mathbb{R})$ axion-dilaton system. Details on this action and subsequent reduction to 4 dimensions can be found in appendix A.2.

This Lagrangian is a consistent truncation of maximal and half-maximal supergravity in $D+1=5$. Upon reduction over time one obtains four-dimensional Euclidean gravity coupled to a set of scalars that span the coset $\mathrm{SO}(1,1)\times\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(2,1)$. The dynamics of the decoupled scalar (the $\mathrm{SO}(1,1)$ part) is trivial and the $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(2,1)$ part differs from the previous example only in that this coset has a different $\mathrm{SO}(2,1)$ isotropy group embedded in $\mathrm{SL}(3,\mathbb{R})$. The effect of this is purely a matter of signs, as can be seen in the metric on the moduli space (neglecting the decoupled scalar, see appendix A.2)

$$\begin{aligned}\tilde{G}_{ij}\mathrm{d}\tilde{\phi}^i\mathrm{d}\tilde{\phi}^j = {}&(\mathrm{d}\phi^1)^2 + (\mathrm{d}\phi^2)^2 + \mathrm{e}^{-\sqrt{3}\phi^1+\phi^2}(\mathrm{d}\chi^0)^2 - \mathrm{e}^{2\phi^2}(\mathrm{d}\chi^2)^2\\ &- \left[\mathrm{e}^{+\sqrt{3}\phi^1+\phi^2} + \mathrm{e}^{2\phi^2}(\chi^0)^2\right](\mathrm{d}\chi^1)^2 - 2\chi^0\mathrm{e}^{2\phi^2}\mathrm{d}\chi^1\mathrm{d}\chi^2\,.\end{aligned} \tag{8.2}$$

This sigma model can be obtained from (7.1) through the analytic continuation

$$\chi^0 \to \mathrm{i}\chi^0\,, \qquad \chi^2 \to \mathrm{i}\chi^2\,. \tag{8.3}$$

The representative $\tilde{L}$ of the full coset $\mathrm{SO}(1,1)\times\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(2,1)$ is then given by

$$\tilde{L} = \mathrm{e}^{\phi^0/\sqrt{6}}L\,, \tag{8.4}$$

where L is the $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(2,1)$ coset representative (B.5) and the decoupled scalar ϕ^0 is related to $\tilde{\varphi}$ of (8.1) by eq. (A.16).

We will again assume that the charge matrix describes only the geodesics that go through the origin. As before we can justify this restriction by proper field redefinitions and coordinate transformations of the general solution. The Cartan involution condition (7.4) implies (cf. (7.5))

$$Q^3 = -Q^6, \qquad Q^4 = Q^7, \qquad Q^5 = -Q^8\,, \tag{8.5}$$

so that

$$Q = Q^\Lambda T_\Lambda = \begin{pmatrix} Q^0 - \frac{Q^1}{\sqrt{3}} - Q^2 & -Q^6 & -Q^8 \\ Q^6 & Q^0 + \frac{2Q^1}{\sqrt{3}} & Q^7 \\ Q^8 & Q^7 & Q^0 - \frac{Q^1}{\sqrt{3}} + Q^2 \end{pmatrix}, \tag{8.6}$$

where now $\Lambda = 0,\ldots,8$, T_0 is the three-dimensional identity matrix generating the decoupled $\mathrm{SO}(1,1)$ part and the remaining generators are, as previously, given by (B.4). The parameters Q^6 and Q^8 can be identified with the electric charges in $D+1=5$.

To obtain the first-order velocity field for the effective action with the black hole potential one needs to eliminate χ^1 and χ^2 in terms of the remaining scalars using the

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constraint equations, which can be concisely written as

$$\dot{\chi}^0 = \mathrm{e}^{+\sqrt{3}\phi^1-\phi^2}\left(Q^7 - Q^8\chi^1\right), \tag{8.7}$$

$$\dot{\chi}^1 = \mathrm{e}^{-\sqrt{3}\phi^1-\phi^2}\left(Q^6 - Q^8\chi^0\right), \tag{8.8}$$

$$\dot{\chi}^2 = \mathrm{e}^{-2\phi^2}Q^8 - \chi^0\dot{\chi}^1 = \mathrm{e}^{-2\phi^2}Q^8 - \mathrm{e}^{-\sqrt{3}\phi^1-\phi^2}\chi^0\left(Q^6 - Q^8\chi^0\right), \tag{8.9}$$

where the left-hand side is understood to be expressed by eq. (6.21) and does not contain derivatives. Unlike in the dilatonic black hole example, there are more constraints than variables to eliminate, unless specific choices for the charges make fewer of them independent. Using different combinations of constraint equations to eliminate χ^1 and χ^2 leads to different velocity fields in five dimensions. Although they become equivalent upon using the Hamiltonian constraint (which is exactly the remaining constraint equation), the expression for the curl is not unique. One preferred form should however distinguish itself, namely that not containing second-order integration constants. Finding such a combination of constraint equations is a technically complex task, as it involves relaxing the boundary conditions ($M(0) = 1$) in order to distinguish first- and second-order integration constants.[11] For this reason we have not pursued it further.

Regardless of which combination of constraint equations should serve to eliminate the extraneous scalars χ^1 and χ^2, the resulting expression for the curl in five dimensions will be non-trivial and will not involve Q^0, hence the condition for the curl to vanish is independent of extremality, which in turn amounts to (cf. remarks preceding equations (6.7) and (6.19))

$$\mathrm{Tr}(Q\cdot Q) = 3(Q^0)^2 + 2\left[(Q^1)^2 + (Q^2)^2 - (Q^6)^2 + (Q^7)^2 - (Q^8)^2\right] = 0\,. \tag{8.10}$$

We conclude that among both extremal and non-extremal solutions there exist examples that admit a generalised superpotential, but also examples that do not.

9 Discussion

9.1 Summary of results

For theories of gravity coupled to neutral scalar fields and Abelian vector fields, we have presented the most general form of first-order flow equations consistent with rewriting the effective action as a sum (or difference) of squares. The derivatives of the scalars with respect to the radial parameter are given by the gradient of a generalised superpotential on the scalar manifold (equations (3.8), (3.9)). The generalised superpotential is related to the black hole potential by eq. (3.5). The above gradient flow equations are equally applicable to extremal (whether supersymmetric or not) as well as non-extremal black holes (necessarily non-supersymmetric). They naturally encompass previously known partial results, although they differ from the form conjectured in [6].

[11] The first-order integration constants in Q and the second-order integration constants in $M(0)$ are intertwined through the involution condition (6.20) making it difficult to distinguish them in the coset matrix formalism.

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We considered theories with scalar manifolds which become symmetric spaces after a timelike dimensional reduction and produced a method to verify when a generalised superpotential exists. We have provided examples of extremal and non-extremal solutions with a generalised superpotential, but also shown that it is possible to find solutions (including extremal ones) for which one cannot exist.

Let us now discuss the examples in which we obtained the above results in more detail. We have applied our formalism to a dilatonic black hole in four dimensions (one scalar field) and a Kaluza-Klein black hole in five dimensions (multiple scalars). For the dilatonic black hole with the dilaton coupling $a = 1$ we were able to show by direct integration that the generalised flow equations exist in all situations. When $a = \sqrt{3}$ we were able to show the same, using group-theoretical tools to integrate the second-order equations of motion to first-order equations. For all other values of a we derived the existence of a fake superpotential in the *extremal* case, using the argument applied for single scalar domain walls [14]. The existence of generalised flow equations for non-extremal black holes with arbitrary dilaton coupling is not known to us. Although the $a = 1$ case was easy to integrate by hand, this is also an example for which we could have constructed the flow using group theory. The reason is that this case is embeddable in an $\mathcal{N} = 4$ action, which has a symmetric moduli space after timelike reduction: $\mathrm{SO}(8, 8+n)/[\mathrm{SO}(6, 2)\times\mathrm{SO}(6+n, 2)]$ (see for instance [22]). The investigation of the Kaluza-Klein black hole in five dimensions, in turn, demonstrated that for both extremal and non-extremal solutions, there are cases where a generalised superpotential exists and where it does not, depending on the values of the scalar and vector charges.

The same techniques can be applied to more complicated examples. Possible further work might include exploring other non-extremal cases, a natural candidate being the STU or the T^3 model in $\mathcal{N} = 2$ supergravity. It would be most interesting to see whether there exists a closed form of the generalised superpotential, universally valid for all cosets. It might be also useful to investigate whether the vanishing curl condition (5.6) could be given a physical interpretation in terms of other black hole properties. A broader problem that suggests itself for study is one of the existence of a generalised superpotential in theories whose scalar manifolds are not symmetric spaces after a timelike dimensional reduction.

9.2 Comparison with domain walls

It has been noticed in the literature that black hole effective actions are very similar to domain-wall (and cosmology) effective actions [5, 41]. Roughly speaking, the only difference is in the expression for the potential in terms of the 'superpotential'

$$V \sim W^2 + \xi(\partial W)^2\,, \tag{9.1}$$

where the constant ξ is positive for black holes and negative for domain walls (and cosmologies). The precise value for ξ depends on the dimension. We would like to point out that this is more then just an analogy as spherically symmetric black holes are nothing but domain walls seen from a $1+1$ dimensional point of view. This is consistent with (9.1) since, in the domain wall case, the usual formula for ξ diverges when $D+1 = 2$ and instead

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one has to use the expression for ξ for black hole effective potentials. This exact correspondence between spherically symmetric black holes and domain walls in two dimensions comes about in the same way as the correspondence between domain wall solutions of gauged supergravity theories in $2 < D + 1 < 10$ and p-brane (and M-brane) solutions in $D+1 = 10$ (and $D+1 = 11$) [42, 43]. There the correspondence was obtained by considering spherical flux reductions of type II theories to some lower-dimensional gauged supergravity. The domain-wall solutions of the latter theory lift to various (distributions of) p-brane solutions, whose metrics possess the required spherical symmetry. Similarly, the construction of the black hole effective potential can effectively be seen as an S^2 reduction of 4d ungauged supergravity to a 2d gauged supergravity. The charges that appear in the black hole effective potential correspond to the flux parameters of the 'flux compactification'.

As in the black hole case, it was also appreciated that the effective action for domain walls can be described both as a free particle action and as a particle subject to a potential. In the case of cosmological solutions this is known as 'cosmology as a geodesic motion' [44] (see also [45]). By virtue of the 'domain wall/cosmology correspondence' [14], the same principle applies to domain wall solutions. It is in this sense that the existence of two different types of effective actions for black holes can be understood: it is the same as 'domain walls as a geodesic motion' [44], but applied to $D = 2$ domain walls. Finally, in a pure mathematical context, this correspondence between particle actions with a force field and an associated action of a free particle in an enlarged target space is what underlies the way the integrability of Toda-Liouville equations is linked to the integrability of the geodesic equations on symmetric spaces [46].

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A Dimensional reductions

A.1 From $D + 2 = 4 + 1$ to $D + 1 = 3 + 1$ to $D = 3$

The reduction of pure gravity in $4+1$ dimensions on a spacelike circle leads to the Einstein-Maxwell-dilaton action (4.1) with $a = \sqrt{3}$. When this is further reduced over a timelike

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circle through the following reduction ansatz[12]

$$\mathrm{d}s_4^2 = \mathrm{e}^{\phi^2}\mathrm{d}s_3^2 - \mathrm{e}^{-\phi^2}(\mathrm{d}t - B_t)^2\,, \tag{A.1}$$

$$A = B_z - \chi^0(\mathrm{d}t - B_t)\,, \qquad \phi^1 = \phi\,, \tag{A.2}$$

where ϕ is the dilaton in the four-dimensional theory, the resulting 3d Euclidean action is given by

$$S_3 = \int \sqrt{g}\Big(\mathcal{R}_3 - \tfrac{1}{2}\partial\varphi\partial\varphi + \tfrac{1}{4}\,\mathrm{Tr}(\partial K\partial K^{-1}) - \tfrac{1}{4}\mathrm{e}^{-\sqrt{3}\varphi}K_{mn}F^mF^n\Big), \tag{A.3}$$

where $m, n = t, z$ and

$$\varphi = \tfrac{1}{2}\phi^1 + \tfrac{\sqrt{3}}{2}\phi^2\,, \tag{A.4}$$

$$K_{tt} = -\mathrm{e}^{\frac{\sqrt{3}}{2}\phi^1 - \frac{1}{2}\phi^2} + (\chi^0)^2\mathrm{e}^{-\frac{\sqrt{3}}{2}\phi^1 + \frac{1}{2}\phi^2}\,, \tag{A.5}$$

$$K_{zz} = \mathrm{e}^{-\frac{\sqrt{3}}{2}\phi^1 + \frac{1}{2}\phi^2}\,, \tag{A.6}$$

$$K_{tz} = -\mathrm{e}^{-\frac{\sqrt{3}}{2}\phi^1 + \frac{1}{2}\phi^2}\chi^0\,. \tag{A.7}$$

If we dualise the 3d vectors to scalars via

$$K_{zn}\star F^n \equiv \mathrm{d}\chi^1\,, \qquad K_{tn}\star F^n \equiv \mathrm{d}\chi^2\,, \tag{A.8}$$

we find the sigma model (7.1).

A.2 From $D+3 = 6+1$ to $D+1 = 4+1$ to $D = 4$

If we reduce pure 7d gravity, given by the action:

$$\int \mathrm{d}^7x\sqrt{|g_7|}\mathcal{R}_7\,, \tag{A.9}$$

over a spacelike two-torus (with coordinates y^m, $m = 1, 2$) via

$$\mathrm{d}s_7^2 = \mathrm{e}^{2\alpha\tilde{\varphi}}\mathrm{d}s_5^2 + \mathrm{e}^{2\beta\tilde{\varphi}}K_{mn}(\mathrm{d}y^m + A^{(m)})(\mathrm{d}y^n + A^{(n)})\,, \quad \alpha = \sqrt{\tfrac{1}{15}}\,, \quad \beta = -\tfrac{1}{2}\sqrt{\tfrac{3}{5}}\,, \tag{A.10}$$

we find the 5d action

$$S = \int \mathrm{d}^5x\sqrt{|g_5|}\Big(\mathcal{R}_5 - \tfrac{1}{2}\partial\tilde{\varphi}\partial\tilde{\varphi} + \tfrac{1}{4}\,\mathrm{Tr}(\partial K^{-1}\partial K) - \tfrac{1}{4}\mathrm{e}^{\sqrt{\frac{5}{3}}\tilde{\varphi}}K_{mn}F^mF^n\Big)\,. \tag{A.11}$$

The $\mathrm{SL}(2,\mathbb{R})$ matrix K parameterises the deformations of the torus through the two scalars φ^0 and χ^0:

$$K = \mathrm{e}^{-\varphi^0}\begin{pmatrix}\mathrm{e}^{2\varphi^0} + (\chi^0)^2 & \chi^0\\ \chi^0 & 1\end{pmatrix}\,. \tag{A.12}$$

In five dimensions, this gives rise to an $\mathrm{SO}(1,1)\times\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$ sigma model, where the decoupled $\mathrm{SO}(1,1)$ is parameterised by φ, as can be seen from the action.

[12]Note that in equation (A.1) ϕ^2 is a scalar with an upper index and not a square.

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A subsequent timelike reduction to $D = 4 + 0$ via the ansatz (we truncate the 4d vectors)

$$\mathrm{d}s_5^2 = -\mathrm{e}^{-2\varphi^1}\mathrm{d}t^2 + \mathrm{e}^{\varphi^1}\mathrm{d}s_4^2\,, \tag{A.13}$$

$$A^{(1)} = \chi^1\mathrm{d}t\,, \qquad A^{(2)} = \chi^2\mathrm{d}t\,, \tag{A.14}$$

gives the 4d action

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{g_4}\Big[&\mathcal{R}_4 - \tfrac{1}{2}\partial\tilde{\varphi}\partial\tilde{\varphi} - \tfrac{1}{2}\partial\varphi^0\partial\varphi^0 - \tfrac{3}{2}\partial\varphi^1\partial\varphi^1 - \tfrac{1}{2}\mathrm{e}^{-2\varphi^0}\partial\chi^0\partial\chi^0 \\ &+ \tfrac{1}{4}\mathrm{e}^{\sqrt{\frac{5}{3}}\tilde{\varphi}+2\varphi^1-\varphi^0}\Big(\big(\mathrm{e}^{2\varphi^0} + (\chi^0)^2\big)\partial\chi^1\partial\chi^1 + 2\chi^0\partial\chi^1\partial\chi^2 + \partial\chi^2\partial\chi^2\Big)\Big]\,. \end{aligned} \tag{A.15}$$

This action describes an $\mathrm{SO}(1,1) \times \frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(2,1)}$ sigma model coupled to gravity. Written in this way, it is not evident how to decouple the $\mathrm{SO}(1,1)$ part. To obtain the form (8.2), we further have to perform the following rotation of the dilatons $\{\tilde{\varphi}, \varphi^0, \varphi^1\} \to \{\phi^0, \phi^1, \phi^2\}$:

$$\phi^0 = -\tfrac{2}{3}\tilde{\varphi} + \sqrt{\tfrac{5}{3}}\varphi^1\,, \tag{A.16}$$

$$\phi^1 = \tfrac{1}{2\sqrt{3}}\Big(\sqrt{\tfrac{5}{3}}\tilde{\varphi} + 3\varphi^0 + 2\varphi^1\Big), \tag{A.17}$$

$$\phi^2 = \tfrac{1}{2}\Big(\sqrt{\tfrac{5}{3}}\tilde{\varphi} - \varphi^0 + 2\varphi^1\Big). \tag{A.18}$$

The action now becomes:

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{g_4}\Big(&\mathcal{R}_4 - \tfrac{1}{2}\partial\phi^0\partial\phi^0 - \tfrac{1}{2}\partial\phi^1\partial\phi^1 - \tfrac{1}{2}\partial\phi^2\partial\phi^2 - \tfrac{1}{2}\mathrm{e}^{-\sqrt{3}\phi^1+\phi^2}\partial\chi^0\partial\chi^0 \\ &+ \tfrac{1}{2}\mathrm{e}^{2\phi^2}\partial\chi^2\partial\chi^2 + \tfrac{1}{2}\big(\mathrm{e}^{\sqrt{3}\phi^1+\phi^2} + \mathrm{e}^{2\phi^2}(\chi^0)^2\big)\partial\chi^1\partial\chi^1 + \chi^0\mathrm{e}^{2\phi^2}\partial\chi^1\partial\chi^2\Big)\,. \end{aligned} \tag{A.19}$$

The scalar ϕ^0 describes the $\mathrm{SO}(1,1)$-part, while the others parameterise a $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(2,1)}$ sigma model.

B The $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(2,1)}$ sigma model

We define the $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(2,1)$ coset element in the Borel gauge

$$L = \exp(\chi^1 E_{12})\exp(\chi^0 E_{23})\exp(\chi^2 E_{13})\exp(\tfrac{1}{2}\phi^1 H_0 + \tfrac{1}{2}\phi^2 H_2)\,, \tag{B.1}$$

where H_1 and H_2 are the Cartan generators of $\mathfrak{sl}(3)$ and the E_α are the three positive root generators. In here we use the fundamental representation of $\mathfrak{sl}(3)$ and choose the following basis for the generators

$$H_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B.2}$$

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and the three positive step operators

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.3}$$

The generators T_Λ, $\Lambda = 1, \ldots, 8$, of $\mathrm{SL}(3,\mathbb{R})$ are given by

$$T_\Lambda = \{H_0, H_1, E_{12}, E_{23}, E_{13}, E_{12}^{\mathrm{T}}, E_{23}^{\mathrm{T}}, E_{13}^{\mathrm{T}}\}. \tag{B.4}$$

Then the coset element is explicitly given by

$$L = \begin{pmatrix} \mathrm{e}^{-\frac{1}{2\sqrt{3}}\phi^1 - \frac{1}{2}\phi^2} & \mathrm{e}^{\frac{\phi^1}{\sqrt{3}}}\chi^1 & \mathrm{e}^{-\frac{\phi^1}{2\sqrt{3}}+\frac{\phi^2}{2}}(\chi^0\chi^1 + \chi^2) \\ 0 & \mathrm{e}^{\frac{\phi^1}{\sqrt{3}}} & \mathrm{e}^{-\frac{\phi^1}{2\sqrt{3}}+\frac{\phi^2}{2}}\chi^0 \\ 0 & 0 & \mathrm{e}^{-\frac{\phi^1}{2\sqrt{3}}+\frac{\phi^2}{2}} \end{pmatrix}. \tag{B.5}$$

To find the metric on the coset we define the symmetric coset matrix M via $M = L\eta L^{\mathrm{T}}$ where η is the matrix whose stabiliser defines the specific isotopy group $\mathrm{SO}(2,1)$ of the coset. To reproduce the sigma model (7.1) we choose

$$\eta = \mathrm{diag}(+1, -1, +1), \tag{B.6}$$

whereas the other sigma model (8.2) has another $\mathrm{SO}(2,1)$ defined by

$$\eta = \mathrm{diag}(-1, +1, +1). \tag{B.7}$$

The metric that is then defined by $\mathrm{d}s^2 = -\frac{1}{2}\,\mathrm{Tr}(\mathrm{d}M\mathrm{d}M^{-1})$ and the Cartan involution for a matrix $A \in \mathfrak{sl}(3,\mathbb{R})$ is

$$\theta(A) = -\eta A^{\mathrm{T}}\eta. \tag{B.8}$$

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