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Transfer of radiance by twisted Gaussian Schell-model beams in paraxial systems

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Abstract. The radiometric brightness theorem for a generalized radiance function is used to study the propagation of twisted Gaussian Schell-model beams in paraxial optical systems. First it is shown that a certain unique correspondence exists between the Gaussian radiance and cross-spectral density functions associated with these beams. The method is then applied to the passage of the twisted beam field through a thin lens, and the analysis yields algebraic expressions for the quantities characterizing the transmitted beam. The phenomenon of focal shift occurring in imaging by wavefields of this kind is examined, and the effects of partial spatial coherence and beam twist are elucidated.

1. Introduction

Within the last few years considerable progress has been made towards clarifying the relation between classical radiometry and statistical wave theory [1–16]. In the course of these investigations several different generalized radiance functions were introduced, which have some but not all the properties of the radiance function of the phenomenological radiometry. However, for fields generated by quasi-homogeneous sources [17], all the different generalized radiances have been found to have the same short-wavelength limit [18] and, moreover, they then possess the properties that one usually attributes to the radiance [19].

An important and very useful property of phenomenological radiometry is expressed by the 'transport law' for radiance, which implies, roughly speaking, that the propagation of radiance through an optical system is governed by the laws of geometrical optics [19]. This feature of the radiance has been justified under special circumstances by a number of authors. An immediate consequence of this propagation law is the so-called brightness theorem which relates the brightness in any two planes perpendicular to the axis of a centred optical system—usually taken to be the object and image planes. Analogues of this theorem in optical systems have also been found to hold for some of the generalized radiance functions, at least within the accuracy of the paraxial approximation [20–24].

In the present paper we use the brightness theorem for a particular generalized radiance function to study the transfer properties of the twisted Gaussian Schell-model (GSM) beams [25–27], propagating through a thin lens. The twisted GSM beams constitute an extension

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of the usual GSM beams which contain a subtle, rotationally invariant phase factor [25]. The propagation of the various GSM beams in first-order optical systems can be treated in terms of field quantities, using the extended $abcd$-law [23, 25, 28–30]. Our method deals instead with radiance functions and it yields expressions for the focal shifts arising in focusing of beams of this kind. Such shifts were previously studied for imaging by fully coherent (monochromatic) beams [31] and also for partially coherent and twisted GSM beams [32, 33], using a different approach. The present analysis illustrates in a novel way the usefulness of the generalized radiometric brightness theorem in problems involving partially coherent light.

2. Transfer of generalized radiance through a paraxial system

We begin by briefly reviewing the main concepts associated with the passage, within the paraxial approximation, of partially coherent wavefields through optical systems.

Consider first the paraxial propagation of monochromatic light of frequency ω according to geometrical optics through a lossless system, whose input and output planes are assumed to be situated in free space. Let Q be a point specified by a transverse position vector $\boldsymbol{\rho}$ and let $\boldsymbol{s}$ denote the unit vector along the direction of the ray through that point (see figure 1). Suppose that after the ray has passed through the system it emerges in a direction specified by a unit vector $\boldsymbol{s}'$ and that it intersects the output plane at a point Q' specified by a transverse vector $\boldsymbol{\rho}'$. The transmission of the ray through the system may be expressed as

$$\begin{pmatrix} \boldsymbol{\rho}' \\ \boldsymbol{s}'_\perp \end{pmatrix} = T \begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{s}_\perp \end{pmatrix}, \tag{1}$$

where $\boldsymbol{s}_\perp$ and $\boldsymbol{s}'_\perp$ are the projections, considered as two-dimensional vectors, of the unit 'ray vectors' $\boldsymbol{s}$ and $\boldsymbol{s}'$ onto the input and output planes, respectively. The quantity T is the so-called ray-transfer matrix, which for a rotationally symmetric system containing no tilted or misaligned elements takes the form [34, ch 15]

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2}$$

where a, b, c, and d are constants, and the determinant $\det T \equiv ad - bc = 1$.

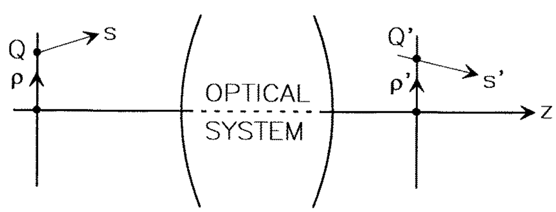

Figure 1. Illustration of the notation relating to paraxial propagation through an optical system.

It should be noted that the (forward) ray directions are fully specified by the transverse components, because the longitudinal (axial) components are given by $s_z = +\sqrt{1 - \boldsymbol{s}_\perp^2}$ and $s'_z = +\sqrt{1 - \boldsymbol{s}'^2_\perp}$. Since we assume that the paraxial approximation adequately describes the propagation, both s_z and s'_z may be considered to have the value unity.

If instead of using geometrical optics one describes the transmission by physical optics, then one has to consider a complex field of the form

$$V(\boldsymbol{r}, t) = U(\boldsymbol{\rho}, \omega) \exp[\mathrm{i}(kz - \omega t)], \tag{3}$$

where $k = \omega/c$ (c being the speed of light in vacuum) is the wavenumber associated with frequency ω, t is the time, and $\boldsymbol{r} = (\boldsymbol{\rho}, z)$ with $\boldsymbol{\rho}$ denoting the transverse (vector) component of $\boldsymbol{r}$ and z the longitudinal component. The complex amplitude factors $U_0(\boldsymbol{\rho}, \omega)$ and $U_1(\boldsymbol{\rho}', \omega)$ in the input and output planes are connected by the relation [35]

$$U_1(\boldsymbol{\rho}', \omega) = \iint G(\boldsymbol{\rho}', \boldsymbol{\rho}, \omega) U_0(\boldsymbol{\rho}, \omega)\, \mathrm{d}^2\rho, \tag{4}$$

where the propagator (Green's function of Fresnel diffraction) is expressible in terms of the elements of the ray matrix as

$$G(\boldsymbol{\rho}', \boldsymbol{\rho}, \omega) = -\frac{\mathrm{i}k}{2\pi b} \exp\left\{ \frac{\mathrm{i}k}{2b} (a\boldsymbol{\rho}^2 - 2\boldsymbol{\rho} \cdot \boldsymbol{\rho}' + d\boldsymbol{\rho}'^2) \right\}. \tag{5}$$

Here we have omitted an unimportant constant phase factor associated with axial propagation. It is also to be noted that if $b = 0$, corresponding to imaging with magnification $m = a = d^{-1}$, expression (5) must be interpreted through an appropriate limiting procedure.

Suppose now that the system transmits partially coherent light. In place of the complex amplitudes $U_0(\boldsymbol{\rho}, \omega)$ and $U_1(\boldsymbol{\rho}', \omega)$ one must then employ a correlation function of the light in the two planes. For our purposes the most convenient correlation function is the cross-spectral density function. According to the coherence theory in the space–frequency domain, the cross-spectral density function may be expressed in terms of ensembles $\{U_0(\boldsymbol{\rho}, \omega)\}$ and $\{U_1(\boldsymbol{\rho}', \omega)\}$ of complex amplitudes in the two planes in the form [36]

$$W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \langle U_0^*(\boldsymbol{\rho}_1, \omega) U_0(\boldsymbol{\rho}_2, \omega)\rangle, \tag{6}$$

$$W_1(\boldsymbol{\rho}_1', \boldsymbol{\rho}_2', \omega) = \langle U_1^*(\boldsymbol{\rho}_1', \omega) U_1(\boldsymbol{\rho}_2', \omega)\rangle, \tag{7}$$

where the angular brackets denote the ensemble average.

We are now in a position to introduce a generalized radiance function (generalized brightness) of partially coherent light which propagates through the system. Of the great variety of such functions we will employ the one introduced by Walther [1], in a paper on radiometry and coherence. It may be defined by the formula

$$B(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = \left(\frac{k}{2\pi}\right)^2 \int W(\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}_d, \boldsymbol{\rho} + \tfrac{1}{2}\boldsymbol{\rho}_d, \omega) \exp(-\mathrm{i}k\boldsymbol{s}_\perp \cdot \boldsymbol{\rho}_d)\, \mathrm{d}^2\rho_d, \tag{8}$$

where we have replaced the usual $s_z = \cos\theta$ factor by unity, consistent with the fact that we are considering a paraxial system. This radiance function is analogous to the Wigner distribution function used extensively in modern optics [37].

Relation (4) between the field amplitude factors implies, through the space–frequency representation of equations (6) and (7), a certain relationship for the cross-spectral densities in the input and output planes. Using that relationship and definition (8) of the generalized radiance one can then show that the generalized radiance functions in the input and output planes are connected by the simple transport law [21, 23, 24]

$$B_1(\boldsymbol{\rho}', \boldsymbol{s}'_\perp, \omega) = B_0(d\boldsymbol{\rho}' - b\boldsymbol{s}'_\perp, -c\boldsymbol{\rho}' + a\boldsymbol{s}'_\perp, \omega). \tag{9}$$

This formula implies that the generalized radiance with arguments $(\boldsymbol{\rho}', \boldsymbol{s}'_\perp)$ in the output plane is equal to the generalized radiance with arguments $(\boldsymbol{\rho}, \boldsymbol{s}_\perp)$ in the input plane, where

$$\boldsymbol{\rho} = d\boldsymbol{\rho}' - b\boldsymbol{s}'_\perp \qquad \boldsymbol{s}_\perp = -c\boldsymbol{\rho}' + a\boldsymbol{s}'_\perp. \tag{10}$$

To appreciate the full significance of this result, let us invert relation (1):

$$\begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{s}_\perp \end{pmatrix} = T^{-1} \begin{pmatrix} \boldsymbol{\rho}' \\ \boldsymbol{s}'_\perp \end{pmatrix}, \tag{11}$$

where, because $\det T$ is unimodular, the inverse matrix T^{-1} is readily found to be

$$T^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{12}$$

Relation (11) together with expression (12) is seen to be precisely the pair of equations (10). Hence the important formula (9) implies that in a lossless, paraxial system, the generalized radiance as defined by equation (8) is invariant along each ray passing through the system. We emphasize that this result is valid for a field of any state of coherence, and it may be referred to as the brightness theorem for the generalized radiance. Within the accuracy of the paraxial approximation this result is an analogue of the brightness theorem of conventional radiometry, where it is usually justified by heuristic arguments which do not take into account the coherence properties of the field [19].

Of particular interest is the case when the source, located in the input plane, $z = 0$ say, is quasi-homogeneous [17], i.e. when the cross-spectral density of the light in that plane has the form

$$W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \begin{cases} S_0[\frac{1}{2}(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2), \omega] g_0(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1, \omega) & \text{when } \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \in \sigma \\ 0 & \text{when } \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \notin \sigma, \end{cases} \tag{13}$$

with σ denoting the source area. Here $S_0(\boldsymbol{\rho}, \omega)$ denotes the spectral density and $g_0(\boldsymbol{\rho}_d, \omega)$ the complex spectral degree of coherence of the light in the source plane (the input plane), and $S_0(\boldsymbol{\rho}, \omega)$ changes much more slowly with $\boldsymbol{\rho}$ than $g_0(\boldsymbol{\rho}_d, \omega)$ changes with $\boldsymbol{\rho}_d$ at each frequency for which $S_0(\boldsymbol{\rho}, \omega)$ is not negligible. Most thermal, secondary, planar sources, including the usual Lambertian radiators are well approximated by the quasi-homogeneous model.

It follows from equations (13) and (8) that for a quasi-homogeneous source, the generalized radiance has the simple form

$$B_0(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = k^2 S_0(\boldsymbol{\rho}, \omega) \tilde{g}_0(k\boldsymbol{s}_\perp, \omega), \tag{14}$$

where

$$\tilde{g}_0(\boldsymbol{v}, \omega) = \frac{1}{(2\pi)^2} \int g_0(\boldsymbol{\rho}_d, \omega) \exp(-\mathrm{i}\boldsymbol{\rho}_d \cdot \boldsymbol{v})\, \mathrm{d}^2\rho_d \tag{15}$$

is the two-dimensional spatial Fourier transform of $g_0(\boldsymbol{\rho}_d, \omega)$ and $k = \omega/c$ is the wavenumber. Expression (14) is known to satisfy all the postulates of traditional radiometry in the source plane [17] and hence it may be identified with the usual source radiance function. In view of this fact and, because of the transport law given by equation (9), the brightness theorem for the generalized radiance stated above reduces to the usual radiometric brightness theorem, having been derived rigorously for lossless paraxial systems whose input is a quasi-homogeneous source.

We note that the same physical conclusion remains valid for any class of incident fields, whose generalized radiance functions in the source plane meet the conditions of conventional radiometry. Examples of such classes are the customary and the twisted Gaussian Schell-model beam fields, which will be discussed below.

3. Application to twisted Gaussian Schell-model beams

We will now illustrate the use of the brightness theorem by applying it to analyse the propagation of a twisted Gaussian Schell-model beam through a lossless lens system. We begin by noting a certain useful one-to-one correspondence between the cross-spectral density of such beams and the Gaussian radiance function.

The cross-spectral density of a twisted GSM beam in a transverse plane $z =$ constant has the form [25–27]

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \sqrt{S(\boldsymbol{\rho}_1, \omega)}\sqrt{S(\boldsymbol{\rho}_2, \omega)}\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega), \tag{16}$$

where the spectral density $S(\boldsymbol{\rho}, \omega)$ and the absolute value of the spectral degree of coherence $\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega)$ are Gaussian functions:

$$S(\boldsymbol{\rho}, \omega) = s(\omega) \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma_s^2(\omega)}\right], \tag{17}$$

$$\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \exp\left[-\frac{(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2}{2\sigma_\mu^2(\omega)}\right] \exp\left[-\mathrm{i}\frac{k(\boldsymbol{\rho}_1^2 - \boldsymbol{\rho}_2^2)}{2R(\omega)}\right] \exp[-\mathrm{i}ku(\omega)\boldsymbol{\rho}_1 \cdot \epsilon\boldsymbol{\rho}_2]. \tag{18}$$

Here $s(\omega)$ is the spectral density at the on-axis point $\boldsymbol{\rho} = 0$, and $\sigma_s(\omega)$, $\sigma_\mu(\omega)$, $R(\omega)$, and $u(\omega)$ are, respectively, the effective beam width, the transverse coherence length, the radius of wavefront curvature, and the strength of the beam twist in the chosen transverse plane. The symbol ϵ denotes the two-dimensional antisymmetric matrix such that $\boldsymbol{\rho}_1 \cdot \epsilon\boldsymbol{\rho}_2 = x_1 y_2 - y_1 x_2$, with (x_j, y_j) being the Cartesian coordinates of $\boldsymbol{\rho}_j$ for $j = 1, 2$. The twist parameter $u(\omega)$ must satisfy the condition $|u(\omega)| \leqslant 1/k\sigma_\mu^2(\omega)$ for the cross-spectral density (16)–(18) to be a non-negative definite function [25]. Finally, we note that $R(\omega)$ is positive or negative according to whether the beam diverges or converges.

In order to determine the generalized radiance which is associated with this cross-spectral density function, we first set $\boldsymbol{\rho}_1 = \boldsymbol{\rho} - \boldsymbol{\rho}_d/2$, $\boldsymbol{\rho}_2 = \boldsymbol{\rho} + \boldsymbol{\rho}_d/2$ and find at once from equations (16)–(18) that

$$W(\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}_d, \boldsymbol{\rho} + \tfrac{1}{2}\boldsymbol{\rho}_d, \omega) = s(\omega) \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma_s^2} - \frac{\boldsymbol{\rho}_d^2}{2\delta^2}\right] \exp\left[\mathrm{i}k\left(\frac{\boldsymbol{\rho}}{R} + u\epsilon\boldsymbol{\rho}\right) \cdot \boldsymbol{\rho}_d\right], \tag{19}$$

where

$$\frac{1}{\delta^2} = \frac{1}{\sigma_\mu^2} + \frac{1}{(2\sigma_s)^2}. \tag{20}$$

To keep the notation as simple as possible, we do not display from now on the argument ω in σ_s, σ_μ, R, u, and δ. On substituting from equation (19) into definition (8), we obtain the important formula

$$B(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = \frac{k^2\delta^2}{2\pi} s(\omega) \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma_s^2} - \frac{k^2\delta^2}{2}\left(\boldsymbol{s}_\perp - \frac{\boldsymbol{\rho}}{R} - u\epsilon\boldsymbol{\rho}\right)^2\right] \tag{21}$$

for the generalized radiance of the twisted GSM beam field in the plane considered.

The one-to-one correspondence between twisted Gaussian Schell-model cross-spectral densities and Gaussian radiance functions is now readily recognized. Given a twisted GSM cross-spectral density with parameters σ_s, δ, R, and u, the corresponding (Gaussian) radiance function can be determined from equation (21). Conversely, suppose that we are given a Gaussian radiance function of the form

$$B(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = A \exp\{-\tfrac{1}{2}[C\boldsymbol{\rho}^2 + D(\boldsymbol{s}_\perp - E\boldsymbol{\rho} - F\epsilon\boldsymbol{\rho})^2]\}, \tag{22}$$

with A, C, and D being non-negative quantities such that $CD \leqslant 4k^2$ and with E and F being constants, we can readily identify the parameters of the corresponding cross-spectral density by comparing equations (21) and (22). This gives

$$\sigma_s^2 = 1/C \qquad \delta^2 = D/k^2 \qquad R = 1/E \qquad u = F, \tag{23}$$

and $s(\omega) = 2\pi A/D$. Clearly σ_μ can be determined from σ_s and δ by the use of equation (20), and one finds that

$$\sigma_\mu^2 = \frac{(2\sigma_s)^2\delta^2}{(2\sigma_s)^2 - \delta^2}. \tag{24}$$

The unique correspondence between Gaussian cross-spectral densities and radiance functions is consistent with the (invertible) Fourier-transform property indicated by equation (8). The main significance of the correspondence in the present context is that it enables us to characterize the twisted GSM beams directly in terms of the radiance. The general properties of the Gaussian radiance function (or Wigner distribution) associated with a twisted GSM beam field are discussed in more detail in [25].

In view of the above correspondence we may analyse the propagation of a twisted GSM beam in a lossless paraxial system by tracing the radiance through the system, using the brightness theorem or, equivalently, the transport law expressed by equation (9). We will illustrate this procedure by considering a simple system consisting of a single thin lens. For systems that contain several components, the method can be used by repeating it from component to component.

Let Π_0 and Π_3 be arbitrary input and output planes, and Π_1 and Π_2 planes immediately in front and immediately behind the thin lens, all the planes being perpendicular to the axis of rotational symmetry (see figure 2). We will specify axial distances on the input side with respect to the input plane Π_0, and those on the output side with respect to the plane Π_2. Transverse locations with respect to the axis will be denoted by the two-dimensional vectors $\boldsymbol{\rho} = (x, y)$ on both sides of the lens. In this notation the lens is located at the distance $z = z_1$ from Π_0 and the output plane is located at distance $z' = z_2$ from Π_2.

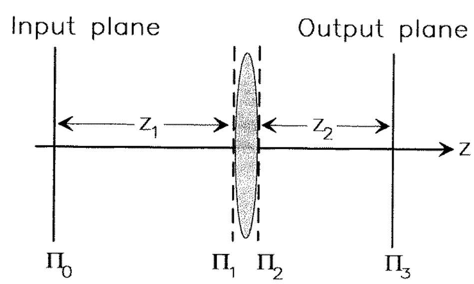

Figure 2. Geometry and notation relating to light propagation through a thin lens.

The radiance may be conveniently traced through the lens in successive stages. We choose the input plane Π_0 to coincide with the waist of the input beam, i.e. the plane in which $|R| = \infty$. The cross-spectral density in the plane Π_0 will then have the form

$$W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = s_0(\omega)\exp\left[-\frac{\boldsymbol{\rho}_1^2 + \boldsymbol{\rho}_2^2}{4\sigma_{s0}^2} - \frac{(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2}{2\sigma_{\mu 0}^2}\right]\exp[-\mathrm{i}ku_0(\boldsymbol{\rho}_1\cdot\epsilon\boldsymbol{\rho}_2)], \tag{25}$$

which readily follows from equations (16)–(18) and the fact that $1/R = 0$ in the plane Π_0. The corresponding (generalized) radiance function at points in that plane is, according to equation (21), given by

$$B_0(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = \frac{k^2\delta_0^2}{2\pi}s_0(\omega)\exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma_{s0}^2} - \frac{k^2\delta_0^2}{2}(\boldsymbol{s}_\perp - u_0\epsilon\boldsymbol{\rho})^2\right]. \tag{26}$$

The subscript zero indicates, of course, that the quantity in question pertains to the field in the input plane Π_0.

We may readily trace the radiance from the plane Π_0 to the plane Π_1 by using the fact that, for propagation through a distance z_1 in free space, the elements of the ray-transfer matrix T are $a = d = 1$, $b = z_1$, and $c = 0$ [38]. The transport law (9) and equation (26) then give at once the following expression for the radiance function at points in the plane Π_1:

$$\begin{aligned} B_1(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) &= B_0(\boldsymbol{\rho} - z_1\boldsymbol{s}_\perp, \boldsymbol{s}_\perp, \omega) \\ &= \frac{k^2\delta_0^2}{2\pi} s_0(\omega) \exp\left\{-\frac{(\boldsymbol{\rho} - z_1\boldsymbol{s}_\perp)^2}{2\sigma_{s0}^2} - \frac{k^2\delta_0^2}{2}[\boldsymbol{s}_\perp - u_0\epsilon(\boldsymbol{\rho} - z_1\boldsymbol{s}_\perp)]^2\right\}. \end{aligned} \tag{27}$$

Making use of the identities $\boldsymbol{\rho}\cdot\epsilon\boldsymbol{\rho} = 0$ and $\epsilon\boldsymbol{\rho}\cdot\epsilon\boldsymbol{s}_\perp = \boldsymbol{\rho}\cdot\boldsymbol{s}_\perp$, and after rearranging the terms in the exponent, this formula can be rewritten in the form of equation (22), i.e. as

$$B_1(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = \frac{k^2\delta_1^2}{2\pi} s_1(\omega) \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma_{s1}^2} - \frac{k^2\delta_1^2}{2}\left(\boldsymbol{s}_\perp - \frac{\boldsymbol{\rho}}{R_1} - u_1\epsilon\boldsymbol{\rho}\right)^2\right], \tag{28}$$

with

$$\sigma_{s1}^2 = \sigma_{s0}^2\Delta_{10}^2, \tag{29}$$

$$\delta_1^2 = \delta_0^2\Delta_{10}^2, \tag{30}$$

$$R_1 = z_1\left[1 + \left(\frac{z_{R0}}{z_1}\right)^2\right], \tag{31}$$

$$u_1 = u_0\Delta_{10}^{-2}, \tag{32}$$

and

$$s_1(\omega) = s_0(\omega)\Delta_{10}^{-2}, \tag{33}$$

where

$$\Delta_{10}^2 = 1 + \left(\frac{z_1}{z_{R0}}\right)^2 \tag{34}$$

and

$$z_{R0}^2 = k^2\sigma_{s0}^2\left(\frac{1}{4\sigma_{s0}^2} + \frac{1}{\sigma_{\mu0}^2} + k^2\sigma_{s0}^2u_0^2\right)^{-1}. \tag{35}$$

The one-to-one correspondence between twisted GSM cross-spectral densities and Gaussian radiance functions allows us to draw several physical conclusions from these expressions.

Formula (29) shows that on propagation from plane Π_0 to plane Π_1 the beam width has increased by a factor of Δ_{10}. For this reason we will refer to Δ_{10} as the beam expansion coefficient. Formula (31) implies that the beam has acquired a 'phase curvature', given by the reciprocal of expression (31). Moreover, according to equation (33), the spectral density along the axis has decreased by a factor of Δ_{10}^2. Further, since according to equation (30) the δ-factor has increased in the same proportion as σ_s, it follows from equation (24) that

$$\sigma_{\mu1}^2 = \sigma_{\mu0}^2\Delta_{10}^2. \tag{36}$$

We see at once from this formula and from equation (29) that the ratio $\alpha = \sigma_\mu/\sigma_s$, sometimes called the degree of global coherence, is invariant on propagation in free space not only for ordinary GSM beams but also for twisted GSM beams. From equations (32) and (36) it follows as well that the quantity $\eta = k\sigma_\mu^2 u$ [25], known as the (normalized) twist parameter [26, 27], remains unchanged on the beam's passage from plane Π_0 to plane Π_1. Finally we note that the parameter z_{R0}, given by equation (35), may evidently be identified with the so-called Rayleigh range [34, ch 17] of the incident twisted GSM beam. It is seen to depend, in addition to the beam width and transverse coherence, on the twist condition of the wavefield across the source plane Π_0.

The radiance $B_2(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega)$ at points in plane Π_2 immediately behind the lens is obtained from the radiance $B_1(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega)$ at points in plane Π_1 right in front of the lens by means of the following formula, which follows at once from the transfer law (9) and the fact that the elements of the ray-transfer matrix T for a thin lens of focal length f are $a = d = 1$, $b = 0$, and $c = -1/f$ [38]:

$$B_2(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = B_1\left(\boldsymbol{\rho}, \boldsymbol{s}_\perp + \frac{1}{f}\boldsymbol{\rho}, \omega\right). \tag{37}$$

On substituting for $B_1(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega)$ from equation (28), we obtain for $B_2(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega)$ the expression

$$B_2(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = \frac{k^2\delta_2^2}{2\pi} s_2(\omega) \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma_{s2}^2} - \frac{k^2\delta_2^2}{2}\left(\boldsymbol{s}_\perp - \frac{\boldsymbol{\rho}}{R_2} - u_2\epsilon\boldsymbol{\rho}\right)^2\right], \tag{38}$$

where

$$\sigma_{s2} = \sigma_{s1}, \tag{39}$$
$$\delta_2 = \delta_1, \tag{40}$$
$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}, \tag{41}$$
$$u_2 = u_1 \tag{42}$$

and

$$s_2(\omega) = s_1(\omega). \tag{43}$$

On comparing equations (38) and (28) we see that relations (39)–(43) imply that the lens only changes the beam's radius of wavefront curvature and leaves all the other parameters unaltered. In particular, the transverse coherence properties and the twist condition of the field are in no way modified in passage through the thin lens. We will assume that the focal power $1/f$ of the lens is high enough to ensure that $R_2 < 0$. The beam will then converge to the right of the lens, forming a (real) waist of the transmitted beam in the image space. The condition $R_2 < 0$ is satisfied when $z_1 > f$.

Finally we consider the propagation of the radiance from plane Π_2 to a plane Π_3 in the image space (see figure 2). By the same argument as given in connection with equation (27) we have, for the radiance in plane Π_3,

$$B_3(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = B_2(\boldsymbol{\rho} - z_2\boldsymbol{s}_\perp, \boldsymbol{s}_\perp, \omega), \tag{44}$$

or, using equation (38),

$$\begin{aligned} B_3(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = \frac{k^2\delta_2^2}{2\pi} s_2(\omega) \exp\Bigg\{ &- \frac{(\boldsymbol{\rho} - z_2\boldsymbol{s}_\perp)^2}{2\sigma_{s2}^2} \\ &- \frac{k^2\delta_2^2}{2}\left[\boldsymbol{s}_\perp - \frac{(\boldsymbol{\rho} - z_2\boldsymbol{s}_\perp)}{R_2} - u_2\epsilon(\boldsymbol{\rho} - z_2\boldsymbol{s}_\perp)\right]^2\Bigg\}. \end{aligned} \tag{45}$$

After some algebraic manipulations, and on using the identity $\boldsymbol{\rho} \cdot \epsilon\boldsymbol{s}_\perp = -\boldsymbol{s}_\perp \cdot \epsilon\boldsymbol{\rho}$, this formula may be rewritten in the same form as equation (38), namely

$$B_3(\boldsymbol{\rho}, \boldsymbol{s}_\perp, \omega) = \frac{k^2\delta_3^2}{2\pi} s_3(\omega) \exp\left[-\frac{\boldsymbol{\rho}^2}{2\sigma_{s3}^2} - \frac{k^2\delta_3^2}{2}\left(\boldsymbol{s}_\perp - \frac{\boldsymbol{\rho}}{R_3} - u_3\epsilon\boldsymbol{\rho}\right)^2\right], \tag{46}$$

with

$$\sigma_{s3}^2 = \sigma_{s2}^2 \Delta_{32}^2, \tag{47}$$

$$\delta_3^2 = \delta_2^2 \Delta_{32}^2, \tag{48}$$

$$\frac{1}{R_3} = \left[\frac{1}{R_2}\left(1 + \frac{z_2}{R_2}\right) + \frac{z_2}{z_{R2}^2}\right]\Delta_{32}^{-2}, \tag{49}$$

$$u_3 = u_2 \Delta_{32}^{-2} \tag{50}$$

and

$$s_3(\omega) = s_2(\omega)\Delta_{32}^{-2}, \tag{51}$$

where

$$\Delta_{32}^2 = \left(1 + \frac{z_2}{R_2}\right)^2 + \left(\frac{z_2}{z_{R2}}\right)^2 \tag{52}$$

and

$$z_{R2}^2 = k^2 \sigma_{s2}^2 \left(\frac{1}{4\sigma_{s2}^2} + \frac{1}{\sigma_{\mu 2}^2} + k^2 \sigma_{s2}^2 u_2^2\right)^{-1}, \tag{53}$$

in complete analogy with equation (35). When equations (29)–(32) and (39)–(42) are used, it is readily seen that $z_{R2}^2 = z_{R0}^2 \Delta_{10}^4$.

We note that when $1/R_2 = 0$, the expressions for Δ_{32} and $1/R_3$ are of the same form as those of Δ_{10} and $1/R_1$ (cf equations (34) and (31), respectively). This was to be expected, because the beam propagates from plane Π_0 to plane Π_1 with no initial phase curvature, whereas it propagates from plane Π_2 to plane Π_3 with the initial phase curvature of $1/R_2$. We also observe that according to equations (47), (48), and (24),

$$\sigma_{\mu 3}^2 = \sigma_{\mu 2}^2 \Delta_{32}^2. \tag{54}$$

On combining the results of the three successive stages ($\Pi_0 \to \Pi_1 \to \Pi_2 \to \Pi_3$) we have shown explicitly, using the radiometric brightness theorem for a generalized radiance, that the degree of global coherence $\alpha = \sigma_\mu/\sigma_s$ and the normalized twist parameter $\eta = k\sigma_\mu^2 u$ associated with twisted GSM beams remain invariant in paraxial lens systems.

Formulae (28)–(35), (38)–(43), and (46)–(53), together with equation (26), provide a complete solution to the problem of transmission of a twisted Gaussian Schell-model beam through a thin lens. In figure 3 we illustrate, in various planes of the system, the angular distribution of the generalized radiance associated with the GSM beam generated by a slightly twisted, Gaussian quasi-homogeneous secondary source. The (schematic) drawing pertains to a meridional (yz) plane, though in twisted fields the radiance has also an azimuthal dependence. The graphs are calculated from the general formulae for a particular set of parameters indicated in the figure caption.

4. Focal shifts

Next, let us examine the focusing properties of the transmitted beam. It is readily observed that the waist of the beam in the image space is not formed at the plane onto which the input plane Π_0 is geometrically imaged by the lens, i.e. it is not at the distance $(z_2)_{\text{image}}$ given by the lens formula

$$\frac{1}{(z_2)_{\text{image}}} = \frac{1}{f} - \frac{1}{z_1}. \tag{55}$$

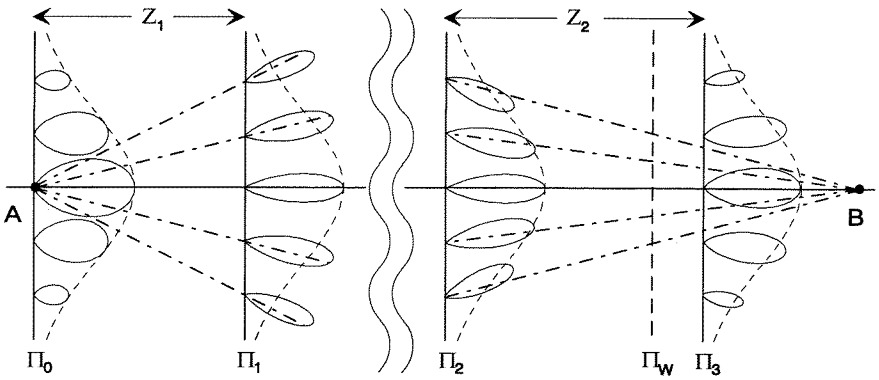

Figure 3. Example which illustrates the changes in the angular distribution of the generalized radiance of a twisted GSM beam on propagation through a thin lens. The symbols Π_0, Π_1, Π_2, and Π_3 have the same meaning as in figure 2, with Π_0 coinciding with the input-beam waist. The plane Π_W contains the waist of the twisted GSM beam on exit from the thin lens. Point B is the geometrical image of point A. For illustrative purposes the plots correspond to the choice $\alpha = 0.1$, $\eta = 0.1$, $k\delta_0 = 3$, $\Delta_{10} = 2$, $\Delta_{32} = 0.55$, and $|R_2| = R_1$. The Gaussian envelope curves (broken lines) indicate the radiance in the forward direction, which is seen to remain invariant on propagation.

This can be seen by determining the value $z_2 = (z_2)_{\min}$ at which the beam width σ_{s3} in the image space takes on its smallest value. We readily find from equations (47) and (52) that

$$\frac{1}{(z_2)_{\min}} = -\frac{1}{R_2} - \frac{R_2}{z_{R2}^2}. \tag{56}$$

The plane $z = (z_2)_{\min}$ is evidently the plane in the image space which contains the waist of the transmitted beam. At this plane the phase curvature $1/R_3$ becomes zero, a fact which can readily be verified from equation (49). Since $R_2 < 0$, equation (56) implies that $(z_2)_{\min} > 0$.

We will now show that

$$(z_2)_{\min} < -R_2 < (z_2)_{\text{image}}. \tag{57}$$

Since $R_2 < 0$, both terms on the right-hand side of equation (56) are positive and so

$$\frac{1}{(z_2)_{\min}} > -\frac{1}{R_2}. \tag{58}$$

Now from equations (31) and (41) it readily follows that

$$-\frac{1}{R_2} = \frac{1}{f} - \frac{1}{z_1}\left[1 + \left(\frac{z_{R0}}{z_1}\right)^2\right]^{-1}, \tag{59}$$

which implies, if we make use of equation (55), that

$$-\frac{1}{R_2} > \frac{1}{(z_2)_{\text{image}}}. \tag{60}$$

Inequality (57) then follows at once on combining equations (58) and (60).

The inequalities of equation (57) imply that the waist of the GSM beam in the image space is always closer to the lens than is the geometrical image of plane Π_0 containing the waist of the beam incident onto the lens (see figure 3). This result has been known in connection with imaging of monochromatic Gaussian beams [39, 40], but now we have shown using the radiance transport law that it holds for imaging of twisted GSM beams of any state of coherence. Some related results were also previously reported for the focusing of both ordinary and twisted GSM beams [32, 33].

To examine more closely the effects of partial coherence and beam twist on the focal shift we may use the explicit formulae for the parameters z_{R0} and z_{R2} in terms of the quantities characterizing the input-beam waist and the optical system. Since $z_{R2}^2 = [1+(z_1/z_{R0})^2]^2 z_{R0}^2$, we find in this way first from equations (31), (41), and (56) that

$$\frac{1}{(z_2)_{\min}} = -\frac{1}{R_2} - \frac{R_2 z_{R0}^2}{(z_1^2 + z_{R0}^2)} \tag{61}$$

and

$$-\frac{1}{R_2} = \frac{1}{f} - \frac{z_1}{z_1^2 + z_{R0}^2}. \tag{62}$$

After some algebra we then obtain for the location of the exit-beam waist the expression

$$(z_2)_{\min} = -\frac{f(fz_1 - z_1^2 - z_{R0}^2)}{f^2 - 2fz_1 + z_1^2 + z_{R0}^2}, \tag{63}$$

where, in view of equations (20) and (35),

$$z_{R0}^2 = \frac{k^2\sigma_{s0}^2\delta_0^2}{1 + k^2\sigma_{s0}^2\delta_0^2 u_0^2}. \tag{64}$$

The various special cases corresponding to the usual (untwisted) GSM beams can be obtained from equations (63) and (64) by setting $u_0 = 0$ and varying δ_0 in relation to σ_{s0}. Evidently $\delta_0 = 2\sigma_{s0}$ for a Gaussian laser beam, while $\delta_0 \approx \sigma_{\mu 0}$ for a Gaussian quasi-homogeneous beam.

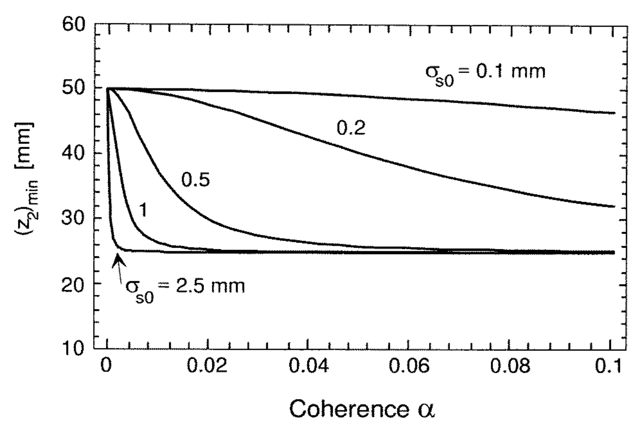

Figure 4. Location of the waist of a focused GSM beam as a function of the global degree of coherence $\alpha = \sigma_{\mu 0}/\sigma_{s0}$ for selected input-beam widths σ_{s0}. The focal length of the lens is $f = z_1/2 = 25$ mm and the wavelength corresponds to a HeNe laser, $\lambda = 0.633$ μm.

In figures 4 and 5 the focal shifts are illustrated in a typical situation corresponding to a symmetric geometrical imaging at equal conjugates $z_1 = (z_2)_{\text{image}} = 50$ mm. The finiteness of the lens aperture is ignored. The curves in figure 4 concern untwisted ($u_0 = 0$) GSM fields; they show the position of the beam waist behind the lens, $(z_2)_{\min}$, as a function of the global degree of coherence $\alpha = \sigma_{\mu 0}/\sigma_{s0}$ for some chosen values of σ_{s0}. It is seen, first, that in the incoherent limit as $\alpha \to 0$ the output beam waist is located in the geometrical image

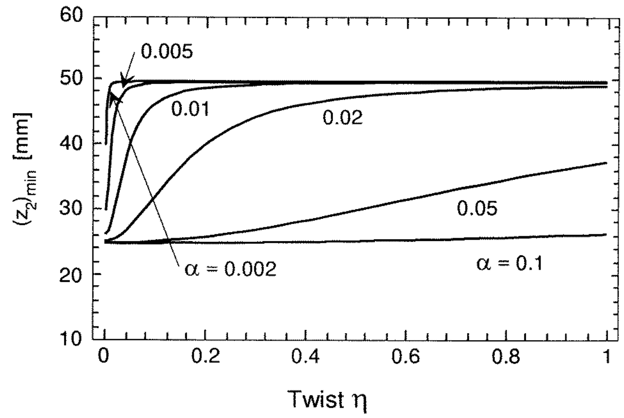

Figure 5. Location of the focused, twisted GSM beam waist as a function of the normalized twist parameter $\eta = k\sigma_{\mu0}^2 u_0$. The curves are characterized by the global coherence degree $\alpha = \sigma_{\mu0}/\sigma_{s0}$, with a fixed input-beam width of $\sigma_{s0} = 1$ mm. The system parameters are as in figure 4.

plane (i.e. $(z_2)_{\mathrm{min}} = (z_2)_{\mathrm{image}}$); second, that in the coherent limit as $\alpha \to \infty$ the output beam-waist location approaches the back focal plane of the lens (i.e. $(z_2)_{\mathrm{min}} = f$); and third, that this transition from $(z_2)_{\mathrm{image}}$ to f is the faster the wider the input beam is. For moderate beam sizes in the range of $\sigma_{s0} \approx 1$ mm or larger the focused beam waist is, except for nearly incoherent light, approximately at $z_2 = f$ as predicted by plane-wave imaging.

The curves in figure 5 pertain to focused twisted GSM beams. In each case the input beam size is $\sigma_{s0} = 1$ mm, the global degree of coherence α (and hence the transverse coherence width $\sigma_{\mu0}$ as well) is fixed, and the twist parameter varies from $u_0 = 0$ (representing an ordinary GSM beam with $\eta = 0$) to $u_0 = k\sigma_{\mu0}^2$ (corresponding to maximum twist $\eta = 1$). It is observed that as the beam twist increases, the plane of the best focus moves from its untwisted position towards the geometrical image plane $z_2 = (z_2)_{\mathrm{image}}$, i.e. the focal shift is reduced. For relatively coherent GSM beams this change is seen to be minimal. However, in the quasi-homogeneous region represented by values $\alpha \ll 1$ the variation in the focused-beam waist location, as a function of η, is found to cover nearly the entire range of $f < (z_2)_{\mathrm{min}} < (z_2)_{\mathrm{image}}$. This illustrates that inclusion of the twist may considerably alter the behaviour of a GSM beam in optical systems.

5. Conclusions

The radiance transport law associated with partially coherent fields takes on a particularly simple form in paraxial optical systems. We have demonstrated that this law may be used efficiently to analyse the propagation of twisted GSM beams. We have shown, in particular, that the method readily leads to algebraic expressions for the various quantities characterizing such beams in free space and in the passage through a thin lens. The results also allow the assessment of the role of partial coherence and of twist effects in the focal shifts occurring in imaging by Gaussian beams. An increase in the transverse degree of coherence generally shifts the location of the best focus away from the corresponding geometrical image plane, while an increase in the beam twist moves it back towards the geometrical image. This is consistent with the notion that the twist phenomenon reduces the beam's 'effective' degree of partial coherence.

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