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D-brane instantons on the $T^6/\mathbb{Z}_3$ orientifold

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Abstract: We give a detailed microscopic derivation of gauge and stringy instanton generated superpotentials for gauge theories living on D3-branes at $\mathbb{Z}_3$-orientifold singularities. Gauge instantons are generated by D(-1)-branes and lead to Affleck, Dine and Seiberg (ADS) like superpotentials in the effective $\mathcal{N} = 1$ gauge theories with three generations of bifundamental and anti/symmetric matter. Stringy instanton effects are generated by Euclidean ED3-branes wrapping four-cycles on $T^6/\mathbb{Z}_3$. They give rise to Majorana masses, Yukawa couplings or non-renormalizable superpotentials depending on the gauge theory. Finally we determine the conditions under which ADS like superpotentials are generated in $\mathcal{N} = 1$ gauge theories with adjoints, fundamentals, symmetric and antisymmetric chiral matter.

Keywords: Brane Dynamics in Gauge Theories, Solitons Monopoles and Instantons, Nonperturbative Effects, D-branes.

Contents

1. Introduction

Our understanding of non perturbative effects in four dimensional supersymmetric gauge theories (SYM) has dramatically improved in recent years. This is due mainly to the observation that integrals over the moduli space of gauge connections localize around a finite number of points [1]. These techniques have been applied to the study of multi-instanton corrections to $\mathcal{N} = 1, 2, 4$ supersymmetric gauge theories in $\mathbb{R}^4$ [2−10] (see [11, 12] for reviews of multi-instanton techniques before localization and complete lists of references). In the D-brane language language, the dynamics of the gauge theory around the instanton background is described by an effective theory governing the interactions of the lowest energy excitations of open strings ending on a bound state of Dp-D(p+4) branes. For the case of $\mathcal{N} = 2, 4$ SYM the multi-instanton action has been derived via string techniques in [13, 14].

In [15], D-brane techniques have been applied to the computation of the Affleck, Dine and Seiberg (ADS) superpotential [16, 17] for $\mathcal{N} = 1$ SQCD with gauge group SU(N_c) and $N_f = N_c - 1$ massless flavours and Sp($2N_c$) with $2N_f = 2N_c$ flavours. The $\mathcal{N} = 1$ gauge theory is realized on the four-dimensional intersection of N_c coloured and N_f flavour

D6 branes. Chiral matter comes from strings connecting the flavor and color D6 branes. Instantons in the $\mathrm{U}(N_c)$ gauge theory are realized in terms of ED2 branes parallel to the stack of N_c D6-branes. By carefully integrating the supermoduli (massless strings with at least one end on the ED2) the precise form of the ADS superpotential was reproduced in the low energy, field theory limit $\alpha' \to 0$. In the recent literature ED2-brane instantons in intersecting D6-brane models have received particular attention in connection with the possibility of generating a Majorana mass for right handed neutrinos and their superpartners [18 − 22]. The field theory interpretation of this new instanton effect is far from clear and it is the subject of active investigation. In this paper we present a detailed derivation of these new non perturbative superpotentials in $\mathcal{N} = 1$ $\mathbb{Z}_3$-orientifold models. Investigations of stringy instantons on $\mathcal{N} = 1$ $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold singularities appeared recently in [23].

We study SYM gauge theories living on D3 branes located at a $\mathbb{Z}_3$-orientifold singularity. There are two choices for the orientifold projection [24 − 28] realized by two types of O3-planes.[1] They lead to anomaly free[2] chiral $\mathcal{N} = 1$ gauge theories with gauge groups $\mathrm{SO}(N-4) \times \mathrm{U}(N)$ or $\mathrm{Sp}(N+4) \times \mathrm{U}(N)$ and three generations of chiral matter in the bifundamental and anti/symmetric representation of $\mathrm{U}(N)$. The archetype of this class can be realized as a stack of $3N + 4$ D3-branes and one $\mathrm{O3}^-$-plane sitting on top of an $\mathbb{R}^6/\mathbb{Z}_3$ singularity. This system can be thought of as a T-dual local description (near the origin) of the $T^6/\mathbb{Z}_3$ type I string vacuum found in [40]. The lowest choices of N lead to $\mathrm{U}(4)$ or $\mathrm{U}(5)$ gauge theories with three generations of chiral matter in the $\mathbf{6}$ and $\mathbf{10} + \mathbf{5}^*$ that are clearly of phenomenological interest in unification scenarios [45 − 48].[3]

In [49] the $\mathrm{U}(4)$ case was studied and the form of the ADS-like superpotential was determined combining holomorphicity, $\mathrm{U}(1)$ anomaly, dimensional analysis and flavour symmetry. Stringy instanton effects were also considered. Very much as for worldsheet instantons in heterotic strings [50 − 54], these genuinely stringy instantons give rise to superpotentials that do not vanish at large VEV's of the open string (charged) 'moduli'.

Here we derive the non-perturbative superpotentials from a direct integration over the D-instanton super-moduli space. Gauge instantons are described in terms of open strings ending on D(-1) branes while stringy instantons are given by open strings ending on euclidean ED3 branes wrapping a four cycle inside the Calabi Yau . The open strings connecting the stack of D3 branes to D(-1) and ED3 branes have four and eight mixed Neumann-Dirichelet directions respectively. This ensures that the bound state is supersymmetric. The superpotential receives contribution from disk, one-loop annulus and Möbius amplitudes ending on the D(-1) or ED3 branes. We find that ADS superpotentials are generated only for two gauge theory choices $\mathrm{U}(4)$ and $\mathrm{Sp}(6) \times \mathrm{U}(2)$ inside the $\mathbb{Z}_3$-orientifold class. Stringy instantons leads to Majorana masses in the $\mathrm{U}(4)$ case, Yukawa couplings in the $\mathrm{U}(6) \times \mathrm{SO}(2)$ gauge theory and non-renormalizable couplings for

[1]We will only consider $\mathrm{O3}^{\pm}$-planes, not the more exotic $\widetilde{\mathrm{O3}}^{\pm}$-planes [29 − 31].

[2]Factorizable $\mathrm{U}(1)$ anomalies are cancelled by a generalization of the Green-Schwarz mechanism [32 − 38, 37] that may require the introduction of generalized Chern-Simons couplings [39].

[3]Only the $\mathrm{U}(4)$ case can be realized in the compact $\mathbb{Z}_3$ orientifold. In general the Chan-Paton group is $\mathrm{SO}(8-2n) \times \mathrm{U}(12-2n) \times H_n$ where $H_n = \mathrm{U}(n)^3, \mathrm{SO}(2n), \mathrm{U}(n), \mathrm{U}(1)^n$ depending on the choice of Wilson lines [28, 41 − 44].

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$\mathrm{U}(2N+4) \times \mathrm{SO}(2N)$ gauge theories with $N > 3$.

The plan of the paper is as follows.

In section 2 we review the gauge theories coming from a stack of D3 branes at a $\mathbb{C}^3/\mathbb{Z}_3$ orientifold singularity. In section 3 we consider non-perturbative effects generated by D(-1) gauge instantons, corresponding to ADS-like superpotential in the low energy limit. A detailed analysis of one-loop vacuum amplitudes and the integrals over the supermoduli is presented for SYM theories with gauge groups $\mathrm{Sp}(6) \times \mathrm{U}(2)$ and $\mathrm{U}(4)$. In section 5, we consider stringy instanton effects generated by ED3-branes. Once again a detailed analysis of the the one-loop string amplitudes and the integrals over the supermoduli is presented. In section 6 we present a "complete" list of $\mathcal{N}=1$ SYM theories with matter in the adjoint, fundamental, symmetric and antisymmetric representation of the gauge groups (U, SO, Sp) which exhibit a non perturbatively generated ADS superpotential.

We conclude with some comments and directions for future investigation in section 7.

2. The Gauge theory

The low energy dynamics of the open strings living on a stack of N D3-branes in flat space is described by a $\mathcal{N}=4$ $\mathrm{U}(N)$ SYM gauge theory. In the $\mathcal{N}=1$ language the fields are grouped into a vector multiplet $V = (A_\mu, \lambda_\alpha, \bar\lambda_{\dot\alpha})$ and 3 chiral multiplets $\Phi^I = (\phi^I, \psi^I_\alpha)$, $I = 1, 2, 3$, all in the adjoint of the gauge group.

We consider the D3-brane system at a $\mathbb{R}^6/\mathbb{Z}_3$ singularity. At the singularity the N D3-branes group into stacks of N_n fractional branes with $n = 0, 1, 2$ labelling the conjugacy classes of $\mathbb{Z}_3$. The gauge group $\mathrm{U}(N)$ decomposes as $\prod_n \mathrm{U}(N_n)$. More precisely, denoting by $\gamma_{\theta,N}$ the projective embedding of the orbifold group element $\theta \in \mathbb{Z}_3$ in the Chan-Paton group and imposing $\gamma^3_{\theta,N} = 1$ and $\gamma^\dagger_{\theta,N} = \gamma^{-1}_{\theta,N}$ one can write

$$\gamma_{\theta^h,N} = (\mathbf{1}_{N_0\times\bar N_0}, \omega^h\, \mathbf{1}_{N_1\times\bar N_1}, \bar\omega^h\, \mathbf{1}_{N_2\times\bar N_2}) \tag{2.1}$$

with $N = \sum_n N_n$. The resulting gauge theory can be found by projecting the $\mathcal{N}=4$ $\mathrm{U}(N)$ gauge theory under the $\mathbb{Z}_3$ orbifold group action:

$$V \to \gamma_{\theta,N}\, V\, \gamma^{-1}_{\theta,N} \qquad \Phi^I \to \omega\, \gamma_{\theta,N}\, \Phi^I\, \gamma^{-1}_{\theta,N} \qquad \omega = e^{2\pi i/3} \tag{2.2}$$

Keeping only invariant components under (2.2) one finds the $\mathcal{N}=1$ quiver gauge theory

$$\begin{aligned} V:&\ \ \mathbf{N_0\bar N_0} + \mathbf{N_1\bar N_1} + \mathbf{N_2\bar N_2} \\ \Phi^I:&\ \ 3\times\left[\mathbf{N_0\bar N_1} + \mathbf{N_1\bar N_2} + \mathbf{N_2\bar N_0}\right] \end{aligned} \tag{2.3}$$

with gauge group $\prod_n \mathrm{U}(N_n)$ and three generations of bifundamentals. More precisely V and Φ^I are $N\times N$ block matrices ($N = \sum_n N_n$) with non trivial $N_n \times \bar N_m$ blocks given by (2.3). Under $\mathbb{Z}_3$ a block $N_n \times \bar N_m$ transform as ω^{n-m}. These non-trivial transformation properties are compensated by the space-time eigenvalues of the corresponding field (ω^0 for V and ω for Φ^I) making the corresponding component invariant under $\mathbb{Z}_3$.

Next we consider the effect of introducing an O3$^{\pm}$-plane. Woldsheet parity Ω flips open string orientations and act on Chan-Paton indices as $N_n \leftrightarrow \bar{N}_{-n}$ where subscripts are always understood mod 3. This prescription leads to

$$\Omega: \qquad N_0 \leftrightarrow \bar{N}_0 \qquad N_1 \leftrightarrow \bar{N}_2 \tag{2.4}$$

The choices of O3$^{\pm}$-planes correspond to keep states with eigenvalues $\Omega = \pm 1$ and lead to symplectic or orthogonal gauge groups.[4]

We start by considering the O3^{-} case. Keeping $\Omega = -$ components from (2.3) one finds

$$\begin{aligned} V:&\ \ \frac{\mathbf{1}}{\mathbf{2}}\mathbf{N_0}(\mathbf{N_0}-\mathbf{1}) + \mathbf{N_1}\bar{\mathbf{N}}_\mathbf{1} \\ \Phi^I:&\ \ 3\times\left[\mathbf{N_0}\bar{\mathbf{N}}_\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{N_1}(\mathbf{N_1}-\mathbf{1})\right] \end{aligned} \tag{2.5}$$

This follows from (2.3) after identifying the mirror images $\bar{N}_0 = N_0$, $\bar{N}_2 = N_1$, and anti-symmetrizing the resulting block matrix. (2.5) describes the field content of a $\mathcal{N}=1$ SYM with gauge group $\mathrm{SO}(N_0)\times\mathrm{U}(N_1)$ and three chiral multiplets in the $\left[(\Box,\bar{\Box}) + (\bullet,\begin{smallmatrix}\Box\\ \Box\end{smallmatrix})\right]$.

For general N_0, N_1 the $\mathrm{U}(N_1)$ gauge theory is anomalous. The anomaly is a signal of the presence of a twisted RR tadpole [34, 35]. Focusing on a local description near the orientifold singularity one can relax the global tadpole cancellation condition [55, 56]. These models can be thought of as local descriptions of a more complicated Calabi Yau near a $\mathbb{Z}_3$ sigularity. Cancellation of the twisted RR tadpole can be written as [40]

$$\mathrm{tr}\,\gamma_{\theta,N} = -4 \qquad \Rightarrow \qquad N_0 = N_1 - 4 \tag{2.6}$$

and ensures the cancellation of the irreducible four-dimensional anomaly

$$I(F) \sim [-N_0 + (N_1 - 4)]\,\mathrm{tr}\,F^3 = 0 \tag{2.7}$$

Finally the running of the gauge coupling constants is governed by the β functions with one-loop coefficients[5]

$$\begin{aligned} \beta_0 &= 3\,\ell\left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{N_0}(\mathbf{N_0}-\mathbf{1})\right) - 3N_1\,\ell(\mathbf{N_0}) \\ &= \frac{3}{2}(N_0 - N_1 - 2) = -9 \qquad \text{(IR free)} \\ \beta_1 &= 3\,\ell(\mathbf{N_1}\bar{\mathbf{N}}_\mathbf{1}) - 3N_0\,\ell(\bar{\mathbf{N}}_\mathbf{1}) - 3\,\ell\left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{N_1}(\mathbf{N_1}-\mathbf{1})\right) \\ &= \frac{3}{2}(-N_0 + N_1 + 2) = +9 \qquad \text{(UV free)} \end{aligned} \tag{2.8}$$

[4]In the compact case, realized in terms of D9-branes and O9-plane on $T^6/\mathbb{Z}_3$, the orthogonal choice is dictated by global tadpole cancellation. Turning on a quantized NS-NS antisymmetric tensor [28, 41, 29] leads to symplectic groups.

[5]Here $\mathrm{tr}_{\mathcal{R}}T^aT^b = \ell(\mathcal{R})$, i.e. $\ell(\mathbf{N}) = \frac{1}{2}$, $\ell(\mathbf{N}\bar{\mathbf{N}}) = N$ and $\ell(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{N}(\mathbf{N}\pm\mathbf{1})) = \frac{1}{2}(N\pm 2)$.

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with β_n refering to the n^{th}-gauge group. The last equalities arise after imposing the anomaly cancellation (2.6). As expected, $\beta_0 + \beta_1 = 0$ since the ten-dimensional dilaton does not run.

The case $\Omega = +$ works in a similar way. The resulting $\mathcal{N} = 1$ quiver has gauge group $\mathrm{Sp}(N_0) \times \mathrm{U}(N_1)$ and three chiral multiplets in the $[(\square, \bar{\square}) + (\bullet, \square\square)]$. The $\mathrm{U}(N_1)$ is anomaly free for $N_0 = N_1 + 4$ and the one-loop β function coefficients are given by $\beta_0 = +9$ (UV free) and $\beta_1 = -9$ (IR free).

3. D(-1) Instantons

There are two sources of supersymmetric instanton corrections in the D3 brane gauge theory: D(-1)-instantons and Euclidean ED3-branes wrapping four cycles on $T^6/\mathbb{Z}_3$. Both are point-like configurations in the space-time and can be thought of as D(-1)-D3 and ED3-D3 bound states with four and eight directions with mixed Neumann-Dirichlet boundary conditions.

3.1 D3-D(-1) in flat space

Gauge instantons in SYM can be efficiently described in terms of D(-1)-branes living on the world-volume of D3-branes [57]. As before, we start from the $\mathcal{N} = 4$ case: a bound state of N D3 and k D(-1) branes in flat space. In this formalism, instanton moduli are described by the lowest energy modes of open strings with at least one end on the D(-1)-brane stack. The gauge theory dynamics, around the instanton background, can be described in terms of the $\mathrm{U}(k) \times \mathrm{U}(N)$ 0-dimensional matrix theory living on the D-instanton world-volume. In particular, the ADHM constraints [58], defining the moduli space of self-dual YM connections, follow from the F- and D- flatness condition in the matrix theory [57].

The instanton moduli space is given by the D(-1)D3 field content

$$\begin{aligned}
(a_\mu, \theta^A_\alpha, \chi_a, D^c, \bar{\theta}_{A\dot{\alpha}}) &\quad \mathbf{k\bar{k}} \\
(w_{\dot{\alpha}}, \nu^A) &\quad \mathbf{k\bar{N}} \\
(\bar{w}_{\dot{\alpha}}, \bar{\nu}^A) &\quad \mathbf{N\bar{k}}
\end{aligned} \tag{3.1}$$

with $\mu = 1, \ldots, 4$, $\alpha, \dot{\alpha} = 1, 2$ (vector/spinor indices of SO(4)), $a = 1, \ldots, 6$, $A = 1, \ldots, 4$ (vector/spinor indices of $\mathrm{SO}(6)_R$), $c = 1, \ldots, 3$. The matrices a_μ, χ_a describe the positions of the instanton in the directions parallel and perpendicular to the D3-brane respectively, w_α is given by the NS open D3-D(-1) string (instanton sizes and orientations), D^c are auxiliary fields and $\theta^A_\alpha, \bar{\theta}_{A\dot{\alpha}}, \nu^A$ are the fermionic superpartners.

The D3-D(-1) action can be written as [59]

$$S_{k,N} = \mathrm{tr}_k \left[\frac{1}{g_0^2} S_G + S_K + S_D \right] \tag{3.2}$$

with

$$\begin{aligned}
S_G &= -[\chi_a, \chi_b]^2 + i\bar{\theta}_{\dot{\alpha}A}[\chi^\dagger_{AB}, \bar{\theta}^{\dot{\alpha}}_B] - D^c D^c \\
S_K &= -[\chi_a, a_\mu]^2 + \chi_a \bar{w}^{\dot{\alpha}} w_{\dot{\alpha}} \chi_a - i\theta^{\alpha A}[\chi_{AB}\theta^B_\alpha] + 2i\chi_{AB}\bar{\nu}^A \nu^B \\
S_D &= i\left(-[a_{\alpha\dot{\alpha}}, \theta^{\alpha A}] + \bar{\nu}^A w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}}\nu^A\right)\bar{\theta}^{\dot{\alpha}}_A + D^c\left(\bar{w}\sigma^c w - i\bar{\eta}^c_{\mu\nu}[a^\mu, a^\nu]\right)
\end{aligned} \tag{3.3}$$

with $\chi_{AB} \equiv \frac{1}{2}\mathcal{T}^a_{AB}\chi_a$, $\mathcal{T}^a_{AB} = (\eta^c_{AB}, i\bar{\eta}^c_{AB})$ given in terms of the t'Hooft symbols and $g_0^2 = 4\pi(4\pi^2\alpha')^{-2}\, g_s$. The action (3.3) follows from the dimensional reduction of the D5-D9 action in six dimensions down to zero dimension. As a consequence, our subsequent results hold up to some computable non vanishing numerical constant.

In the presence of a v.e.v., for the six $\mathrm{U}(N)$ scalars φ_a in the D3-D3 open string sector, we must add to $S_{k,N}$

$$S_\varphi = \mathrm{tr}_k \left[\bar{w}^{\dot{\alpha}}(\varphi_a\varphi_a + 2\chi_a\varphi_a)w_{\dot{\alpha}} + 2i\bar{\nu}^A\varphi_{AB}\nu^B\right] \tag{3.4}$$

The multi-instanton partition function is

$$\mathcal{Z}_{k,N} = \int_{\mathfrak{M}} e^{-S_{k,N}-S_\varphi} = \frac{1}{\mathrm{Vol}\, U(k)} \int_{\mathfrak{M}} d\chi\, dD\, da\, d\theta\, d\bar{\theta} dw\, d\nu\, e^{-S_{k,N}-S_\varphi}$$

In the limit $g_0 \sim (\alpha')^{-1} \to \infty$, gravity decouples from the gauge theory and the contributions coming from S_G are suppressed. The fields $\bar{\theta}_{\dot{\alpha}A}$, D^c become Lagrange multipliers implementing the super ADHM constraints

$$\begin{aligned} \bar{\theta}_{\dot{\alpha}A}: &\ \ \bar{\nu}^A w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}}\nu^A - [a_{\alpha\dot{\alpha}}, \theta^{\alpha A}] = 0 \\ D^c: &\ \ \bar{w}\sigma^c w - i\bar{\eta}^c_{\mu\nu}[a^\mu, a^\nu] = 0 \end{aligned} \tag{3.5}$$

3.2 D(-1)-D3 at the $\mathbb{C}^3/\mathbb{Z}_3$-orientifold

Let us now consider in turn the Ω and then the $\mathbb{Z}_3$ projection.

The effect of introducing an O3$^\pm$-plane in the D(-1)-D3 system corresponds to keep open string states with eigenvalue $\Omega I = \pm$. Ω is the worldsheet parity and I is a reflection along the Neumann-Dirichlet directions of the Dp-O3 system [8]. On D(-1) string modes, I acts as a reflection in the spacetime plane

$$I: \ a_\mu \to -a_\mu \qquad \theta^A_\alpha \to -\theta^A_\alpha \tag{3.6}$$

leaving all other moduli invariant. In addition, consistency with the D3-O3 projection requires that the D(-1) strings are projected in the opposite way with respect to the D3-branes [60] . From the gauge theory point of view, this corresponds to the well known fact that $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ gauge instantons have ADHM constraints invariant under $\mathrm{Sp}(k)$ and $\mathrm{SO}(k)$ respectively.

We start by considering the O3$^-$ case. After the ΩI projection the surviving fields are

$$\begin{aligned} (a_\mu, \theta^A_\alpha) &\ \ \frac{\mathbf{1}}{\mathbf{2}}\mathbf{k}(\mathbf{k}-\mathbf{1}) \\ (D^c, \chi^I, \bar{\chi}_I, \bar{\theta}_{A\dot{\alpha}}) &\ \ \frac{\mathbf{1}}{\mathbf{2}}\mathbf{k}(\mathbf{k}+\mathbf{1}) \\ (w_{\dot{\alpha}}, \nu) &\ \ \mathbf{kN} \quad . \end{aligned} \tag{3.7}$$

Since we are dealing with a $\mathrm{SO}(N)$ gauge theory, the D^c moduli are projected in the adjoint of $\mathrm{Sp}(k)$. This is also the case for all the other moduli even under I, while the odd ones, (a_μ, θ^A_α), turn out to be antisymmetric.

Let us now consider the $\mathbb{Z}_3$ projection. Out of the six χ_a one can form three complex fields χ^I with eigenvalues ω under $\mathbb{Z}_3$ and their conjugate $\bar{\chi}_I$. To embed the $\mathbb{Z}_3$ projection into SU(4) we decompose the spinor index $A = (0, I)$, with $I = 1, \ldots, 3$ and the zeroth direction along the surviving $\mathcal{N} = 1$ supersymmetry. The D3 and D(-1) gauge groups, SO(N) and Sp(k), break into SO(N_0) $\times$ U(N_1) and Sp(k_0) $\times$ U(k_1) respectively. N_0 (k_0) is the number of fractional D3 (D(-1)) branes invariant under $\mathbb{Z}_3$ and N_1 (k_1) are the branes transforming with eigenvalue ω. More precisely, the projective embedding of the $\mathbb{Z}_3$ basic orbifold group element θ in the Chan-Paton group can be written

$$
\begin{aligned}
\gamma_{\theta^h,N} &= (\mathbf{1}_{N_0\times N_0}, \omega^h\, \mathbf{1}_{N_1\times \bar{N}_1}, \bar{\omega}^h\, \mathbf{1}_{\bar{N}_1\times N_1}) \\
\gamma_{\theta^h,k} &= (\mathbf{1}_{k_0\times k_0}, \omega^h\, \mathbf{1}_{k_1\times \bar{k}_1}, \bar{\omega}^h\, \mathbf{1}_{\bar{k}_1\times k_1})
\end{aligned} \tag{3.8}
$$

After projecting under $\mathbb{Z}_3$, the symmetric/antisymmetric matrices in (3.7) break into $k_m \times \bar{k}_n$, $k_m \times \bar{N}_n$ or $N_m \times \bar{k}_n$, each transforming with eigenvalue ω^{m-n}. In addition fields with up(down) index I transform like $\omega(\bar{\omega})$. Keeping only the invariant components one finds

$$
\begin{array}{rl}
(a_\mu; \theta^0_\alpha) & \frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_0}(\mathbf{k_0}-\mathbf{1}) + \mathbf{k_1}\bar{\mathbf{k}}_\mathbf{1} \\
\theta^I_\alpha & \frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_1}(\mathbf{k_1}-\mathbf{1}) + \mathbf{k_0}\bar{\mathbf{k}}_\mathbf{1} \\
(D^c; \bar{\theta}_{0\dot{\alpha}}) & \frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_0}(\mathbf{k_0}+\mathbf{1}) + \mathbf{k_1}\bar{\mathbf{k}}_\mathbf{1} \\
(\bar{\chi}_I; \bar{\theta}_{I\dot{\alpha}}) & \frac{\mathbf{1}}{\mathbf{2}}\bar{\mathbf{k}}_\mathbf{1}(\bar{\mathbf{k}}_\mathbf{1}+\mathbf{1}) + \mathbf{k_0}\mathbf{k_1} \\
\chi^I & \frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_1}(\mathbf{k_1}+\mathbf{1}) + \mathbf{k_0}\bar{\mathbf{k}}_\mathbf{1} \\
(w^{\dot{\alpha}}; \nu^0) & \mathbf{k_0}\mathbf{N_0} + \mathbf{k_1}\bar{\mathbf{N}}_\mathbf{1} + \bar{\mathbf{k}}_\mathbf{1}\mathbf{N_1} \\
\nu^I & \mathbf{k_0}\bar{\mathbf{N}}_\mathbf{1} + \bar{\mathbf{k}}_\mathbf{1}\mathbf{N_0} + \mathbf{k_1}\mathbf{N_1}
\end{array} \tag{3.9}
$$

Notice that the $\mathbb{Z}_3$ eigenvalues of the Chan-Paton indices in the r.h.s. of (3.9) compensate for those of the moduli in the l.h.s., making the field invariant under $\mathbb{Z}_3$. In addition (odd)even components under I are (anti)symmetrized ensuring the invariance under ΩI.

The multi-instanton action follows from that of $N = N_0 + 2N_1$ D3 branes and $k = k_0 + 2k_1$ D(-1) instanton in flat space (3.2) with U(N) and U(k) matrices restricted to the invariant blocks (3.9).

The results for $\mathrm{O3}^+$ can be read off from (3.9) by exchanging symmetric and antisymmetric representations.

4. ADS-like superpotential

4.1 D3-D(-1) one-loop vacuum amplitudes

Non-perturbative superpotentials can be computed from the instanton moduli space integral [11, 13, 18]

$$
S_W = e^{\langle \mathbb{1}\rangle_{\mathcal{D}} + \langle \mathbb{1}\rangle_{\mathcal{A}} + \langle \mathbb{1}\rangle_{\mathcal{M}}}\, \mu^{\beta_n k_n} \int_{\mathfrak{M}} e^{-S_{k,N} - S_\varphi} \tag{4.1}
$$

The integration is over the instanton moduli space, $\mathfrak{M}$, $\langle \mathbb{1} \rangle_{\mathcal{D}}$ is the disk amplitude and $\langle \mathbb{1} \rangle_{\mathcal{A},\mathcal{M}}$ are the one-loop vacuum amplitudes with at least one end on the D(-1)-instanton. The factor $\mu^{\beta_n k_n}$, μ being the energy scale, comes from the quadratic fluctuations around the instanton background and as we will see it combines with a similar contribution coming from the moduli measure to give a dimensionless S_W.

The terms in front of the integral in (4.1) combine into

$$S_W = \Lambda^{\beta_n k_n} \int_{\mathfrak{M}} e^{-S_{k,N} - S_\varphi} \tag{4.2}$$

with

$$\Lambda^{k_n \beta_n} = e^{2\pi i k_n \tau_n(\mu)} \mu^{\beta_n k_n} \qquad \tau_n(\mu) = \tau_n - \frac{\beta_n}{2\pi i} \ln \frac{\mu}{\mu_0} \tag{4.3}$$

the one-loop renormalization group invariant and the running coupling constant. τ_n refers to the complexified coupling constant of the n^{th} gauge group. μ_0 is a reference scale.

More precisely, the disk amplitude and one-loop amplitudes yield

$$\begin{aligned} e^{\langle \mathbb{1} \rangle_{\mathcal{D}}} &= e^{2\pi i k_n \tau_n} \qquad \tau_n = \frac{\theta_n}{2\pi} + \frac{4\pi i}{g_n^2} \\ e^{\langle \mathbb{1} \rangle_{\mathcal{A}} + \langle \mathbb{1} \rangle_{\mathcal{M}}} &= \left(\frac{\mu}{\mu_0} \right)^{-\beta_n \kappa_n} + \ldots \end{aligned} \tag{4.4}$$

with dots refering to threshold corrections that will not be considered here.

To verify (4.4) we should compute the following one-loop amplitudes

$$\begin{aligned} \langle \mathbb{1} \rangle_{\mathcal{A}} &= -\int_{\mu_0}^{\mu} \frac{dt}{t} \frac{1}{12} \mathrm{Tr}[(1 + (-)^F)(1 + \theta + \theta^2)\, q^{L_0 - a}] \\ &= -\int_{\mu_0}^{\mu} \frac{dt}{t} \mathcal{A}_{D(-1)D3} = -\mathcal{A}_{0,D(-1)D3} \ln \frac{\mu}{\mu_0} + \ldots \\ \langle \mathbb{1} \rangle_{\mathcal{M}} &= -\int_{\mu_0}^{\mu} \frac{dt}{t} \frac{1}{12} \mathrm{Tr}[\, \Omega I\, (1 + (-)^F)(1 + \theta + \theta^2)\, q^{L_0 - a}] \\ &= -\int_{\mu_0}^{\mu} \frac{dt}{t} \mathcal{M}_{D(-1)} = -\mathcal{M}_{0,D(-1)} \ln \frac{\mu}{\mu_0} + \ldots \end{aligned} \tag{4.5}$$

In the above formula μ enters as a UV regulator in the open string channel (see [61] for details) and $\mathcal{A}_0, \mathcal{M}_0$ are the massless contributions to the amplitudes.

We start by considering the $O3^-$ projection. It is important to notice that only the annulus with one end on the D(-1) and one on the D3 contributes to these amplitudes. In fact, D(-1)-D(-1) amplitudes cancel due to the Riemann identity. Finally one finds

$$\begin{aligned} \mathcal{A}_{D(-1),D3} &= \frac{4}{12} \mathrm{tr}\gamma_{\theta,k} \mathrm{tr}\gamma_{\theta,N} \sum_{\alpha,\beta} c_{\alpha\beta} \frac{\eta^3}{\vartheta[^\alpha_\beta]} \frac{\vartheta[^{\alpha+\frac{1}{2}}_{\beta}]^2}{\vartheta[^0_{\frac{1}{2}}]^2} \prod_{i=1}^{3} \frac{\vartheta[^{\alpha}_{\beta+h_i}]}{\hat{\vartheta}[^{\frac{1}{2}}_{\frac{1}{2}+h_i}]} \\ &= \frac{3}{2}(k_0 - k_1)(N_0 - N_1) + \ldots \\ \mathcal{M}_{D(-1)} &= \frac{2}{12} \mathrm{tr}\gamma_{\theta^2,k} \sum_{\alpha,\beta} c_{\alpha\beta} \frac{\eta^3}{\vartheta[^\alpha_\beta]} \frac{\vartheta[^{\alpha}_{\beta+\frac{1}{2}}]^2}{\hat{\vartheta}[^{\frac{1}{2}}_0]^2} \prod_{i=1}^{3} \frac{\vartheta[^{\alpha}_{\beta+h_i}]}{\hat{\vartheta}[^{\frac{1}{2}}_{\frac{1}{2}+h_i}]} \\ &= -3(k_0 - k_1) + \ldots \end{aligned} \tag{4.6}$$

The sum runs over the even spin structures and $c_{\alpha\beta} = (-)^{2(\alpha+\beta)}$. The term $\frac{\eta^3}{\vartheta[^\alpha_\beta]}$ comes from the (b,c) and (β,γ) ghosts while the extra five thetas in the numerator and denominator describe the contributions of the ten fermionic and bosonic worldsheet degrees of freedom. We adopt the shorthand notation $\hat{\vartheta}[^{\frac{1}{2}}_{h}] \equiv \vartheta[^{\frac{1}{2}}_{h}]/(2\cos\pi h)$ to describe the massive contribution of a periodic boson to the partition function. $h_i = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ denote the $\mathbb{Z}_3$-twists while the extra $\frac{1}{2}$-shifts in the annulus account for the D(-1)-D3 open string twist along Neuman-Dirichlet directions while $\frac{1}{2}$ twists in the Möbius come from the I-projection. In addition we used the fact that the contribution of the unprojected sector is zero after using the Riemann identity while that of the θ- and θ^2-projected sectors are identical explaining the overall factor of 2. The extra factor of 2 in the annulus comes from the two orientations of the string. The second line displays the massless contributions. We use the Chan Paton traces

$$\begin{aligned} \mathrm{tr}\gamma_{\mathbb{1},k} &= k_0 + 2k_1 \qquad & \mathrm{tr}\gamma_{\theta,k} &= k_0 - k_1 \\ \mathrm{tr}\gamma_{\mathbb{1},N} &= N_0 + 2N_1 \qquad & \mathrm{tr}\gamma_{\theta,N} &= N_0 - N_1 \end{aligned} \tag{4.7}$$

which follow from (2.1) and the first few terms in the theta expansions

$$\begin{aligned} \vartheta[^0_h] &= 1 + q^{\frac{1}{2}}\, 2\cos 2\pi h + \ldots \qquad \vartheta[^{\frac{1}{2}}_h] = q^{\frac{1}{8}}\, 2\cos\pi h + \ldots \\ \eta &= q^{\frac{1}{24}} + \ldots \end{aligned} \tag{4.8}$$

From (4.6) one finds

$$\mathcal{A}_0 + \mathcal{M}_0 = \frac{3}{2}(k_0 - k_1)(N_0 - N_1 - 2) = k_n \beta_n \tag{4.9}$$

with β_n the one-loop β coefficients given in (2.8). Plugging (4.9) into (4.5) results into (4.4). The fact that the β function coefficients are reproduced by the instanton vacuum amplitudes is a nice test of the instanton field content (3.9).

Now let us determine the dependence of the instanton measure on the string scale $M_s \sim \alpha'^{-1/2}$. The scaling of the various instanton moduli follows from (3.3):

$$\begin{aligned} D, g_0 &\sim M_s^2 \qquad \chi_a, \varphi_a \sim M_s \qquad w_{\dot{\alpha}}, a_\mu \sim M_s^{-1} \\ \nu^A,\ \theta^A_\alpha &\sim M_s^{-1/2} \qquad \bar{\theta}_{A\dot{\alpha}} \sim M_s^{3/2} \end{aligned} \tag{4.10}$$

Collecting from (3.9) the number of components of the various moduli entering in the instanton measure one finds[6]

$$\begin{aligned} \int_{\mathfrak{M}} e^{-S_{k,N} - S_\varphi} &\sim M_s^{-\beta_n k_n} \\ k_n \beta_n &= -2n_D - n_\chi + n_a + n_w + \frac{3}{2} n_{\bar{\theta}} - \frac{1}{2} n_\theta - \frac{1}{2} n_\nu \\ &= \frac{3}{2}(k_0 - k_1)(N_0 - N_1 - 2) \end{aligned} \tag{4.11}$$

[6]We recall that fermionic differentials scale as the inverse of the dimension of the fermion itself. This explains the extra minus sign in (4.11).

Notice that this factor precisely combines with that in (4.2) leading to a dimensionless S_W as expected. This simple dimensional analysis can be used to determine the form of the allowed ADS superpotentials in the gauge theory. A superpotential is generated if and only if the integral over the instanton moduli space reduces to an integral over x_0^μ describing the center of the instanton and θ_α its superpartner. More precisely

$$S_W = \Lambda^{k_n\beta_n} \int_{\mathfrak{M}} e^{-S_{k,N}-S_\varphi} = c \int d^4x_0 d^2\theta \, \frac{\Lambda^{k_n\beta_n}}{\varphi^{k_n\beta_n-3}} \tag{4.12}$$

where c is a numerical constant. Whether c is zero or not depends on the presence or not of extra fermionic zero modes besides θ. Notice that the power of φ is completely fixed requiring that S_W is dimensionless. The precise form of the superpotential requires the evaluation of the moduli space integral and will be the subject of the next section. The superpotential follows from (4.12) after promoting φ^I to the chiral superfield Φ^I and x_0, θ_α to the measure of the superspace

$$S_W = c \int d^4x d^2\theta \, \frac{\Lambda^{k_n\beta_n}}{\Phi^{k_n\beta_n-3}} \tag{4.13}$$

A superpotential of type (4.13) is generated whenever [63, 64, 16, 17]

$$\langle \lambda^2 \, \varphi^{k_n\beta_n-3} \rangle \neq 0 \tag{4.14}$$

Each scalar φ soaks two fermionic zero-modes and each gaugino λ one zero mode.[7] The condition (4.14) translates into

$$\dim\mathfrak{M}_F = 2k_n\beta_n - 4 \tag{4.15}$$

with $\dim\mathfrak{M}_F$ the fermionic dimension of the instanton super-moduli space. The number of fermionic zero modes can be read off from (3.9)

$$\begin{aligned}
\dim\mathfrak{M}_F &= n_\theta + n_\nu - n_{\bar\theta} \\
&= k_0(3N_1 + N_0 - 2) + k_1[2N_1 + 3(N_0 + N_1 - 2)] \\
&= k_0(4N_0 + 10) + k_1(8N_0 + 14)
\end{aligned} \tag{4.16}$$

where we used the fact that $\bar\theta_{\dot\alpha A}$ plays the role of a Lagrangian multiplier imposing the fermionic ADHM constraint and therefore subtracts degrees of freedom. The last line in (4.16) follows from using the anomaly cancellation condition $N_1 = N_0 + 4$. The result (4.16) is consistent with the Atiyah-Singer index theorem that states

$$\begin{aligned}
\dim\mathfrak{M}_F &= 2k_0 \left[\ell\left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{N_0}(\mathbf{N_0} - \mathbf{1})\right) + 3N_1\ell(\mathbf{N_0}) \right] \\
&\quad + 2k_1 \left[\ell(\mathbf{N_1}\bar{\mathbf{N}}_\mathbf{1}) + 3N_0\ell(\mathbf{N_1}) + 3\ell\left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{N_1}(\mathbf{N_1} - \mathbf{1})\right) \right] \\
&= k_0(3N_1 + N_0 - 2) + k_1[2N_1 + 3(N_0 + N_1 - 2)]
\end{aligned} \tag{4.17}$$

[7] This can be seen by explicitly solving the equations of motion of the gaugino and the φ-field in the instanton background [59]. In particular the source for the scalar field comes from the Yukawa coupling $L_{\rm Yuk} = g_{\rm YM}\varphi^\dagger\psi\lambda$ in the gauge theory action.

Combining (4.15) and (4.16) one finds

$$N_0 = \frac{k_1 - 7k_0 - 1}{k_0 + 2k_1} \tag{4.18}$$

One can easily see that the only non-negative solution for N_0 is

$$N_0 = 0 \qquad k_0 = 0 \qquad k_1 = 1$$

We conclude that in the class of $\mathrm{U}(N_0+4)\times\mathrm{SO}(N_0)$ SYM theories describing the low-energy dynamics of D3-branes on the $\mathbb{Z}_3$ orientifold only the $\mathrm{U}(4)$ theory with three chiral multiplets in the antisymmetric leads to an ADS-like superpotential generated by gauge instantons.

The counting can be easily repeated for the $\mathrm{Sp}(N_1+4)\times\mathrm{U}(N_1)$ cases by exchanging symmetric and antisymmetric representations in (3.9) as required by the presence of the O3$^+$-plane. The results are

$$\begin{aligned} k_n\beta_n &= 9(k_0 - k_1) \\ \dim\mathfrak{M}_F &= k_0(4N_1+6) + k_1(8N_1+18) \\ N_1 &= \frac{3k_0 - 9k_1 - 1}{k_0 + 2k_1} \end{aligned} \tag{4.19}$$

One can easily see that the only non-negative solution is

$$N_1 = 2 \qquad k_0 = 1 \qquad k_1 = 0$$

We conclude that in this class, only the gauge theory $\mathrm{Sp}(6)\times\mathrm{U}(2)$ with three chiral multiplets in the $(\square,\bar{\square})+(\bullet,\begin{smallmatrix}\square\\\square\end{smallmatrix})$ admits an ADS-like superpotential generated by instantons.

The aim of the rest of this section is to compute S_W. The integral (4.12) will be evaluated for the $\mathrm{Sp}(6)\times\mathrm{U}(2)$ and $\mathrm{U}(4)$ cases in the following.

4.2 $\mathrm{Sp}(6)\times\mathrm{U}(2)$ superpotential

We first consider the O3$^+$ case, i.e. the $\mathrm{Sp}(6)\times\mathrm{U}(2)$ gauge theory with three chiral multiplets in the $[(\mathbf{6},\bar{\mathbf{2}})+(\mathbf{1},\mathbf{3})]$. The instanton moduli is given by (3.9) after flipping symmetric/antisymmetric representations in order to deal with the symplectic projection. Plugging $k_0=1, k_1=0, N_0=6, N_1=2$ into (3.9) one finds the surviving fields

$$a_\mu,\ w^{\dot\alpha}_{u_0}, \theta^0_\alpha,\ \nu^{0u_0},\ \nu^{Iu_1} \tag{4.20}$$

with $u_0 = 1,..6$, and $u_1 = 1,2$. In this section, for notational convenience, subscripts and superscripts indices u_0, u_1 will be switched. in this section for notational convenience as we will momentarily see. In particular both $\bar\theta_{0\dot\alpha}$ and D^c are projected out (the D(-1) "gauge" group is $O(1)\approx\mathbb{Z}_2$ in this case) and therefore no ADHM constraint survives. The instanton action reduces then to

$$S = S_K + S_\varphi = w^{u_0}_{\dot\alpha}\bar\varphi_{Iu_0u_1}\varphi^{I\,u_1v_0}w^{\dot\alpha}_{v_0} + \nu^{Iu_1}\nu^{0u_0}\bar\varphi_{Iu_0u_1} \tag{4.21}$$

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Here and below we omit numerical coefficients that can be always reabsorbed at the end in the definition of the scale. The integrations over $w^{\dot\alpha}_{u_0}$, $\nu^0_{u_0}, \nu^I_{u_1}$ are gaussian and the final result, up to a non vanishing numerical constant, can be written as

$$\begin{aligned} S_W &= \Lambda^9 \int d^4a d^2\theta \, \frac{\det_{6\times 6}\left(\bar\varphi_{Iu_1,u_0}\right)}{\det_{6\times 6}\left(\bar\varphi_{Iu_1,u_0}\varphi^{Iu_1,v_0}\right)} \\ &= \int d^4a d^2\theta \, \frac{\Lambda^9}{\det_{6\times 6}\left(\varphi^{Iu_1,u_0}\right)} \end{aligned} \tag{4.22}$$

where we have exploited the possibility of combining I and u_1 in one 'bi-index' Iu_1 so as to get a range of six values. For the sake of simplicity we have dropped the subscript 0 denoting bare scalar fields. In the following scalar fields entering in formulae involving Λ will be always understood to be bare. The last step makes use of $\det(AB) = \det(A)\det(B)$.

4.3 U(4) superpotential

We now consider the $\mathrm{O3}^-$ case, i.e. the U(4) gauge theory with three chiral multiplets in the **6**. Setting $k_0 = 0, k_1 = 1, N_0 = 0, N_1 = 4$ in (3.9) the surviving fields can be written as

$$\begin{aligned} &\varphi^{I[uv]}_{(0)} \,,\ \bar\varphi_{I[uv](0)} \,,\ a_{\mu(0)} \,,\ \bar\chi_{I(-2)} \,,\ \chi^I_{(+2)} \,,\ D^c_{(0)} \,,\ w^{\dot\alpha}_{u(+1)} \,,\ \bar w^u_{\dot\alpha(-1)} \\ &\theta^0_{\alpha(0)} \,,\ \bar\theta_{0\dot\alpha(0)} \,,\ \bar\theta_{\dot\alpha I(-1)} \,;\ \nu^0_{u(+1)} \,,\ \bar\nu^{0u}_{(-1)} \,,\ \nu^{Iu}_{(+1)} \end{aligned} \tag{4.23}$$

with $u = 1, ..4$ and the charge q under $\mathrm{U}(1)_{k_1}$ is denoted in parentheses. Plugging into (3.3) (after taking $\alpha' \to 0$) one finds

$$S = S_B + S_F \tag{4.24}$$

where

$$\begin{aligned} S_F &= \left(\bar\nu^{0u} w_{u\dot\alpha} + \bar w^u_{\dot\alpha}\nu^0_u\right)\bar\theta^{\dot\alpha}_0 + \nu^{Iu}\, w_{u\dot\alpha}\,\bar\theta^{\dot\alpha}_I + \bar\chi_I\nu^0_u\nu^{Iu} + \nu^{Iu}\bar\varphi_{Iuv}\bar\nu^{0v} \\ S_B &= \bar w^u_{\dot\alpha}\bar\varphi_{Iuw}\varphi^{Iwv}w^{\dot\alpha}_v + \varphi^{Iuv}w^{\dot\alpha}_u w_{v\dot\alpha}\bar\chi_I + \bar\varphi_{Iuv}\bar w^{u\dot\alpha}\bar w^v_{\dot\alpha}\chi^I + \bar w^{u\dot\alpha}w_{u\dot\alpha}\bar\chi_I\chi^I + D^c\,\bar w\sigma_c w \end{aligned} \tag{4.25}$$

As before we omit numerical coefficients. The integral over D^c leads to a δ function on the ADHM constraints

$$\int d^8w d^8\bar w \delta^3(\bar w\sigma_c w) = \int d\rho\, \rho^9 d^{12}\mathcal{U} \tag{4.26}$$

In the r.h.s of (4.26) we have solved the ADHM constraints in favour of w and $\mathcal{U}$ defined by

$$w_{u\dot\alpha} = \rho\,\mathcal{U}_{u\dot\alpha} \qquad \bar w^{u\dot\alpha} = \rho\,\bar{\mathcal{U}}^{u\dot\alpha} \qquad \bar{\mathcal{U}}^{u\dot\alpha}\mathcal{U}_{u\dot\beta} = \delta^{\dot\alpha}_{\dot\beta} \tag{4.27}$$

The coset representatives $\mathcal{U}_{\dot\alpha u}$ parameterizes the SU(4)/SU(2) orientations of the instanton inside the gauge group. The fermionic integrations lead to the determinant

$$\Delta_{\mathrm{F}} = \rho^8\, \epsilon^{u_1u_2u_3u_4}\epsilon^{v_1v_2u_5u_6}\epsilon^{v_3v_4v_5v_6} X_{u_1v_1u_2v_2}\, X_{u_3v_3u_4v_4}\, Y_{u_5v_5}\, Y_{u_6v_6} \tag{4.28}$$

with

$$\begin{aligned} X_{u_1v_1u_2v_2} &= \epsilon^{I_1I_2I_3}\bar\chi_{I_1}\bar\varphi_{I_2u_1v_1}\bar\varphi_{I_3u_2v_2} \\ Y_{uv} &= \mathcal{U}^{\dot\alpha}_u\mathcal{U}_{\dot\alpha u} \end{aligned} \tag{4.29}$$

The bosonic integrals are more involved. For arbitrary choices of the scalar VEV's $\bar{\varphi}_I$ and φ^I, even along the flat directions of the potential, the integration over $\mathcal{U}$ represents a challenging if not a prohibitive task. Fortunately choosing $\varphi^{Iuv} = \varphi\eta^{Iuv}$, $\bar{\varphi}_{Iuv} = \bar{\varphi}\eta^{Iuv}$, the full φ-dependence can be factorized. $SU(4)$ gauge and $SU(3)$ 'flavor' invariance can then be used to recover the full answer. After the rescaling

$$\rho^2 \to \rho^2/(\varphi\bar{\varphi}) \qquad \chi^I \to \varphi\chi^I \qquad \bar{\chi}_I \to \bar{\varphi}\bar{\chi}_I \tag{4.30}$$

The integral becomes

$$S_W = \Lambda^9\, I \int d^4x_0 d^2\theta\, \frac{1}{\varphi^6} \tag{4.31}$$

with I the φ-independent integral

$$\begin{aligned} I &= \int d\rho\rho^9\, d^{12}\mathcal{U}\, d^3\chi d^3\bar{\chi}\, \Delta_{\mathrm{F}}\, e^{-\tilde{S}_B} \\ \tilde{S}_B &= -\rho^2(1 + \eta^{Iuv}Y_{uv}\chi_I + \bar{\eta}_{Iuv}\bar{Y}^{uv}\chi^I + \bar{\chi}_I\chi^I) \end{aligned} \tag{4.32}$$

and Δ_{F} given again by (4.28) but now in terms of

$$X_{u_1v_1u_2v_2} = \epsilon^{I_1I_2I_3}\bar{\chi}_{I_1}\bar{\eta}_{I_2u_1v_1}\bar{\eta}_{I_3u_2v_2}$$

Finally one can restore the gauge covariance of (4.31) by noticing that there is a unique $\mathrm{SU}(4)_c \times \mathrm{SU}(3)_f$ singlet in the symmetric tensor of six φ^I

$$\det{}_{3\times3}[\epsilon_{u_1..u_4}\varphi^{Iu_1u_2}\varphi^{Ju_3u_4}]$$

Therefore one can replace φ^6 in (4.31) by this singlet. The superpotential follows after replacing $\varphi^I \to \Phi^I$

$$S_W = c\int d^4x d^2\theta\, \frac{\Lambda^9}{\det_{3\times3}[\epsilon_{u_1..u_4}\Phi^{Iu_1u_2}\Phi^{Ju_3u_4}]} \tag{4.33}$$

where c is a computable non-zero numerical coefficient.

5. ED3-instantons

Let us now consider the ED3-D3 system. We restrict ourselves to the compact case $T^6/\mathbb{Z}_3$ and consider the ED3 fractional instanton wrapping a four-cycle $\mathcal{C}_n$ inside $T^6/\mathbb{Z}_3$. We start by considering the O3$^-$-orientifold projection. The zero modes of the Yang-Mills fields in the instanton background can be described as before in terms of open strings with at least one end on the ED3. Open strings connecting ED3 and D3 branes have 8 Neumann-Dirichlet directions therefore the zero-mode dynamics of the ED3-D3 system is equivalent to that of the D7-D(-1) bound state. The instanton action can be found starting from that of the $\mathcal{N} = (8, 0)$ sigma model describing the low energy dynamics of a D1-D9 bound state in type I [65] reduced down to zero dimensions. In flat space the D(-1)-D7 action reads

$$S = \mathrm{tr}_k\left[\frac{1}{g_0^2}\, S_g + S_K + S_D\right] \tag{5.1}$$

with

$$\begin{aligned}
S_g &= -[\chi,\bar\chi]^2 + \tilde\Theta^{\dot a}\chi\tilde\Theta^{\dot a} + D^c D^c \\
S_K &= -[\chi, X_m][\bar\chi, X_m] + \Theta^a\bar\chi\Theta^a + \nu(\chi+\varphi)\nu \\
S_D &= \tilde\Theta^{\dot a} X_m \Gamma^m_{\dot a a}\Theta^a + D^c\hat\Gamma^c_{mn}[X_m, X_n]
\end{aligned} \tag{5.2}$$

with $m = 1,\ldots,8_v$, $a = 1,\ldots,8_s$, $\dot a = 1,\ldots,8_c$, $c = 1,,\ldots,7$. We denote by $\varphi = m_I(\mathcal{C}_n)\varphi^I$, the gauge scalar parametrizing the position of the D3-brane along the direction perpendicular to the 4-cycle $\mathcal{C}_n$. Here $\Gamma^m_{\dot a a}$, $\hat\Gamma^c_{mn}$ are gamma matrices of SO(8) and SO(7) respectively. The introduction of the auxiliary fields D^c has broken the manifest SO(8) invariance of the action that will be further broken by the $\mathbb{Z}_3$-projection. In (5.2), X_m and $\chi,\bar\chi$ describe the position of the D(-1)-instanton in the directions longitudinal and perpendicular to the D7-brane respectively while $\Theta^a, \tilde\Theta^{\dot a}$ are the fermionic superpartners grouped according to the their chirality along the Dirichlet-Dirichlet χ-plane. Unlike the D(-1)-D3 case, in the case of 8 Neumann-Dirichlet directions Ω acts in the same way on the D(-1) and D7 Chan-Paton indices. This implies that D^c transform in the adjoint of SO(k) if we take the D7 gauge symmetry to be SO(N). In addition I acts as

$$I: \qquad X_m \to -X_m \qquad \Theta^a \to -\Theta^a \tag{5.3}$$

Fields with eigenvalues $\Omega I = -$ are then in the following representations of SO(k)$\times$SO(N)

$$\begin{aligned}
(\chi,\bar\chi,D^c,\tilde\Theta^{\dot a}) &\quad \tfrac{\mathbf{1}}{\mathbf{2}}\mathbf{k}(\mathbf{k}-\mathbf{1}) \\
(X_m,\Theta^a) &\quad \tfrac{\mathbf{1}}{\mathbf{2}}\mathbf{k}(\mathbf{k}+\mathbf{1}) \\
\nu &\quad \mathbf{kN}
\end{aligned} \tag{5.4}$$

Fields even under I transform in the adjoint of SO(k) while odd fields tranform in the symmetric representation. For $k = 1$, $N = 32$ the D(-1)-D7 system or equivalently the D1-D9 bound state describes the S-dual version of the fundamental heterotic string on T^2. $k > 1$ bound states correspond to multiple windings of the heterotic string [65].

The field D^c implements the one-real D and three complex F flatness conditions

$$V = -\frac{1}{g_0^2}\sum_{c=1}^{7} D^c D^c = -g_0^2 \sum_{m,n=1}^{8}[X_m, X_n]^2 = 0 \tag{5.5}$$

with

$$D^c = -\frac{1}{2}g_0^2\Gamma^c_{mn}[X_m, X_n] \tag{5.6}$$

An explicit choice of Γ matrices in $D = 7$ is given by ($a = 1, 2, 3$)

$$\Gamma^a_{8\times8} = i\sigma_1\otimes\eta^a_{4\times4} \quad \Gamma^{a+3}_{8\times8} = i\sigma_3\otimes\bar\eta^a_{4\times4} \quad \Gamma^7_{8\times8} = i\sigma_2\otimes\mathbf{1}_{4\times4} \tag{5.7}$$

As in section 3, $\mathbb{Z}_3$ acts on both spacetime and Chan-Paton indices. Chan-Paton indices decompose as $N \to N_0 + N_1 + \bar N_1$ and $k \to k_0 + k_1 + \bar k_1$. Spacetime indices on the other hand decompose as

$$\begin{aligned} 8_v &= 4 + 2_\omega + 2_{\bar\omega} \\ 8_s &= 2 + 2_\omega + 4_{\bar\omega} \\ 8_c &= 2 + 2_{\bar\omega} + 4_\omega \\ 7 &= 3 + 2_\omega + 2_{\bar\omega} \end{aligned} \tag{5.8}$$

In addition χ, ν transform with eigenvalue ω under $\mathbb{Z}_3$. Combining with (5.4) one finds the $\mathbb{Z}_3$-invariant components

$$\begin{array}{rl} \chi, \bar\chi & \frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_1}(\mathbf{k_1}-\mathbf{1}) + \mathbf{k_0}\bar{\mathbf{k}}_\mathbf{1} + \text{h.c.} \\ D^c & 3\left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_0}(\mathbf{k_0}-\mathbf{1}) + \mathbf{k_1}\bar{\mathbf{k}}_\mathbf{1}\right) + 2\left[\frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_1}(\mathbf{k_1}-\mathbf{1}) + \mathbf{k_0}\bar{\mathbf{k}}_\mathbf{1} + \text{h.c.}\right] \\ \tilde\Theta^{\dot a} & 2\left[\frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_0}(\mathbf{k_0}-\mathbf{1}) + \mathbf{k_1}\bar{\mathbf{k}}_\mathbf{1}\right] + 2\left[\frac{\mathbf{1}}{\mathbf{2}}\bar{\mathbf{k}}_\mathbf{1}(\bar{\mathbf{k}}_\mathbf{1}-\mathbf{1}) + \mathbf{k_0}\mathbf{k_1})\right] \\ & +4\left[\left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_1}(\mathbf{k_1}-\mathbf{1}) + \mathbf{k_0}\bar{\mathbf{k}}_\mathbf{1}\right)\right] \\ X_m & 4\left[\frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_0}(\mathbf{k_0}+\mathbf{1}) + \mathbf{k_1}\bar{\mathbf{k}}_\mathbf{1}\right] + 2\left[\frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_1}(\mathbf{k_1}+\mathbf{1}) + \mathbf{k_0}\bar{\mathbf{k}}_\mathbf{1} + \text{h.c.}\right] \\ \Theta^a & 2\left[\frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_0}(\mathbf{k_0}+\mathbf{1}) + \mathbf{k_1}\bar{\mathbf{k}}_\mathbf{1}\right] + 2\left[\frac{\mathbf{1}}{\mathbf{2}}\mathbf{k_1}(\mathbf{k_1}+\mathbf{1}) + \mathbf{k_0}\bar{\mathbf{k}}_\mathbf{1})\right] \\ & +4\left[\left(\frac{\mathbf{1}}{\mathbf{2}}\bar{\mathbf{k}}_\mathbf{1}(\bar{\mathbf{k}}_\mathbf{1}+\mathbf{1}) + \mathbf{k_0}\mathbf{k_1}\right)\right] \\ \nu & \mathbf{k_0}\bar{\mathbf{N}}_\mathbf{1} + \mathbf{k_1}\mathbf{N_1} + \bar{\mathbf{k}}_\mathbf{1}\mathbf{N_0} \end{array} \tag{5.9}$$

5.1 D3-ED3 one-loop vacuum amplitudes

ED3 generated superpotentials can be computed following the same steps as in section 4.1. The disk amplitude can be written as

$$e^{\langle \mathbb{1} \rangle_{\mathcal{D}}} = e^{2\pi i k_n \tilde\tau_n} \qquad \tilde\tau_n = i\,\frac{4\pi V_4(\mathcal{C}_n)}{g_n^2\,\alpha'^{-2}} + \int_{\mathcal{C}_n} (C_4 + C_0 \wedge R \wedge R) \tag{5.10}$$

$\tilde\tau_n$ describes the coupling of closed string moduli to the ED3 instanton wrapping the 4-cycle $\mathcal{C}_n$ with volume $V_4(\mathcal{C}_n)$. We remark that closed string states in the $\mathbb{Z}_3$-twisted sectors flow in the ED3-ED3 cylinder amplitude and therefore $\tilde\tau_n$ is function of both untwisted and twisted closed twisted moduli. This is not surprising since the volume of the cycle depends also on the volume of the exceptional cycles that the ED3 wraps.

The annulus and Möbius amplitudes are given by

$$\begin{aligned}
\mathcal{A}_{ED3,D3} &= \frac{2}{12}\sum_{\alpha,\beta} c_{\alpha\beta}\frac{\eta^3}{\vartheta\left[{}^{\alpha}_{\beta}\right]}\frac{\vartheta\left[{}^{\alpha+\frac{1}{2}}_{\beta}\right]^2}{\vartheta\left[{}^{0}_{\frac{1}{2}}\right]^2}\times \\
&\times\left(2\mathrm{tr}\gamma_{\theta,k}\mathrm{tr}\gamma_{\theta,N}\frac{\vartheta\left[{}^{\alpha+\frac{1}{2}}_{\beta+h_1}\right]^2}{\vartheta\left[{}^{0}_{\frac{1}{2}+h_1}\right]^2}\frac{\vartheta\left[{}^{\alpha}_{\beta-2h_1}\right]}{\hat{\vartheta}\left[{}^{\frac{1}{2}}_{\frac{1}{2}-2h_1}\right]}+\mathrm{tr}\gamma_{\mathbb{1},k}\mathrm{tr}\gamma_{\mathbb{1},N}\frac{\vartheta\left[{}^{\alpha+\frac{1}{2}}_{\beta}\right]^2}{\vartheta\left[{}^{0}_{\frac{1}{2}}\right]^2}\frac{\vartheta\left[{}^{\alpha}_{\beta}\right]}{\eta^3}\right) \\
&= -\frac{1}{2}k_0N_1-\frac{1}{2}k_1(N_0+N_1)+\ldots \\
\mathcal{M}_{ED3} &= -\frac{1}{12}\sum_{\alpha,\beta} c_{\alpha\beta}\frac{\eta^3}{\vartheta\left[{}^{\alpha}_{\beta}\right]}\frac{\vartheta\left[{}^{\alpha}_{\beta+\frac{1}{2}}\right]^2}{\hat{\vartheta}\left[{}^{\frac{1}{2}}_{0}\right]^2}\times \\
&\times\left(2\mathrm{tr}\gamma_{\theta^2,k}\frac{\vartheta\left[{}^{\alpha}_{\beta+\frac{1}{2}+h_1}\right]^2}{\hat{\vartheta}\left[{}^{\frac{1}{2}}_{0+h_1}\right]^2}\frac{\vartheta\left[{}^{\alpha}_{\beta-2h_1}\right]}{\hat{\vartheta}\left[{}^{\frac{1}{2}}_{\frac{1}{2}-2h_1}\right]}+\mathrm{tr}\gamma_{\mathbb{1},k}\frac{\vartheta\left[{}^{\alpha}_{\beta+\frac{1}{2}}\right]^2}{\hat{\vartheta}\left[{}^{\frac{1}{2}}_{0}\right]^2}\frac{\vartheta\left[{}^{\alpha}_{\beta}\right]}{\eta^3}\right) \\
&= 3k_0+k_1+\ldots
\end{aligned} \tag{5.11}$$

The origin of the various contributions is the same as those in the D(-1)-D3 system. Now the D3-ED3 open strings have 8 Neumann-Dirichlet directions explaining the extra $\frac{1}{2}$ twists in the annulus amplitude. On the other side, the I projection accounts for the $\frac{1}{2}$-shift in the Möbius amplitude. Notice that unlike the D(-1)-D3 case, the unprojected amplitude $\mathrm{tr}\mathbb{1}$ now gives a non-trivial contribution.

Collecting the contributions from (5.11) one finds

$$\tilde{\Lambda}^{k_n b_n} = \mu^{k_n b_n}\, e^{\langle\mathbb{1}\rangle_{\mathcal{D}}+\langle\mathbb{1}\rangle_{\mathcal{A}}+\langle\mathbb{1}\rangle_{\mathcal{M}}} = \mu^{k_n b_n}\, e^{2\pi i k_n \tilde{\tau}_n(\mu)} \tag{5.12}$$

with

$$\tilde{\tau}_n(\mu) = \tilde{\tau}_n - \frac{b_n}{2\pi i}\ln\frac{\mu}{\mu_0} \tag{5.13}$$

and

$$\mathcal{A}_0+\mathcal{M}_0 = k_n b_n = \frac{1}{2}k_0(6-N_1)+\frac{1}{2}k_1(2-N_0-N_1) \tag{5.14}$$

The interpretation of the b_n as the one-loop β function coefficients of the $\tilde{\tau}_n$ coupling, though tantalizing, is not clear to us. We will now check that $k_n b_n$ reproduces the right scale dependence of the instanton measure. The scaling of the various instanton moduli follows from (5.2):

$$\begin{aligned}
&D, g_0 \sim M_s^2 \qquad \chi,\bar{\chi},\varphi \sim M_s \qquad X_m \sim M_s^{-1} \\
&\nu,\ \Theta^a \sim M_s^{-1/2} \qquad \tilde{\Theta}^{\dot{a}} \sim M_s^{3/2}
\end{aligned} \tag{5.15}$$

Collecting from (5.9) the number of degrees of freedom entering in the instanton supermoduli measure one finds

$$\begin{aligned}\int_{\mathfrak{M}} e^{-S_{k,N}} &\sim M_s^{-k_n b_n} \\ k_n b_n &= -2n_D - n_\chi + n_X + \frac{3}{2}n_{\tilde{\Theta}} - \frac{1}{2}n_\Theta - \frac{1}{2}n_\nu \\ &= \frac{1}{2}k_0(6-N_1) + \frac{1}{2}k_1(2-N_0-N_1)\end{aligned} \tag{5.16}$$

As in the previous case we write the instanton generated superpotential as the moduli space integral

$$S_W = \tilde{\Lambda}^{k_n b_n} \int_{\mathfrak{M}} e^{-S_{k,N}-S_\varphi} = \int d^4x_0 d^2\theta \, \tilde{\Lambda}^{k_n b_n} \, \varphi^{-k_n b_n + 3} \tag{5.17}$$

After promoting $\varphi \to \Phi$ and x_0, θ_α to the measure of the superspace one finds the ED3 generated superpotential

$$S_W = \int d^4x d^2\theta \, \tilde{\Lambda}^{k_n b_n} \, \Phi^{-k_n b_n + 3} \tag{5.18}$$

The main difference with respect to the D(-1) instantons is that now φ enters into S_φ (5.2) only through the coupling to the ν-fermions. This implies that in order to get a non zero result from the fermionic integral in (5.17) only the ν's and the two fermionic zero modes $\theta_\alpha \in \Theta^a$ should survive the orientifold projections. From (5.9) one can easily see that this implies $k_0 = 1$, $k_1 = 0$. The same counting shows that no solutions are allowed in the Sp(N) case.

5.2 The superpotential

Here we evaluate the instanton moduli space integral for the SO(N_0) $\times$ U(N_1) case. From our analysis above the relevant cases are $k_0 = 1$, $k_1 = 0$.

The surviving fields in (5.9) are

$$\theta_\alpha \in \Theta^a \qquad x_0^\mu \in X_m \qquad \nu_u \tag{5.19}$$

with $u = 1, \dots N_1$. The instanton action reduces to

$$S = \nu_u \varphi^{uv} \nu_v \tag{5.20}$$

The superpotential is then given by the integral

$$S_W = \tilde{\Lambda}^{-\frac{N_1}{2}+3} \int d^4x \, d^2\theta \, d^{N_1}\nu \, e^{-\nu\varphi\nu} \tag{5.21}$$

After integration over ν and lifting $\varphi \to \Phi$ to the superfield one finds

$$S_W = c \, \tilde{\Lambda}^{-\frac{N_1}{2}+3} \int d^4x \, d^2\theta \, \epsilon_{u_1 \dots . u_{N_1}} \Phi^{u_1 u_2} \Phi^{u_3 u_4} \dots \Phi^{u_{N_1-1} u_{N_1}} \tag{5.22}$$

where c is a non vanishing numerical constant. Notice that the result is non-trivial only when N_1 is even. The superpotentials (5.22) are non-renormalizable for $N_1 > 6$ and grow for large vacuum expectation values where the low energy approximation breaks down. The only exceptions are

$$
\begin{aligned}
&\text{Majorana masses} && \mathrm{U}(4) + 3\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \\
&\text{Yukawa couplings} && \mathrm{SO}(2) \times \mathrm{U}(6) + 3\,(\Box,\bar{\Box}) + 3\,(\bullet,\begin{smallmatrix}\Box\\\Box\end{smallmatrix})
\end{aligned} \tag{5.23}
$$

Notice that both instanton generated Yukawa couplings involve only the matter in the antisymmetric representation.

6. ADS superpotentials: a general analysis

Here we consider a general $\mathcal{N}=1$ gauge theory with gauge group $\mathrm{U}(N)$ and a n_{Adj}, $n_f/\bar{n}_f$, $n_S/\bar{n}_S$, $n_A/\bar{n}_A$ number of chiral multiplets in the adjoint, fundamental, symmetric and anti-symmetric representations (and their complex conjugates) respectively.

The cubic chiral anomaly, one-loop β function and number of fermionic zero modes in the instanton background of the gauge theory can be written as

$$
\begin{aligned}
I_{\mathrm{anom}} &= n_{f-} + n_{S-}(N+4) + n_{A-}(N-4) = 0 \\
\beta_{1-\mathrm{loop}} &= 3N - N n_{\mathrm{Adj}} - \frac{1}{2} n_{f+} - \frac{1}{2} n_{S+}(N+2) - \frac{1}{2} n_{A+}(N-2) \\
\dim \mathfrak{M}_F &= k\,[2N + 2N n_{\mathrm{Adj}} + n_{f+} + n_{S+}(N+2) + n_{A+}(N-2)]
\end{aligned} \tag{6.1}
$$

with

$$
n_{f\pm} = n_f \pm \bar{n}_f \qquad n_{S\pm} = n_S \pm \bar{n}_S \qquad n_{A\pm} = n_A \pm \bar{n}_A \tag{6.2}
$$

The condition for an Affleck, Dine and Seiberg like superpotential [16, 17] to be generated was determined in section 4.1 to be

$$
\dim \mathfrak{M}_F = 2k\beta - 4 \tag{6.3}
$$

Combining (6.1) and (6.3) one finds

$$
\begin{aligned}
\beta_{1-\mathrm{loop}} &= 2N + \frac{1}{k} \\
n_{f-} &= -n_{S-}(N+4) - n_{A-}(N-4) \\
n_{f+} &= 2N - \frac{2}{k} - 2N n_{\mathrm{Adj}} - n_{S+}(N+2) - n_{A+}(N-2)
\end{aligned} \tag{6.4}
$$

Remarkably the β function in a theory admitting an instanton generated superpotential depends only on the rank of the gauge group. A simple inspection shows that a superpotential is generated only for $k=1$ and $n_{\mathrm{Adj}}=0$. The complete list follows from a scan of any choice of $n_{S\pm}$,$n_{A\pm}$ such that $n_+ \geq |n_-|$ and $n_+ \geq 0$. One finds

$$
\begin{aligned}
&\mathrm{U}(N) + N_f\,(\Box + \bar{\Box}) && N_f \leq N-1 \\
&\mathrm{U}(N) + \begin{smallmatrix}\Box\\\Box\end{smallmatrix} + (N-4)\bar{\Box} + N_f\,(\Box + \bar{\Box}) && N_f \leq 2 \\
&\mathrm{U}(4) + 2\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + N_f\,(\Box + \bar{\Box}) && N_f \leq 1 \\
&\mathrm{U}(4) + 3\begin{smallmatrix}\Box\\\Box\end{smallmatrix} && \\
&\mathrm{U}(5) + 2\begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 2\bar{\Box} &&
\end{aligned} \tag{6.5}
$$

The inequalities are saturated for gauge theories satisfying (6.3) and (6.4), while the lower cases are found by decoupling quark-antiquark pairs via mass deformations.

The generalization to $\mathrm{SO}(N)/\,\mathrm{Sp}(N)$ gauge groups is straightforward. In these cases there is no restriction coming from anomalies since representations are real. The β function and the number of fermionic zero modes in the instanton background are given by

$$\begin{aligned}\beta_{1-\mathrm{loop}} &= \frac{3}{2}(N \pm 2) - \frac{1}{2}n_f - \frac{1}{2}n_S(N+2) - \frac{1}{2}n_A(N-2) \\ \dim\mathcal{M}_F &= k\,[N \pm 2 + n_f + n_S(N+2) + n_A(N-2)]\end{aligned}$$

with upper sign for $\mathrm{Sp}(N)$ and lower sign for $\mathrm{SO}(N)$ gauge groups. Imposing (6.3) one finds

$$\begin{aligned}\beta_{1-\mathrm{loop}} &= N \pm 2 + \frac{1}{k} \\ n_f &= N \pm 2 - \frac{2}{k} - n_S(N+2) - n_A(N-2)\end{aligned} \tag{6.6}$$

The list of solutions is even shorter

$$\begin{array}{lll} \mathrm{SO}(N) + N_f\square & N_f \leq N-3 & k=2 \\ \mathrm{Sp}(N) + N_f\square & N_f \leq N & k=1 \\ \mathrm{Sp}(N) + \begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix} + 2\square & & k=1 \end{array} \tag{6.7}$$

Notice that $k = 1$, respectively $k = 2$, are the basic instantons in $\mathrm{Sp}(N)$, respectively $\mathrm{SO}(N)$, since the instanton symmetry groups are in these cases $\mathrm{SO}(k)$, respectively $\mathrm{Sp}(k)$.

7. Conclusions

In the present paper, we have given a detailed microscopic derivation of non-perturbative superpotentials for chiral $\mathcal{N} = 1$ D3-brane gauge theories living at $\mathbb{Z}_3$-orientifold singularities. We considered both unoriented projections leading to $\mathrm{SO}(N_1 - 4) \times \mathrm{U}(N_1)$ and $\mathrm{Sp}(N_1 + 4) \times \mathrm{U}(N_1)$ gauge theories with three generations of chiral matter in the representations $(\square,\bar{\square}) + (\bullet,\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix})$ and $(\square,\bar{\square}) + (\bullet,\square\square)$ respectively.

The $\mathrm{U}(4)$ case was studied in details in [49] and describes the local physics of type I theory near the origin of $T^6/\mathbb{Z}_3$ with $\mathrm{SO}(8) \times \mathrm{U}(12)$ gauge group broken by Wilson lines. In the present T-dual setting, there are two sources of non-perturbative effects: D(-1) and ED3 instantons. The former realize the standard gauge instantons and lead to Affleck, Dine and Seiberg like superpotentials. The latter lead to Majorana masses or non-renormalizable superpotentials and were ignored till very recently [18 − 23, 15, 49].

Our explicit instanton computations confirm the form of ADS and stringy superpotentials proposed in [49] on the basis of holomorphicity, dimensional analysis $\mathrm{U}(1)$ anomaly and flavour symmetry. We show that ADS superpotentials are generated only for the $\mathrm{U}(4)$ and $\mathrm{Sp}(6) \times \mathrm{U}(2)$ gauge theories in the $\mathbb{Z}_3$-orientifold list. The precise form of the superpotential is derived from an integration over the instanton super-moduli space. Like in [15], the β function running of gauge couplings are reproduced from vacuum amplitudes given in terms of annulus and Möbius amplitudes ending on the instantons. The same analysis is performed for "stringy instantons" generated by Euclidean ED3-branes (dual to ED1-strings in type I theory) wrapping holomorphic four-cycles on $T^6/\mathbb{Z}_3$. A detailed microscopic analysis of the multi-instanton super-moduli space encompasses massless open string states with a least one end on the ED3-instanton. We show the generation of Majorana mass terms for the open string chiral multiplets in the $\mathrm{U}(4)$ case, Yukwa couplings for the $\mathrm{SO}(2) \times \mathrm{U}(6)$ gauge theory and non-renormalizable superpotentials for $\mathrm{SO}(N_0) \times \mathrm{U}(N_0+4)$ gauge theories. The field theory interpretation of the β function coefficients generated by the one-loop vacuum amplitudes for open strings ending on the ED3-instantons is one of the most interesting open question left by our instanton super-moduli space analysis. As previously observed, the invariance under anomalous $\mathrm{U}(1)$'s results from a detailed balance between the charges of the open strings involved and the axionic shift of a closed string R-R modulus from the twisted sector.

Our present analysis has some analogies with the recent ones performed in [18 – 23, 15] which have focussed on ED2-branes at D6-brane intersections. As stressed in [49], one immediate advantage of the viewpoint advocated here is the consistency of the local description. Indeed, imposing twisted tadpole cancellation [34, 35] the models presented here and all closely related settings of D-branes at singularities (not necessarily of the $\mathbb{Z}_n$ kind) give rise to anomaly free theories, while this is not necessarily the case for the 'local' models with intersecting D-branes. We can envisage the possibility of extending our analysis to other $\mathbb{Z}_n$ singularities [55, 56] or even to Gepner models [66 – 68] where many if not all ingredients, such as the brane actions from gauge kinetic functions including one-loop threshold effects [69 – 72], are available.

In the present paper we have not addressed phenomenological implications of the stringy instanton effects we have analyzed in detail. We hope to be able to investigate these issues in this or similar contexts with D-branes at singularities, where the rigidity of the cycles is well understood and allows for the correct number of fermionic zero-modes. Clearly additional (closed string) fluxes neeeded for moduli stabilization [73, 74] may change some of our present conclusions.

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