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Relativistic theory of tunnel ionization

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Abstract

We present a single-electron-response theory of tunnel ionization of atomic ions in relativistic laser fields, as described by the three-dimensional Klein–Gordon and Dirac equations. The ionization rates are derived analytically for hydrogen-like ions taking into account the presence of the Coulomb potential and are then generalized to arbitrary atomic ions by using quantum defect theory. The resulting ionization rates allow for the first time a quantitative prediction of tunnel ionization of atomic ions in relativistic laser fields.

1. Introduction

The great progress in laser technology during the last decade has allowed the realization of 20–30 fs laser pulses with peak intensities of 10^{21} W cm^{-2} [1, 2]. The next generation of Ti:S laser sources will supply sub-10 fs pulses with peak intensities up to 10^{23} W cm^{-2} [3]. Similar intensities can be realized during collisions of relativistic heavy ions with atomic and molecular targets [4]. At such intensities ions up to charge states $Z = 40$ can be ionized. The motion of free electrons in a Ti:S laser field (centre wavelength 800 nm) becomes relativistic at an intensity of $\approx 10^{18}$ W cm^{-2}. Therefore, ionization of highly charged ions is anticipated to exhibit strong relatvistic effects. Currently, there exists no quantitative theory of relativistic tunnel ionization.

Field ionization is the primary process in matter exposed to high-intensity radiation responsible for the generation of plasmas. Ionization has not appeared to be important for high-field plasma physics so far, because it saturates at the leading edge of the previously used subpicosecond or longer pulses. Therefore, the dominant part of the pulse interacts with a fully ionized plasma. The availability of ultrashort, few-cycle laser pulses [2, 3] has changed the situation, opening a novel parameter regime of plasma physics. Saturation of ionization is shifted to considerably higher intensities and ionization continues playing a role until instants close to the pulse peak. This makes the understanding of relativistic ionization dynamics of highly charged ions an essential issue of strong laser field plasma physics.

Ionization introduces an electron density profile that acts like a defocusing lens and reduces the peak intensity, thus setting a limit to the maximum interaction length (for the definition of interaction length and more details, see [5]). Optimization of the interaction length is of detrimental importance for the performance of laser plasma x-ray lasers [6]. During tunnelling ionization the electron acquires ponderomotive energy and the Lorentz force pushes the electrons in the direction of the laser wavevector [11]. At an intensity of 10^{23} W cm^{-2} the ponderomotive energy is in the gigaelectronvolt energy range presenting an alternative concept to existing electron acceleration mechanisms relying on collective effects, such as wake field acceleration [7]. In contrast to existing schemes the strong nonlinear dependence of tunnelling ionization will result in the generation of relativistic electron pulses on a few-fs to sub-fs timescale. Due to the large kinetic energies of the electrons and the ions, laser induced plasmas have also gained importance as a tool for nuclear physics [8].

Whereas ionization of the valence electrons of atoms is described by the Schrödinger equation [10], nonrelativistic theories are no longer applicable to ionization of highly charged ions. There are two physical aspects responsible for the breakdown of nonrelativisitic theories. (i) The kinetic energy of the electron quivering in the electromagnetic field becomes comparable to the rest mass. (ii) The energy of the bound state becomes comparable to the rest mass. Relativistic effects scale with the ratio of the electron velocity v over light velocity c. To first order in v/c the Lorentz force, caused by the magnetic field, pushes the electron in direction of the laser wavevector. To second and higher orders in v/c relativistic effects, such as the velocity dependence of the electron mass, appear.

The wealth of interesting applications has driven the quest for a theoretical understanding of relativistic optical field ionization. So far relativistic ionization and electron spectra were calculated [11–14] by solving the Dirac equation with exponential accuracy[3]. Recently, the Klein–Gordon equation for π^- atoms was solved including the pre-exponential factor and the Coulomb correction by using the imaginary time method [15–17]. However, so far it has not been possible to find a quantitatively correct solution of the Dirac equation, which is necessary to investigate most of the applications discussed above.

The goal of our paper is to derive an analytical theory for relativistic tunnel ionization of atomic ions in strong laser fields, which is based on a single-electron-response approximation, as described by the Dirac equation. We start out by reviewing ionization theory in the nonrelativistic limit in section 2, which is based on WKB (Wentzel, Kramers, Brillouin) theory. In the remaining sections WKB theory is generalized to relativistic intensities. In sections 3 and 5 the Klein–Gordon equation in the presence of static electric and electromagnetic fields is solved, respectively, describing the ionization of π^- atoms. We obtain an expression including the pre-exponential and Coulomb correction factors in agreement with previous work [15–17]. Finally, in sections 4 and 6 the Dirac equation with static electric and with electromagnetic fields is solved, respectively, including pre-exponential and Coulomb correction factors. This is to the best of our knowledge the first theory allowing a quantitative prediction of ionization of atomic ions. The analytical ionization formulae will help us to gain a better understanding of relativistic laser plasma processes and will help us to design and optimize new experiments, such as the one discussed above.

Our calculations show that the difference in ionization rates between static electric fields and electromagnetic fields is negligible. This shows that magnetic field effects on the sub-barrier motion of the electron during tunnelling are small. The main difference between relativistic and nonrelativistic theory arises from the different binding energies of the ground

[3] Generally speaking, the tunnel ionization probability shows strong exponential dependence which dominates the behaviour of the ionization process. To calculate the ionization with exponential accuracy means to describe it qualitatively, by calculating its exponential dependence only.

state, which starts to play a role for ions with charge states $Z > 10$. The difference increases with increasing Z and is up to an order of magnitude for $Z = 60$. We find that the agreement between relativistic and nonrelativistic theory becomes better with increasing field strength. This is because the forces of the laser/electric field become stronger than the Coulomb forces that are mainly responsible for the relativistic corrections.

2. Nonrelativistic tunnel ionization

Tunnel ionization of quantum systems in a constant electric field is commonly calculated in parabolic coordinates. Then, the Schrödinger equation can be separated into two one-dimensional equations, which can be solved by using the WKB method. However, the separation of coordinates is no longer possible for laser intensities, for which relativistic effects start to play a role. Therefore, we develop in this section a tunnelling theory for cylindrical coordinates in the nonrelativistic limit, which will be generalized to the relativistic case in the following sections [17].

Our derivation starts with a three-dimensional short-range potential and a static electric field. Subsequently, the ionization rates for the short-range potential will be corrected for the presence of the Coulomb potential giving the well known ionization rate of a hydrogen atom in a laser electric field in the dipole approximation. In this section we use the atomic systems of units $e = m = \hbar = 1$, where m is the mass of the particle, and e is its charge.

The unperturbed wavefunction of the ground state in a short-range potential is given by

$$\Psi_{sr}(r) = \frac{A\sqrt{\kappa}}{r}\exp(-\kappa r), \tag{1}$$

where $\kappa = \sqrt{2I_p}$, and I_p is the ionization potential of the ground state. In atomic units the Bohr radius is given by $a_0 = 1/\kappa$. Equation (1) is the exact solution for a δ-function potential with normalization constant $A = 1/\sqrt{2\pi}$. The wavefunction (1) is also a solution of an arbitrary short-range potential in the range $r > r_0$, where $r_0 \ll a_0$ is a finite radius. However, the normalization constant will be different, $A \neq 1/\sqrt{2\pi}$.

We assume that the electric field F is polarized parallel to the z-direction. Our calculation of tunnel ionization relies on the WKB theory that utilizes the quasiclassical solution of the Schrödinger equation [18]. The wavefunction under the barrier, built by the combined potential of the nucleus and the electric field, is exponentially decaying. The outer turning point $a = \kappa^2/2F$ determines the border between the classically forbidden region under the barrier and the classically allowed region, where the wavefunction exhibits oscillatory behaviour. The quasiclassical solution under the barrier is given by

$$\Psi_{qc}(z, p_\perp) = \frac{C}{\sqrt{p_z}}\exp\left(\mathrm{i}\int_{z_1}^{z} p_z\,\mathrm{d}z + \mathrm{i}\frac{\pi}{4}\right). \tag{2}$$

In order to determine the constant C, the solution (2) must be connected to the ground state wavefunction (1) at the matching point z_1. This must be done in the range $a_0 \ll z_1 \ll a$, where two conditions are fulfilled:

(i) the electric field is negligible so that the unperturbed wavefunction (1) is valid;
(ii) we are far from the left classical turning point so that the quasiclassical solution (2) is valid, too.

The electron momentum in the z-direction, p_z, in equation (1) must be determined by solution of Newton's equations of motion subject to the following boundary conditions: the electron is at the birth time, $t = 0$, at the outer turning point $z = a$, $x = 0$, $y = 0$, and at some imaginary initial time $t = t_0$ at point $z(t_0) = x(t_0) = y(t_0) = 0$ with an energy $-\kappa^2/2 = -I_p$. Solution of the classical equations of motion gives the trajectory, $z(t) = z(t_0)+\kappa^2/2F+Ft^2/2$, $x(t) = y(t) = 0$. The first derivative of the trajectory gives after eliminating the time the classical momenta $p_{zc} = \mathrm{i}\sqrt{\kappa^2 + p_{\perp c}^2 - 2Fz}$, $p_{xc} = p_{yc} = 0$ with $p_{\perp c}^2 = p_{xc}^2 + p_{yc}^2$. Note that the momentum under the barrier is imaginary, which comes from the fact that the electron penetrates a classically forbidden region. The use of p_{zc} in equation (1) is sufficient to obtain the ionization rate with exponential accuracy. However, in order to determine the pre-exponential factor, the quantum mechanical uncertainty around the classical trajectory must be taken into account. This is why in equation (2) the momentum $p_z = \mathrm{i}\sqrt{\kappa^2 + p_\perp^2 - 2Fz}$ was introduced containing the transversal momentum uncertainty, $p_\perp = p_{\perp c} + \delta p_\perp = \delta p_\perp$, and $\Psi_{qc} = \Psi_{qc}(z, \boldsymbol{p}_\perp)$ in equation (2).

The connection is performed by bringing the wavefunction (1) into a form similar to the WKB solution (2). For that purpose we perform a Fourier transformation of equation (1) with respect to the transversal coordinates x, y that yields

$$\Psi_{sr}(z, \boldsymbol{p}_\perp) = \frac{2\pi A\sqrt{\kappa}}{\tilde{p}_z} \exp(-\tilde{p}_z z). \tag{3}$$

Here, $\tilde{p}_z = \sqrt{p_\perp^2 + \kappa^2} \approx \kappa + p_\perp^2/(2\kappa)$, where we used the fact that the ionized electron is emitted mainly in the z-direction with a small transversal momentum. We will see below that $p_\perp \ll \kappa$. Using this approximation equation (3) can be rewritten into the WKB form

$$\Psi_{sr}(z, \boldsymbol{p}_\perp) = \frac{2\pi A}{\sqrt{\tilde{p}_z}} \exp\left(-\int_0^z \tilde{p}_z \,\mathrm{d}z\right). \tag{4}$$

Note that for the derivation of the pre-exponential factor in equation (4) the relation $\sqrt{\tilde{p}_z} \approx \sqrt{\kappa}$ was used which is valid only in the limit of negligible transversal momentum $p_\perp \to 0$. Finally, at the point z_1, where $2Fz_1 \ll \kappa^2$, equations (4) and (2) can be connected by requiring $\Psi_{sr}(z_1, \boldsymbol{p}_\perp) = \Psi_{qc}(z_1, \boldsymbol{p}_\perp)$. The resulting prefactor of the quasiclassical wavefunction is

$$\Psi_{qc}(z, \boldsymbol{p}_\perp) = \frac{2\pi A}{\sqrt{p_z}} \exp\left(\mathrm{i}\int_0^z p_z \,\mathrm{d}z\right). \tag{5}$$

This expression is valid for all values of z, except the vicinity of the classical turning points.

From equation (5) the ionization rate is determined by calcuating the current, $j_z(\boldsymbol{p}_\perp) = p_z(z > a)|\Psi_{sr}(z > a, \boldsymbol{p}_\perp)|^2$, leaving the potential barrier along the (field strength) z-direction

$$w_{sn} = \frac{1}{2\pi}\int_0^\infty \mathrm{d}p_\perp\, p_\perp j_z(\boldsymbol{p}_\perp) = 2\pi A^2 \int \mathrm{d}p_\perp\, p_\perp \exp\left(-2\,\mathrm{Im}\int_0^a p_z \,\mathrm{d}z\right). \tag{6}$$

Here we have used the identity $\int \mathrm{d}\delta\,\boldsymbol{p}_\perp = \int \mathrm{d}\boldsymbol{p}_\perp$. Next, p_z is expanded with respect to $p_\perp$. The typical value of the transversal momentum distribution is in Gaussian units $p_\perp^2 \sim \hbar eF/\sqrt{mI_p}$. The transversal momentum is proportional to Planck's constant which shows that it is of quantum mechanical origin and does not appear in the classical analysis, where $p_{\perp c} = 0$. After performing the integration over z and $p_\perp$ we obtain the well known result

$$w_{sn} = \frac{\pi A^2 F}{\kappa} \exp\left(-\frac{2\kappa^3}{3F}\right). \tag{7}$$

In the remaining part of this section the ionization rate (7) is corrected for the presence of the Coulomb potential by using 'the method of Coulomb correction' [19]. The normalized wavefunction of the ground state of a hydrogen-like atom is

$$\Psi_c(r) = \sqrt{\frac{\kappa^3}{\pi}} \exp(-\kappa r), \tag{8}$$

where $\kappa = Z$ denotes the charge of the atom/ion. The ratio of the Coulomb wavefunction, equation (8) to the short-range wavefunction (1) is

$$\left.\frac{\Psi_c}{\Psi_{sr}}\right|_{z=z_1} = \frac{\kappa r}{A\sqrt{\pi}} \approx \frac{1}{A\sqrt{\pi}} \exp(\ln(\kappa z_1)), \tag{9}$$

where $r \approx z_1$ was used. This is justified, because the electron is moving mainly along the direction of the electric field. For the Coulomb correction we assume again that z_1 fulfils the inequality $a_0 \ll z_1 \ll a$, where the quasiclassical approximations apply. Then, the contribution of the Coulomb potential can be calculated taking into account the electric field semiclassically [19, 20]. For the calculation of the correction factor we start from the quasiclassical wavefunction in the time domain, $\exp(\mathrm{i}\int \Delta E\,\mathrm{d}t)$, that is equivalent to the space domain representation, $\exp(\mathrm{i}\int p_z\,\mathrm{d}z)$, used so far. This is because $\int \sqrt{p_z^2 + 2\Delta E}\,\mathrm{d}z = \int p_z\,\mathrm{d}z + \int (\Delta E/p_z)\,\mathrm{d}z = \int p_z\,\mathrm{d}z + \int \Delta E\,\mathrm{d}t$. Here ΔE represents the change in energy caused by the presence of the Coulomb potential or by any other long-range potential. Although $\int \Delta E\,\mathrm{d}t \ll \int p_z\,\mathrm{d}z$, it changes the pre-exponential factor significantly, since $\int \Delta E\,\mathrm{d}t \gg 1$ and therefore has to be taken into account.

Inserting the Coulomb potential gives the additional phase factor $Q_c = \exp(\mathrm{i}\int(\kappa/r)\,\mathrm{d}t)$ picked up by the electron on its way through the barrier. By using the relation $\mathrm{d}t = (\mathrm{d}t/\mathrm{d}z)\,\mathrm{d}z = \mathrm{d}z/P_z = -\mathrm{i}\,\mathrm{d}z/\sqrt{\kappa^2 - 2Fz}$ the factor Q_c can be transformed back into an integral over the classical trajectory,

$$Q_c = \exp\left(\kappa \int_{z_1}^{a} \frac{\mathrm{d}z}{z\sqrt{\kappa^2 - 2Fz}}\right). \tag{10}$$

The transversal coordinates x, y, $p_\perp$ give higher-order corrections and were therefore neglected in the derivation of Q_c. Calculation of the integral in equation (10) yields

$$Q_c = \exp\left(\ln\frac{2\kappa^2}{Fz_1}\right). \tag{11}$$

By multiplying equation (9) with (11) we obtain the Coulomb correction factor

$$Q = \left.\frac{\Psi_c}{\Psi_{sr}}\right|_{z=z_1} Q_c = \frac{1}{A\sqrt{\pi}} \exp\left(\ln\frac{2\kappa^3}{F}\right) = \frac{2\kappa^3}{AF\sqrt{\pi}}, \tag{12}$$

where the arbitrary distance z_1 drops out. Finally, the correct tunnelling ionization rate of a hydrogen-like atom can be found by multiplying the tunnelling ionization rate (7) from the short-range potential by the absolute square of equation (12) yielding

$$w_{cn} = \frac{4\kappa^5}{F} \exp\left(-\frac{2\kappa^3}{3F}\right). \tag{13}$$

Note that equation (13) could also have been obtained in a more direct way by matching the Coulomb wavefunction (8) to the quasiclassical sub-barrier wavefunction (5) instead of the ground state of the short-range potential. The way we have chosen here offers the advantage of yielding the ionization rates for a broader class of potentials without additional effort.

So far we have calculated the ionization rate from the ground state of a hydrogen atom. We now generalize our calculation to arbitrary s states of general atoms, which requires two modifications. (i) Instead of the ground state wavefunctions (1) and (8) the asymptotic expansion [18] of the radial s-state wavefunction must be used. (ii) In contrast to hydrogen atoms, arbitrary atoms exhibit a nonzero quantum defect. The quantum defect is taken into account by substituting the principal quantum number n with an effective principal quantum number $n^* = Z/\kappa$, where $\kappa = \sqrt{2I_p}$ and I_p is the experimentally measured value of the binding energy. For a detailed derivation of quantum defect theory see [10]. Inclusion of the quantum defect corrections gives

$$w_{cn} = \left(\frac{F}{4Z}\right)\left(\frac{4\kappa^4}{ZF}\right)^{2Z/\kappa} \exp\left(-\frac{2\kappa^3}{3F}\right) \tag{14}$$

instead of equation (13). The ionization rate w_c is valid for s states as well as for the p states, when averaging over the magnetic quantum numbers is performed [10].

3. Relativistic tunnel ionization of spinless particles in a static electric field

Spinless particles are described by the Klein–Gordon (relativistic) wave equation. In this section tunnel ionization in a static electric field is calculated following the approach of [17]. The derivation presents the relativistic generalization of the nonrelativistic ionization theory presented in the previous section. First, tunnelling in a short-range potential is considered and then the Coulomb correction is calculated.

The wavefunction of the ground state,

$$\Psi_{sr}(r) = \frac{A\sqrt{\kappa}}{r} \exp(-\kappa r), \tag{15}$$

is similar to the nonrelativistic ground state (1). Here $\kappa = \sqrt{1-\varepsilon^2}$ denotes the total relativistic energy in relativistic units ($\varepsilon < 1$). In the remainder of the paper we use the relativistic system of units $c = m = \hbar = 1$, which is more convenient for relativistic calculations. The final results (ionization rates) will be given in Gaussian units.

In analogy to section 2 the quasiclassical solution is connected with the ground state wavefunction (15) giving the WKB solution for the electron wavefunction under the barrier. From that the current at the outer turning point $a = (1-\varepsilon)/eF$ is found to be

$$j_z(\boldsymbol{p}_\perp) = p_z(z > a)|\Psi_{sr}(z > a, \boldsymbol{p}_\perp)|^2 = 4\pi^2 A^2 \exp\left(2\mathrm{i}\int_0^a p_z\,\mathrm{d}z\right) \tag{16}$$

with $p_z = \mathrm{i}\sqrt{1-(\varepsilon+eFz)^2+p_\perp^2}$. Next the integrand in equation (16) is expanded with respect to $p_\perp$ and the integration over dz is performed yielding

$$j_z(\boldsymbol{p}_\perp) = 4\pi^2 A^2 \exp\left(-\frac{1}{eF}\left(\arccos\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right) - \frac{\arccos\varepsilon}{eF} p_\perp^2\right). \tag{17}$$

The resulting Gaussian transversal momentum distribution has a width $p_\perp \propto \sqrt{F} \ll p_z \approx 1$ justifying the validity of the expansion. Note that $F = 1$ in relativistic units corresponds to $F = F_s$ in Gaussian units, where $F_s = m^2c^3/e\hbar = 1.32 \times 10^{16}$ V cm^{-1} is the Schwinger field strength [21] corresponding to an intensity $I_s = 4.7 \times 10^{29}$ W cm^{-2}. From the resulting expression for the current we obtain the Klein–Gordon tunnelling ionization rate for a short-range potential by integrating over the transversal momentum distribution

$$w_{sk} = \int_0^\infty j_z(\boldsymbol{p}_\perp)\frac{p_\perp\,\mathrm{d}p_\perp}{2\pi} = \frac{\pi A^2 eF}{\arccos\varepsilon} \exp\left[-\frac{1}{eF}\left(\arccos\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right)\right]. \tag{18}$$

The result in Gaussian units is

$$w_{sk} = \frac{\pi A^2 mc^2}{\hbar \arccos\varepsilon}\frac{F}{F_S} \exp\left[-\frac{F_S}{F}\left(\arccos\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right)\right], \tag{19}$$

where F_s is the Schwinger field strength defined below equation (17). Finally, in the nonrelativistic limit, equation (19) goes over into equation (7).

Next the Coulomb correction is included which applies to the relativistic (spinless) π^--atom with highly charged nucleus $Z \gg 1$. The total relativistic energy of the ground state of the Klein–Gordon equation is given by [22]

$$\varepsilon = \left[1 + \frac{\mu^2}{\left(1 + \sqrt{1-4\mu^2}\right)^2}\right]^{-1/2} < 1, \tag{20}$$

where $\mu = Ze^2$ in relativistic units. In this solution the charge of the nucleus Z is limited to 68, for which the binding energy is $\varepsilon = 1/\sqrt{2}$. The normalized ground state wavefunction is [22]

$$\Psi_c = Br^s \exp(-\kappa r) \tag{21}$$

with $s = -1/2 + 1/2\sqrt{1-4\mu^2}$, $\kappa = \sqrt{1-\varepsilon^2}$ and $B = \sqrt{(2\kappa)^{2s+3}/(4\pi\Gamma(2s+3))}$. In the nonrelativistic limit equation (21) reduces to the solution of the hydrogen-like atom (8).

The Coulomb factor $Q_c = \exp(\mathrm{i}\int(e^2Z/r)\,\mathrm{d}t)$ is calculated analogously to the nonrelativistic case by using $\mathrm{d}t = \mathrm{d}z(\mathrm{d}t/\mathrm{d}z)$, where the velocity $\mathrm{d}z/\mathrm{d}t$ is connected to the momentum by $p_z = (\varepsilon + eFz)\,\mathrm{d}z/\mathrm{d}t$. Using this relation in Q_c we obtain

$$Q_c = \exp\left(Ze^2 \int_{z_1}^{a} \frac{\mathrm{d}z(\varepsilon + eFz)}{z\sqrt{1-(\varepsilon+eFz)^2}}\right). \tag{22}$$

Similar to the nonrelativistic case, the quasi-static correction (22) is valid as long as the lower limit of the integral is in the range $a_0 \ll z_1 \ll a$. Calculation of the integral yields

$$Q_c = \exp\left\{\frac{e^2Z\varepsilon}{\sqrt{1-\varepsilon^2}} \ln\left[\frac{2(1-\varepsilon^2)}{eFz_1}\right] + e^2Z \arccos\varepsilon\right\}. \tag{23}$$

In order to obtain the Coulomb corrected ionization rate, Q_c must be multiplied with the ratio of the Coulomb (21) and of the short-range potential (15) wavefunctions

$$\left.\frac{\Psi_c}{\Psi_{sr}}\right|_{z=z_1} = \frac{B}{A(1-\varepsilon^2)^{1/4}} \exp[(s+1)\ln z_1]. \tag{24}$$

Performing the multiplication and using the relation $s + 1 = e^2Z\varepsilon/\sqrt{1-\varepsilon^2}$ the arbitrary quantity z_1 cancels and we obtain [15]

$$w_{ck} = D\exp\left[-\frac{F_s}{F}\left(\arccos\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right) + 2\mu\arccos\varepsilon\right], \tag{25}$$

with $\eta = \mu\varepsilon/\sqrt{1-\varepsilon^2}$ and

$$D = \frac{mc^2}{\hbar}\frac{2(1-\varepsilon^2)^{3\eta}}{\Gamma(2\eta+1)\arccos\varepsilon}\left(\frac{4F_s}{F}\right)^{2\eta-1}. \tag{26}$$

In the nonrelativistic limit equation (25) reduces to (13). When the quantum defect is nonzero, equation (20) does not give the correct binding energy, and the experimental binding energy ε has to be used instead [10]. The pre-exponential factor containing the quantum defect is calculated analogously to section 2 and is

$$D = \frac{mc^2}{\hbar}\frac{2}{\Gamma(2s+3)\arccos\sqrt{1+s}}\left(\frac{(1-\varepsilon^2)^4}{|s|}\right)^{\eta}\left(\frac{4F_s}{F}\right)^{2\eta-1}. \tag{27}$$

Equations (25) and (27) describe ionization of arbitrary atomic ions in a static electric field starting from s or p states averaged over the magnetic quantum number. The nonrelativistic limit is given by equation (14).

Finally, if we do not integrate over the transversal momentum in equation (17), we obtain the transversal momentum distribution at the time of birth as

$$w_{ck}(p_\perp) = w_{ck}\frac{4\pi\arccos\varepsilon}{eF}\exp\left(-\frac{\arccos\varepsilon}{eF}p_\perp^2\right). \tag{28}$$

4. Relativistic tunnel ionization of hydrogen-like atoms in a static electric field

In this section we generalize the nonrelativistic approach of section 2 to tunnel ionization of hydrogen-like atoms in a constant electric field, as described by the Dirac equation. Again, we start with the short-range potential and then correct for the presence of the Coulomb potential. The ground state of this particle is determined by the quantum numbers [23]: $j = 1/2$, $m = 1/2$, $K = -1$. The ground state bispinor wavefunction of a short-range Dirac atom is given by [23]

$$\Psi_{sr}(\boldsymbol{r}) = \frac{A\sqrt{\kappa}}{r}\exp(-\kappa r)\left(1, 0, \frac{\kappa + 1/r}{1+\varepsilon}\cos\theta, \frac{\kappa+1/r}{1+\varepsilon}\sin\theta\exp(\mathrm{i}\varphi)\right). \quad (29)$$

Here ε is the binding energy of the particle including its rest mass and $\kappa = \sqrt{1-\varepsilon^2}$. The spherical coordinates r, θ, φ are related to the Cartesian coordinates by $z = r\cos\theta$ and $x + \mathrm{i}y = r\sin\theta\exp(\mathrm{i}\varphi)$. It should be noted here that, unlike the nonrelativistic case, the Dirac equation with a δ-function potential contains divergencies. Therefore, we assume an arbitrary short-range potential and cannot specify the normalization constant A.

Tunnelling is determined by the asymptotic part of the wavefunction, $r \gg 1/\kappa = a_0$. The continuation of the wavefunction into the sub-barrier region is done as in sections 2 and 3 and gives

$$\begin{aligned}\Psi_{sr}(z > a, \boldsymbol{p}_\perp) = {} & \frac{2\pi A}{\sqrt{(1+\varepsilon)p_z(z>a)}}\left(\sqrt{1+\varepsilon}, 0, -\sqrt{1-\varepsilon}\cos\theta, \sqrt{1-\varepsilon}\sin\theta\exp(\mathrm{i}\varphi)\right)\\ & \times\exp\left[-\frac{1}{2eF}\left[\arccos\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right] - \frac{\arccos\varepsilon}{2eF}p_\perp^2\right]. \end{aligned} \quad (30)$$

The momentum in the z-direction is again given by $p_z = \mathrm{i}\sqrt{1-(\varepsilon+eFz)^2+p_\perp^2}$. Calculation of the electron current emitted from the barrier is performed as in the previous sections and results in

$$j_z(p_\perp) = \frac{8\pi^2A^2}{1+\varepsilon}\exp\left(-\frac{\arccos\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}}{eF}\right)\exp\left(-\frac{\arccos\varepsilon}{eF}p_\perp^2\right). \quad (31)$$

After making a transformation to Gaussian units the ionization rate

$$w_{sd} = \frac{2\pi A^2mc^2}{\hbar(1+\varepsilon)\arccos\varepsilon}\frac{F}{F_s}\exp\left[-\frac{F_s}{F}\left(\arccos\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}\right)\right] \quad (32)$$

is obtained. In the nonrelativistic limit equation (32) goes over into equation (7).

In order to calculate the Coulomb correction we consider the ground state solution of the Dirac equation for the hydrogen-like atom with charge Z

$$\Psi_c(\boldsymbol{r}) = B(2\mu r)^{\varepsilon-1}\exp(-\mu r)\left(\sqrt{1+\varepsilon}, 0, -\sqrt{1-\varepsilon}\cos\theta, \sqrt{1-\varepsilon}\sin\theta\exp(\mathrm{i}\varphi)\right) \quad (33)$$

with quantum numbers $j = 1/2$, $m = 1/2$, $K = -1$. The coefficient $B = \mu^{3/2}\sqrt{1/(\pi\Gamma(2\varepsilon+1))}$ and the energy of the ground state ($Z < 137$) is $\varepsilon = \sqrt{1-\mu^2}$, where $\mu = Ze^2$. The binding energy can take values lying in the interval $0 < \varepsilon < 1$. Now we can divide equation (33) by (29), using the fact that in the case of a static electric field the electron tunnels predominantly in the z-direction (i.e. $a_0 \ll r \approx z_1 \ll a$), and find the ratio

$$\left.\frac{\Psi_c}{\Psi_{sr}}\right|_{z=z_1} = \frac{B(2\mu)^{\varepsilon-1}\sqrt{1+\varepsilon}}{A(1-\varepsilon^2)^{1/4}}\exp(\varepsilon\ln z_1), \quad (34)$$

with $a_0 = 1/\mu$ the Bohr radius and $a = (1-\varepsilon)/(eF)$ the outer turning point. Finally, equation (34) is multiplied with the Coulomb factor (23). The absolute square of the resulting factor is multiplied with the ionization rate for the short-range potential, equation (32), followed by a transformation to Gaussian units. This yields the ionization rate from the ground state of a hydrogen-like ion

$$w_{cd} = \frac{mc^2}{\hbar}\frac{2^{4\varepsilon-1}\mu^{6\epsilon}}{\Gamma(2\varepsilon+1)\arccos\varepsilon}\left(\frac{F_s}{F}\right)^{2\varepsilon-1}\exp\left[-\frac{F_s}{F}\left(\arccos\varepsilon-\varepsilon\sqrt{1-\varepsilon^2}\right)+2\mu\arccos\varepsilon\right] \tag{35}$$

with $\mu = e^2 Z/\hbar c$ in Gaussian units. In the nonrelativistic limit the ionization rate (35) reduces to equation (13). The generalization of equation (35) to the case of arbitrary s-state atomic ions is obtained by using in the above derivation the experimental value of the binding energy instead of $\varepsilon = \sqrt{1-\mu^2}$ [10]. The result is

$$w_{cd} = \frac{mc^2}{\hbar}\frac{\mu^{-2\eta}2^{4\eta-1}(1-\varepsilon^2)^{4\eta}}{\Gamma\left(2\sqrt{1-\mu^2}+1\right)\arccos\left(\sqrt{1-\mu^2}\right)}\left(\frac{F_s}{F}\right)^{2\eta-1} \times \exp\left[-\frac{F_s}{F}\left(\arccos\varepsilon-\varepsilon\sqrt{1-\varepsilon^2}\right)+2\mu\arccos\varepsilon\right] \tag{36}$$

with $\eta = e^2 Z\varepsilon/\hbar c\sqrt{1-\varepsilon^2}$ in Gaussian units. In the nonrelativistic limit equation (36) goes over into equation (14). The momentum distribution is the same as for the Klein–Gordon equation, see equation (28), but with w_{cd} given by equation (28).

In the final part of this section we will characterize the parameter range in which ionization is modified by relativistic effects. In figure 1(a) the ratio of equation (35) to (13) is depicted as a function of F/F_{bs} for various charge states Z. Here,

$$F_{bs} = \frac{F_s}{4\mu}\left(1-\sqrt{1-\mu^2}\right)^2 \tag{37}$$

is the relativistic generalization of the barrier suppression field strength [2]. This is the field strength at which the maximum of the effective Coulomb barrier is equal to the binding energy, i.e. $V(z) = -\mu/z_m - eFz_m = \varepsilon - 1$. The position z_m at which the barrier is maximum is determined by $(\mathrm{d}/\mathrm{d}z)V(z) = 0$. For the charges $Z = 1, 10, 20, 40, 60$, the barrier suppression intensities are $I_{bs} = 1.4\times10^{14}$, 1.4×10^{20}, 9.0×10^{21}, 6.3×10^{23} and 8.1×10^{24} W cm^{-2}, respectively. Although our tunnelling theory is strictly speaking only valid in the range $F < F_{bs}$, we have plotted until $F/F_{bs} = 1.5$. It is known from exact numerical, nonrelativistic calculations that the tunnelling theory overestimates above the barrier ionization, but gives a rough estimate [24].

Figure 1(a) shows that for $Z = 1$ the nonrelativistic and the relativistic theory agree, which corroborates the validity of our analysis. For $Z \geqslant 10$ relativistic effects start to appear. The agreement between nonrelativistic and relativistic theory becomes better with increasing field strength. This, at first sight, counter-intuitive behaviour can be understood by recalling that there are two sources for relativistic effects. (i) Relativistic effects originate from the relativistic motion of electrons in the static electric field. (ii) In the case of high charge states, deeply bound electrons move in the binding potential with velocities close to c causing relativistic effects in the bound state wavefunctions. As the distance under the barrier is extremely short and the electron is born with zero velocity in the continuum, field-induced relativistic effects are expected to be weak. However, it is well known that for charge states $Z \geqslant 10$ relativistic effects must be taken into account to model the bound state dynamics properly. Therefore, with increasing field strength the electric field dominates the electron dynamics more and more and the influence of the Coulomb potential is reduced, which explains the increasingly better agreement in figure 1(a). The form of the curves for $F/F_{bs} \ll 1$ is determined by the factor $(F/F_{bs})^{\mu^2}$, which comes from the difference between relativistic and nonrelativistic binding

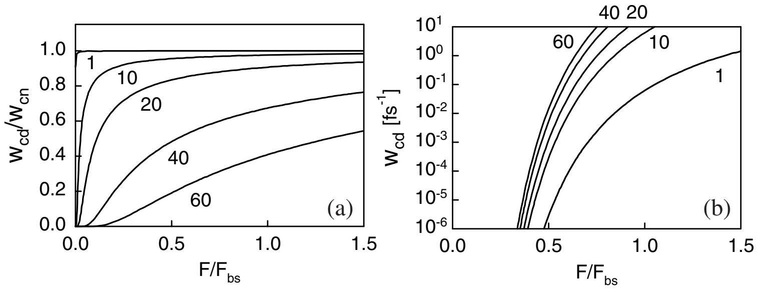

Figure 1. (a) The ratio of relativistic (Dirac) to nonrelativistic instantaneous ionization rates in a static electric field, as determined by the ratio of equations (35)–(13) for ionic charge states $Z = 1, 10, 20, 40, 60$. The field strength is normalized to the barrier suppression field strength, F_{bs}, given in equation (37). For $Z = 1, 10, 20, 40, 60$, the barrier suppression intensities are $I_{bs} = 1.4 \times 10^{14}, 1.4 \times 10^{20}, 9.0 \times 10^{21}, 6.3 \times 10^{23}$ and 8.1×10^{24} W cm^{-2}, respectively. (b) The relativistic ionization rate (35) in a parameter range relevant for experiments.

energy. Although $\mu^2 \ll 1$, it causes $w_{cd}/w_{cn} \to 0$ for $F \to 0$. Note, however, that the range $F \to 0$ is experimentally irrelevant as the ionization rate disappears.

In figure 1(b) the relativistic ionization rate (35) is plotted for the same charge states as in figure 1(a). For field strengths $0.3F_{bs} < F < 0.8F_{bs}$ appreciable ionization takes place. In this range, relativistic effects are negligible up to charge states $Z = 20$. For higher charge states the influence of relativistic dynamics grows quickly and makes up to an order of magnitude difference at $Z = 60$.

Finally, the ionization theory developed for quasi-static electric fields can in special cases also be applied to tunnel ionization in laser fields. In pure electric fields or in nonrelativistic laser fields, the magnetic-field-induced Lorentz force is negligible. As a result the electron can during its quiver motion in the continuum recollide with its parent nucleus. At relativistic laser intensities, the magnetic-field-induced Lorentz force pushes the electron trajectory in the direction of the laser wavevector, thus prohibiting recollision [25]. Realization of recollisions at relativistic intensities would be of great interest for a number of reserach fields. The huge current and the large kinetic energy of the returning electron would allow for example efficient realization of nuclear processes. One way to eliminate the Lorentz force is the use of counter-propagating laser fields, where at the peak of each laser cycle magnetic field effects cancel out [9]. In this case the ionization theory developed above becomes relevant.

5. Relativistic tunnel ionization of spinless particles in an electromagnetic field

In the nonrelativistic limit tunnelling takes place only as long as the tunnelling time is much shorter than the oscillation period of the electromagnetic field. This is the so-called low-frequency or quasi-static limit of field ionization, where the electromagnetic field may be considered constant during tunnelling. The range of validity of the nonrelativistic quasi-static approach is determined by the Keldysh parameter [26]. Here we introduce the relativistic generalization of the Keldysh parameter which is in Gaussian and SI units

$$\gamma = \frac{\hbar\omega}{mc^2}\frac{F_s}{F}\sqrt{1-\left(\frac{E}{mc^2}\right)^2}, \tag{38}$$

which reduces to the usually used Keldysh parameter in the nonrelativisitic limit. Here, F and ω denote the electric field strength and the circular frequency of the laser field, respectively, and E is the relativistic binding energy.

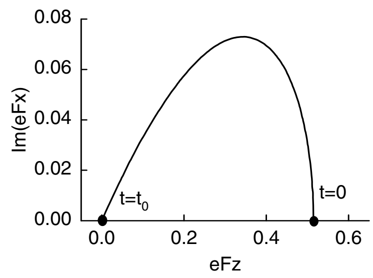

Figure 2. The subbarrier trajectory for relativistic tunnelling in the case of an electromagnetic field, where the case $\varepsilon = 1/2$ is considered. The presence of the magnetic field induces a motion along the x-direction, which is imaginary under the barrier. At time $t = 0$ the electron is born in the continuum at the right turning point eFa. The electron starts at the imaginary initial time t_0 at position $eFz(t_0) = eFx(t_0) = 0$.

In the following we will derive the tunnel ionization of π^- atoms in electromagnetic fields, as described by the Klein–Gordon equation. In the quasi-static limit, linear and circular laser polarization may be substituted by perpendicular, constant electric and magnetic fields of equal amplitude so that the resulting ionization rates apply to both cases [17]. The derivation is performed following the pattern of the previous sections. Ionization in a short-range potential is calculated followed by a correction for the presence of the Coulomb potential. Electric, magnetic field and wavevector are chosen along the z-, y- and x-axis, respectively. Similar to the case of the static electric field, the tunnelling electron moves along the polarization (z-) axis of the electric field; however, due to the presence of the magnetic field the Lorentz force introduces an additional motion into the (x-) direction of the laser wavevector, which breaks the cylindrical symmetry. In this case, determination of the momenta of the two-dimensional quasiclassical wavefunction $\exp(\mathrm{i}\int p_z\,\mathrm{d}z + \mathrm{i}\int p_x\,\mathrm{d}x)$ is no longer obvious. Therefore, first the complex two-dimensional trajectories of the electron sub-barrier motion must be determined. The classical relativistic equations of motion are solved subject to the following boundary conditions: birth time $t = 0$, outer turning point $z(t = 0) = a = 3(\lambda^2 - 1)/(2F\lambda)$, $z(t_0) = 0$, $x(t_0) = 0$ with t_0 a complex time at which the electron energy is equal to the binding energy ε. The parameter λ is given by $\lambda = (1/2)(\sqrt{\varepsilon^2 + 8} - \varepsilon)$. The electron trajectories are found in terms of the parametric equations [17, 27]

$$\begin{aligned} x &= \frac{\mathrm{i}}{2F\lambda}\left[(\lambda^2 - 1)u - \frac{1}{3}u^3\right] \\ z &= \frac{1}{2F\lambda}[3(\lambda^2 - 1) - u^2] \\ t &= \frac{\mathrm{i}}{2F\lambda}\left[(\lambda^2 + 1)u - \frac{1}{3}u^3\right], \end{aligned} \tag{39}$$

where $0 \leqslant u \leqslant \sqrt{3(\lambda^2 - 1)}$ is the parametric variable. A sample trajectory is plotted in figure 2 for the case of $\varepsilon = 1/2$. The appearance of complex space coordinates, momenta and times is associated with the classically forbidden sub-barrier motion of electrons. After the electron leaves the barrier, all quantities become real and the electron is, due to the Lorentz force, predominantly accelerated in the direction of the laser wavevector [12, 28]

From equation (39) the total classical energy of the electron is found to be $E(u) = (\lambda^2 - u^2 + 1)/(2\lambda)$. The relativistic classical kinetic momentum is obtained by $p_{zc} = (\mathrm{d}z/\mathrm{d}t)E = \mathrm{i}\sqrt{1 + p_{xc}^2 - (\varepsilon + eFz)^2}$, where $p_{xc} = E(\mathrm{d}x/\mathrm{d}t) = p_{x0} + eFz$ is the classical kinetic momentum in the x-direction and $p_{x0} = p_{xc}(t_0) = (1/4)(3\varepsilon - \sqrt{\varepsilon^2+8}) < 0$ is the initial x-momentum at time t_0. The integral over dx in the quasiclassical wavefunction $\exp(\mathrm{i}\int_0^0 p_{x0}\,\mathrm{d}x)$ gives no contribution. Note that the momentum in the integral is the canonical momentum. The canonical momentum is $p_{x0} = $ const, as the Hamiltonian is independent of the x-coordinate. Therefore, the integral over dx gives zero. In the z-direction the classical momentum p_{zc} is equal to the canonical momentum and we are left with the quasiclassical wavefunction $(1/\sqrt{-\mathrm{i}p_{zc}})\exp(\mathrm{i}\int p_{zc}\,\mathrm{d}z)$, which must be matched to the ground state wavefunction. In the case of a static electric field we obtained, in section 2, $p_{xc} = p_{yc} = 0$ so that in the ground state wavefunction the p_z momentum could be approximated by $\tilde{p}_z = \sqrt{\kappa^2+p_x^2+p_y^2} \approx \kappa$. As a result the pre-exponential factor in the ground state wavefunction (3) simplified to $\sqrt{\kappa}/\tilde{p}_z \approx 1/\sqrt{\tilde{p}_z}$.

In the presence of an electromagnetic field the classical equations of motion determine the z-momentum at the matching point z_1 as $p_z(z_1) = \mathrm{i}\sqrt{\kappa^2+p_{x0}^2}$ $(\kappa \sim p_{x0})$ assuming that the contribution eFz_1 is small at the matching point. As a result, the z-dependence of the ground state and of the quasi-static wavefunction is different. Therefore, in order to match the two functions, we rewrite the pre-exponential factor in (3) as $(\sqrt{\kappa}/\tilde{p}_z) = (1/\sqrt{\tilde{p}_z(z)})\sqrt{\kappa/\tilde{p}_z(z=z_1)}$, where

$$K^2 = \frac{\kappa}{\tilde{p}_z(z=z_1)} = \sqrt{1-\frac{1}{3}\xi^2} \tag{40}$$

with $\xi = \sqrt{1-\varepsilon(\sqrt{\varepsilon^2+8}-\varepsilon)/2}$. As a result of the above considerations the matched quasiclassical solution can be written as

$$\Psi_{qc}(z,\boldsymbol{p}_\perp) = \frac{2\pi AK}{\sqrt{-\mathrm{i}p_z}}\exp\left(\mathrm{i}\int_0^z p_z\,\mathrm{d}z\right), \tag{41}$$

where $p_z = \mathrm{i}\sqrt{1+p_x^2+p_y^2-(\varepsilon+eFz)^2}$, and $p_x = p_{xc}+\delta p_x$, $\delta p_x \ll p_{xc}$. From the WKB solution the current is obtained as

$$j_z(p_x,p_y) = 4\pi^2A^2K^2\exp\left(-2\,\mathrm{Im}\int_0^a p_z\,\mathrm{d}z\right). \tag{42}$$

By using equation (6) we obtain for the ionization rate

$$w_{sk} = A^2K^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-2\,\mathrm{Im}\int_0^a p_z\,\mathrm{d}z\right)\mathrm{d}p_x\,\mathrm{d}p_y. \tag{43}$$

Expansion of p_z with respect to p_x and p_y and performing the integration over dp_x and dp_y followed by a transformation to Gaussian units yields

$$w_{sk} = \frac{mc^2}{\hbar}\frac{\pi A^2}{\sqrt{3}\xi}\sqrt{\frac{3-\xi^2}{3+\xi^2}\frac{F}{F_s}}\exp\left(-\frac{2\sqrt{3}\xi^3}{1+\xi^2}\frac{F_s}{F}\right). \tag{44}$$

In the nonrelativistic limit w_{sr} reduces to equation (7).

The Coulomb correction is determined by the factor

$$Q_c = \exp\left(\mathrm{i}\mu\int_{t_1}^0\frac{\mathrm{d}t}{r}\right) = \exp\left(\mathrm{i}\mu\int_0^{u_1}\frac{(\lambda^2+1-u^2)\,\mathrm{d}u}{(3(\lambda^2-1)-u^2)\sqrt{1-u^2/9}}\right), \tag{45}$$

where $r = \sqrt{z^2-|x|^2}$ and $u_1 = u(t_1)$. The minus in the definition of the r comes from the fact that the trajectory in x-direction is imaginary. The time t_1 is the imaginary time, when the electron is at the matching position z_1 under the barrier and $t = 0$ is the time at which the particle arrives at the outer turning point $z = a$; see figure 2. In equation (45) the integral over dt was transformed into an integral over du by using equation (39). Solution of the integral in equation (45) and multiplication of Q_c with $(\Psi_c/\Psi_{sr})_{z=z_1}$ in equation (24) gives

$$Q = \frac{1}{A}\left[\frac{4\xi^3(3-\xi^2)^2}{F\sqrt{3}(1+\xi^2)}\right]^{\eta}\frac{\exp\left(3\mu\arcsin\frac{\xi}{\sqrt{3}}\right)}{\sqrt{2\pi}\,\Gamma(2\eta+1)} \tag{46}$$

with $\eta = \mu\varepsilon/\sqrt{1-\varepsilon^2}$. Transformation of $|Q|^2$ to Gaussian units followed by a multiplication with equation (44) gives the tunnel ionization rate of a Klein–Gordon particle

$$w_{ck} = \frac{mc^2}{2\sqrt{3}\hbar\Gamma(2\eta+1)\xi}\sqrt{\frac{3-\xi^2}{3+\xi^2}}\left[\frac{4\xi^3(3-\xi^2)^2}{\sqrt{3}(1+\xi^2)}\right]^{2\eta}\left(\frac{F_s}{F}\right)^{2\eta-1}$$
$$\times\ \exp(-\beta(F_s/F) + 6\mu\arcsin(\xi/\sqrt{3})). \tag{47}$$

The coefficient in the exponent is given by $\beta = 2\sqrt{3}\xi^3/(1+\xi^2)$.

If the integration over the transversal momentum is omitted we obtain the transversal momentum distribution of the electron born in the continuum. This is

$$w_{ck}(p_y, p_x) = w(0, p_{x0})\exp[-Mp_y^2 - N(p_x - p_{x0})^2]. \tag{48}$$

Here, $w(0, p_{x0})$ is connected to the ionization rate in equation (47) by integrating over dp_x and dp_y. The parameters M and N are given by

$$M = \frac{\sqrt{3}\xi}{F} \tag{49}$$

and by

$$N = \frac{\xi(\xi^2+3)}{\sqrt{3}F}. \tag{50}$$

In the case of circularly polarized light the cycle-averaged ionization rate is equivalent to the instantaneous ionization rate (47). In order to determine the cycle-averaged ionization rate of linearly polarized light, we substitute $F \to F\cos(\omega t)$ in equation (47) and utilize the fact that ionization takes place predominantly at the peaks of the laser electric field. Therefore, we may expand the laser field as $F\cos(\omega t) = F(1 - \omega^2 t^2/2)$, where ω is the laser circular frequency. The time dependence is dominated by the Gaussian dependence. Averaging over half a laser period gives the additional factor

$$\sqrt{\frac{2F}{\pi F_s\beta}}. \tag{51}$$

6. Relativistic tunnel ionization of hydrogen-like atoms in an electromagnetic field

The solution of the Dirac equation is performed analogous to sections 4 and 5 and yields the ionization rate

$$w_{sd} = \frac{mc^2}{\hbar}\frac{2\pi A^2}{\sqrt{3}\xi}\frac{1}{1+\varepsilon}\sqrt{\frac{3-\xi^2}{3+\xi^2}}\frac{F}{F_s}\exp\left(-\frac{2\sqrt{3}\xi^3}{1+\xi^2}\frac{F_s}{F}\right) \tag{52}$$

for the short-range potential and

$$w_{cd} = \frac{mc^2}{\hbar}\frac{1}{2\sqrt{3}\Gamma(2\varepsilon+1)\xi}\sqrt{\frac{3-\xi^2}{3+\xi^2}}\left[\frac{4\xi^3(3-\xi^2)^2}{\sqrt{3}(1+\xi^2)}\right]^{2\varepsilon}\left(\frac{F_s}{F}\right)^{2\varepsilon-1}$$
$$\times\ \exp(-\beta(F_s/F) + 6\mu\arcsin(\xi/\sqrt{3})) \tag{53}$$

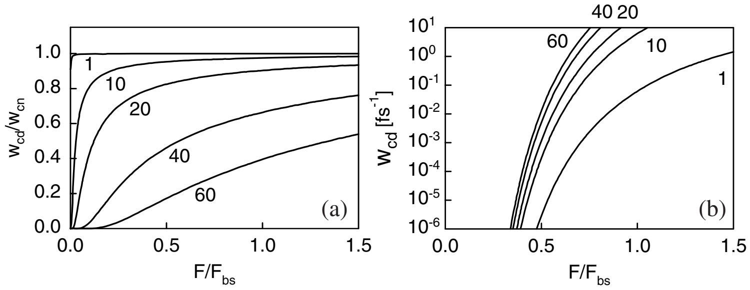

Figure 3. (a) The ratio of relativistic (Dirac) to nonrelativistic instantaneous ionization rates in an electromagnetic field, as determined by the ratio of equations (53)–(13) for ionic charge states $Z = 1, 10, 20, 40, 60$. The field strength is normalized to the barrier suppression field strength, F_{bs}, given in equation (37). (b) The relativistic ionization rate (53) in a parameter range relevant for experiments.

for the Coulomb potential. The expression for the cycle-averaged linear polarization is obtained by multiplying equation (53) with the factor specified in equation (51). The transversal momentum distribution is equivalent to equation (48) derived in the previous section.

Circularly polarized relativistic laser fields accelerate electrons after ionization to light velocity in a direction parallel to the wavevector of the laser field [12, 28]. This is of great interest for the realization of a compact laser driven electron accelerator. With the ultrashort ($<$10 fs) ultrahigh laser intensity (up to 10^{23} W cm^{-2}) sources [3] being designed right now, electron pulses of a few femtosecond duration and with energies up to a few gigaelectronvolts could be realized. A more detailed investigation of this process will be done in a future work. In order to have a chance to investigate this process on a quantitative basis, the ionization rates and momentum distribution derived here are essential.

In the following we will quantify the parameter range in which the commonly used nonrelativistic ionization theory fails and relativisitic effects must be taken into account. In figure 3(a) the ratio of equations (53)–(13) is depicted as a function of F/F_{bs} for various values of Z, respectively. The barrier suppression field strength was defined at the end of section 4. The tunnelling theory loses its validity around $F \approx F_{bs}$, where the barrier disappears. Exact numerical calculations in the nonrelativistic regime have shown that tunnelling theory overestimates above the barrier ionization [24]. The lower limit of our theory with respect to intensity arises from the onset of multiphoton ionization that occurs for $\gamma \approx 1$. Using equation (38) we find for $Z = 10$ that $\gamma = 1$ at a field strength $F/F_{bs} = 0.09$. The value of F/F_{bs} decreases with increasing Z. Therefore, tunnelling theory applies in a range between $0.01 < F/F_{bs} < 1$.

Although equations (53) and (35) for the electromagnetic and the static electric field are different, a comparison of figures 1(a), (b) with 3(a), (b) shows that numerically the difference is negligible. This shows that magnetic field effects during the sub-barrier motion give only a negligible contribution to the tunnelling rate. The only difference introduced by the magnetic field is a birth velocity component in the direction of the laser wavevector (compare equations (48) and (28)).

7. Conclusion

Based on the WKB theory, we have derived analytical formulae for relativistic tunnel ionization, as described by the Klein–Gordon and by the Dirac equations. The analysis was performed for general short-range potentials, as well as for Coulomb potentials by including Coulomb corrections. We find that the difference between relativistic and nonrelativistic ionization theories is smaller than 20% for charge states $Z \leqslant 20$. Significant ionization of ions with $Z = 20$ takes place at intensities of the order of 10^{21} W cm^{-2}. Only for larger charge states do significant deviations up to an order of magnitude (for $Z = 60$) appear. Such charge states can be ionized with laser intensities between 10^{23} and 10^{24} W cm^{-2}. Our ionization formulas make for the first time a quantitative prediction of relativistic tunnelling ionization possible allowing a better understanding of plasma processes in ultrarelativistic laser fields.

Acknowledgments

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