Nonlinearity

An explicit scattering, non-weakly mixing example and weak disjointness

To cite this article: Wen Huang and Xiangdong Ye 2002 *Nonlinearity* **15** 849

INSTITUTE OF PHYSICS PUBLISHING NONLINEARITY

Nonlinearity **15** (2002) 849–862 PII: S0951-7715(02)26828-0

An explicit scattering, non-weakly mixing example and weak disjointness

Wen Huang and Xiangdong Ye

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China

E-mail: wenh@mail.ustc.edu.cn and yexd@ustc.edu.cn

Received 11 July 2001, in final form 15 February 2002
Published 16 April 2002
Online at stacks.iop.org/Non/15/849

Recommended by M J Field

Abstract

By a dynamical system we mean a pair (X, T), where X is a compact metric space and $T : X \to X$ is surjective and continuous. We study weak disjointness in topological dynamics. (X, T) is scattering iff it is weakly disjoint from all minimal systems and (X, T) is strongly scattering iff it is weakly disjoint from all E-systems, i.e. transitive systems having invariant measures with full support. It is clear that a weakly mixing system is strongly scattering and the latter is scattering. An existential proof of scattering and a non-weakly mixing example is obtained by Akin and Glasner (2001 *J. Anal. Math.* **84** 243–86). In this paper, we will give an explicit example which is strongly scattering and not weakly mixing. We also define extreme scattering, weak scattering and study the relationships of the various definitions.

For a dynamical property P stronger than transitivity, let $P^{\curlywedge}$ be the property such that a system has $P^{\curlywedge}$ iff it is weakly disjoint from any system having P. We show that $P^{\curlywedge} = P^{\curlywedge\curlywedge\curlywedge}$. Moreover, we prove that (thickly syndetic-transitive)$^{\curlywedge}$ = piecewise-syndetic-transitive and (piecewise-syndetic-transitive)$^{\curlywedge}$ = thickly syndetic-transitive.

Mathematics Subject Classification: 54H20

1. Introduction

Let (X, T) denote a given dynamical system on a compact metric space X induced by a continuous surjective map T of X onto X. Recall that (X, T) is *transitive* if for each pair of non-empty open subsets U and V, $N(U, V) = \{n \in \mathbb{Z}_+ : T^{-n}V \cap U \neq \emptyset\}$ is infinite. Two dynamical systems are *weakly disjoint* if their product is transitive. (X, T) is an E-*system* if (X, T) is transitive and there is an invariant measure μ with full support, i.e. $supp(\mu) = X$;

(X, T) is *topologically ergodic* (TE, for short) if (X, T) is transitive and for each non-empty open subset U of X, $N(U, U)$ is syndetic, i.e. with bounded gaps. It is known that a minimal system is an E-system and the latter is TE [GW2].

For a dynamical property P stronger than transitivity, let $P^{\curlywedge}$ be the property such that a system (X, T) has $P^{\curlywedge}$ iff it is weakly disjoint from any system having P. It turns out that $P^{\curlywedge} = P^{\curlywedge\curlywedge\curlywedge}$. The chaoticity of a system is an important aspect in the study of dynamical systems and it is known that a weakly mixing system is chaotic in many aspects [HY,BGKM]. To study how the chaoticity of a system is related to open covers of the system, the notion of *scattering* is introduced by Blanchard *et al* [BHM] (using the complexity of open covers) and it can be interpreted as follows: (X, T) is scattering iff it is in minimality$^{\curlywedge}$. It is known that if (X, T) is weakly mixing then it is scattering. Akin and Glasner [AG] give an example which is scattering and not weakly mixing. The existential proof of the fact is based on the existence of monothetic groups whose only minimal actions are trivial [G]. In this paper, we will give an explicit, constructive example which is a subshift and can be verified directly. In the process to construct the example, we naturally define a property which is called *strong scattering*. A dynamical system is strongly scattering iff it is E-system$^{\curlywedge}$. Glasner and Weiss [GW2] show that a weakly mixing system is weakly disjoint from an E-system. Thus a weakly mixing system is strongly scattering, and the latter clearly implies scattering as a minimal system is an E-system. Since our example is in fact strongly scattering and not weakly mixing, strong scattering and weak mixing are different properties and at the same time it suggests that scattering, non-weakly mixing systems are not so 'rare'. This motivates us to define other notions comparable to scattering and to study the relations among them.

A dynamical system (X, T) is *extremely scattering* iff it is in TE$^{\curlywedge}$. As a weakly mixing system is weakly disjoint from any TE system [AG, GW2], weak mixing implies extreme scattering[1]. Since an E-system is TE, extreme scattering implies strong scattering. We show that if (X, T) is extremely scattering, then the access time $N(U, V)$ is piecewise syndetic, i.e. the intersection of a syndetic set with a thick set which by the definition contains arbitrary long intervals. As $N(U, V)$ cannot be piecewise syndetic for an almost equicontinuous system (see [F2, AAB] for the definitions), an almost equicontinuous system cannot be extremely scattering, and this fact implies that extreme scattering and scattering are different properties as there is an almost equicontinuous scattering system [AG]. In fact, a close look at our example shows that strong scattering and extreme scattering are different properties.

It is known that (X, T) is weakly mixing iff the access time $N(U, V)$ is thick [F1]. Using some results in [F2, W], we show that a system is strongly scattering iff the access time $N(U, V)$ is a Poincaré set, and a system is scattering iff the access time $N(U, V)$ is a recurrence set. Though there exists a recurrence set which is not a Poincaré set [W], the question if strongly scattering is equivalent to scattering remains open as here there are some restrictions on those sets.

A dynamical system is *weakly scattering* iff it is in (minimal equicontinuity)$^{\curlywedge}$. Using some result of [BHM] one easily gets that 2-scattering implies weak scattering. At the same time a weakly scattering system is totally transitive, and for an E-system weakly scattering is equivalent to scattering. If scattering and weakly scattering are different properties, then the question of [W, p. 53] if a recurrence sequence for all group rotation is a recurrence set will have a negative answer. To summarize, we have (note that any irrational rotation of the circle is totally transitive and not weakly scattering)

$$\text{weak mixing} \subsetneq \text{extreme scattering} \subsetneq \text{strong scattering}$$
$$\subset \text{scattering} \subset \text{weak scattering} \subsetneq \text{total transitivity}.$$

[1] Huang and Ye showed recently that they are different properties.

Two dynamical properties P_1 and P_2 are *symmetrically dual* if $P_1^{\curlywedge} = P_2$ and $P_2^{\curlywedge} = P_1$. It is clear that for a property P stronger than transitivity, $P^{\curlywedge}$ and $P^{\curlywedge\curlywedge}$ are symmetrically dual. We prove that thickly syndetic-transitive and piecewise-syndetic-transitive are symmetrically dual (see section 4 for the definitions). This is the first pair of symmetrically dual properties (we know) which can be described explicitly using access times.

2. Preliminary

For a dynamical property P stronger than transitivity (write $P \geqslant$ transitivity or $P \subset$ transitivity), we use $P^{\curlywedge}$ to denote the property such that a dynamical system has $P^{\curlywedge}$ iff it is weakly disjoint from all systems satisfying P. Hence we have $P^{\curlywedge} \geqslant$ transitivity. The following fact is easy to prove.

Theorem 2.1. *For a dynamical property P stronger than transitivity, we have $P^{\curlywedge\curlywedge\curlywedge} = P^{\curlywedge}$.*

Proof. It is clear that $P \geqslant P^{\curlywedge\curlywedge}$ and hence $P^{\curlywedge} \leqslant P^{\curlywedge\curlywedge\curlywedge}$. As $P^{\curlywedge}$ is a dynamical property and $P^{\curlywedge} \geqslant$ transitivity, we have $P^{\curlywedge} \geqslant P^{\curlywedge\curlywedge\curlywedge}$. That is, $P^{\curlywedge\curlywedge\curlywedge} = P^{\curlywedge}$. □

For a dynamical system (X, T), $x \in X$ and a pair of non-empty open subsets U, V of X, let $N(x, U) = \{n \in \mathbb{Z}_+ : T^n(x) \in U\}$ and

$$N(U, V) = \{n \in \mathbb{Z}_+ : T^{-n}V \cap U \neq \emptyset\}.$$

Lemma 2.2. *Let (X, S) be a transitive system with a transitive point x. If for any neighbourhood U of x we have property $(*)$,*

$(*)$ *for any $r \in \mathbb{Z}_+$ and any $k \in \mathbb{N}$ there are $p_1, \ldots, p_k > r$ such that for any $1 \leqslant i_1 \leqslant j_1 \leqslant k$, $p_{i_1,j_1} - r \in N(U, U)$, where $p_{i_1,j_1} = \sum_{l=i_1}^{j_1} p_l$,*

then (X, S) is strongly scattering.

Proof. Let (Y, T) be any E-system and T be a homeomorphism. Then there is an invariant measure μ on Y with $supp(\mu) = Y$.

Let U_1, U_2 be any non-empty open subsets of X and V_1, V_2 be any non-empty open subsets of Y. Then by the definition

$$N(U_1 \times V_1, U_2 \times V_2) = \{n \in \mathbb{Z}_+ : (S \times T)^{-n}(U_2 \times V_2) \cap (U_1 \times V_1) \neq \emptyset\}.$$

As (X, S) is transitive, there are $n_0, k \in \mathbb{N}$ such that $U = S^{-(n_0+k)}(U_2) \cap S^{-k}(U_1)$ is a non-empty neighbourhood of x. Thus,

$$\begin{aligned} N(U_1 \times V_1, U_2 \times V_2) &= \{n \in \mathbb{Z}_+ : (S \times T)^{-(n+k)}(U_2 \times V_2) \cap (S \times T)^{-k}(U_1 \times V_1) \neq \emptyset\} \\ &= \{n \in \mathbb{Z}_+ : (S^{-(n+k)}(U_2) \cap S^{-k}(U_1)) \times (T^{-(n+k)}(V_2) \cap T^{-k}(V_1)) \neq \emptyset\} \\ &\supset n_0 + \{m \in \mathbb{Z}_+ : (S^{-m}(U) \cap U) \times (T^{-m}T^{-(n_0+k)}(V_2) \cap T^{-k}(V_1)) \neq \emptyset\}. \end{aligned}$$

Since (Y, T) is transitive and T is a homeomorphism, there is an open $\emptyset \neq V \subset T^{-k}(V_1)$ and $r \in \mathbb{N}$ such that $T^{-r}(V) \subset T^{-(n_0+k)}(V_2)$. Thus

$$N(U_1 \times V_1, U_2 \times V_2) \supset n_0 + N(U \times V, U \times T^{-r}(V)).$$

As $supp(\mu) = Y$, we have $\mu(V) > 1/q > 0$ for some $q \in \mathbb{N}$. For U, r, q using the assumption we get $p_1, \ldots, p_q > r$ such that for any $1 \leqslant i_1 \leqslant j_1 \leqslant q$, $p_{i_1,j_1} - r \in N(U, U)$, where $p_{i_1,j_1} = \sum_{l=i_1}^{j_1} p_l$.

If $V, T^{-p_1}(V), \cdots, T^{-(p_1+\cdots+p_q)}(V)$ are pairwise disjoint, then we have

$$\mu(V \cup T^{-p_1}(V) \cup \cdots \cup T^{-(p_1+\cdots+p_q)}(V)) > 1.$$

This means that there are $1 \leqslant i_1 \leqslant j_1 \leqslant q$ such that $V \cap T^{-p_{i_1,j_1}}(V) \neq \emptyset$. Hence $p_{i_1,j_1} - r \in N(U \times V, U \times T^{-r}(V))$. Thus

$$p_{i_1,j_1} - r + n_0 \in N(U_1 \times V_1, U_2 \times V_2).$$

Now assume that (Y, T) is an E-system, (Y_1, T_1) is the natural extension of (Y, T) and $\pi_i : Y_1 \to Y$ is the projection to the ith coordinate. As

$$\{\pi_i^{-1}(V) : i \in \mathbb{N} \text{ and } V \text{ is open in } Y\}$$

is a basis for the topology of Y_1, it is easy to see that (Y_1, T_1) is transitive. Let μ be an invariant measure of T with full support. For each i, let ν_i be an invariant measure of T_1 with $\pi_i \nu_i = \mu$ (see [DGS]) and $\nu = \sum_{i=1}^{\infty}(1/2^i)\nu_i$. For each non-empty open subset V_1 of Y_1, there are $i \in \mathbb{N}$ and an open subset V of Y such that $V_1 \supset \pi_i^{-1}(V)$ and hence $\nu(V_1) > 0$. That is, (Y_1, T_1) is an E-system. Then by what we have proved, $(X \times Y_1, S \times T_1)$ is transitive. As $(X \times Y, S \times T)$ is a factor of $(X \times Y_1, S \times T_1)$, $(X \times Y, S \times T)$ is transitive. This completes the proof. □

3. The construction

The main purpose of this section is to construct a strongly scattering, not weakly mixing system. To do so, we use lemma 2.2 and the characterization of weak mixing [F1], i.e. a system is weakly mixing iff $N(U, V)$ is thick. Recall that a subset of $\mathbb{Z}_+$ is thick if it contains arbitrary long intervals. Now we are going to show the following theorem.

Theorem 3.1. *There is a strongly scattering, not weakly mixing system.*

Proof. We will construct the system in the one-sided shift on two symbols (Σ, S) and the system is the closure of the orbit of a recurrent point $x = (x_0, x_1, \ldots) \in \Sigma$. To do so, we construct inductively infinitely many finite words C_i such that C_{i+1} begins with C_i and x is just the limit of C_i.

To begin with, let $\{\phi(i)\}$ be a sequence of $\mathbb{Z}_+$ such that for each $i \in \mathbb{Z}_+$, there are infinitely many $j \in \mathbb{Z}_+$ with $\phi(j) = i$ and let

$$C_0 = (0), \qquad C_1 = (0, 0, 1, 0, 0) = (x_0, \ldots, x_4) \qquad \text{and} \qquad k_1 = 5.$$

Set

$$\begin{aligned} &W_1^0 = \{2\} = \{i : x_i = 1, i \leqslant k_1 - 1\}, \\ &W_1^1 = \{0\} = \{j : x_j \ldots x_{j+k_1-1} = C_1\}, \\ &B_1^0 = W_1^0 - W_1^0 = \{0\}, \qquad B_1^1 = W_1^1 - W_1^1 = \{0\}, \end{aligned}$$

where $A - B = \{a - b \geqslant 0 : a \in A,\ b \in B\}$. Inductively we construct C_l. If k_l is the length of C_l, then we define

$$\begin{aligned} &W_l^0 = \{i : x_i = 1, i \leqslant k_l - 1\} \qquad \text{for } 1 \leqslant i \leqslant l, \\ &W_l^i = \{0 \leqslant j \leqslant k_l - k_i : x_j \ldots x_{j+k_i-1} = C_i\}, \\ &B_l^i = W_l^i - W_l^i. \end{aligned} \tag{3.1}$$

Moreover, B_l^0 satisfies:

$(1)_l$ $1 \notin B_l^0$ and there is no $s \in \mathbb{N}$ such that $s, s-1 \in B_l^0$.

$(2)_l$ C_l begins with C_{l-1}.

For $l = 1$, $(1)_1$ and $(2)_1$ are satisfied. Assume that we have constructed C_n for $1 \leqslant n \leqslant l$ with $(1)_n$ and $(2)_n$. We build C_{l+1} as follows. Assume that $p_1^l, p_2^l, \ldots, p_l^l, p_{i,j}^l \geqslant k_l$ $(1 \leqslant i < j \leqslant l+1)$ are positive integers to be defined later and set

$$q_{i,j}^l = p_i^l + \cdots + p_{j-1}^l + (j-(i+1))k_l - \phi(l+1) \tag{3.2}$$

(which will be positive) for $1 \leqslant i < j \leqslant l+1$. Let $A_{i,j}^l = C_l 0^{q_{i,j}^l} C_l$, $1 \leqslant i < j \leqslant l+1$, and

$$\begin{aligned} C_{l+1} = {} & A_{1,2}^l 0^{p_{1,2}^l} A_{1,3}^l 0^{p_{1,3}^l} A_{1,4}^l 0^{p_{1,4}^l} \cdots A_{1,l+1}^l 0^{p_{1,l+1}^l} \\ & A_{2,3}^l 0^{p_{2,3}^l} A_{2,4}^l 0^{p_{2,4}^l} \cdots A_{l,l+1}^l 0^{p_{l,l+1}^l}. \end{aligned}$$

By the construction of C_{l+1} and (3.1) we get

$$\begin{aligned} W_{l+1}^0 = {} & W_l^0 \cup (W_l^0 + (q_{1,2}^l + k_l)) \cup (W_l^0 + (q_{1,2}^l + k_l) + (p_{1,2}^l + k_l)) \\ & \cup \cdots \cup (W_l^0 + (q_{1,2}^l + k_l) + (p_{1,2}^l + k_l) + \cdots + (q_{l,l+1}^l + k_l)). \end{aligned}$$

For $1 \leqslant i \leqslant l$

$$\begin{aligned} W_{l+1}^i = {} & W_l^i \cup (W_l^i + (q_{1,2}^l + k_l)) \cup (W_l^i + (q_{1,2}^l + k_l) + (p_{1,2}^l + k_l)) \\ & \cup \cdots \cup (W_l^i + (q_{1,2}^l + k_l) + (p_{1,2}^l + k_l) + \cdots + (q_{l,l+1}^l + k_l)), \\ W_{l+1}^{l+1} = {} & \{0\}. \end{aligned}$$

Denote

$$(q_{1,2}^l + k_l, p_{1,2}^l + k_l, q_{1,3}^l + k_l, p_{1,3}^l + k_l, \ldots, q_{l,l+1}^l + k_l, p_{l,l+1}^l + k_l)$$

by

$$(a_1^l, a_2^l, a_3^l, \ldots, a_{l(l+1)-1}^l, a_{l(l+1)}^l). \tag{3.3}$$

Then for $0 \leqslant i \leqslant l$,

$$\begin{aligned} B_{l+1}^i &= W_{l+1}^i - W_{l+1}^i = B_l^i \cup \left(\bigcup_{1 \leqslant j_1 \leqslant j_2 \leqslant l(l+1)-1} ((a_{j_1}^l + \cdots + a_{j_2}^l) \pm B_l^i) \right), \\ B_{l+1}^{l+1} &= \{0\}. \end{aligned} \tag{3.4}$$

We can take $p_1^l, p_2^l, \ldots, p_l^l, p_{i,j}^l$ $(1 \leqslant i < j \leqslant l+1)$ such that B_{l+1}^0 satisfies $(1)_{l+1}$. We do this at the end of the proof.

Let $x = \lim C_l$ and X be the orbit closure of x under the shift S. We now prove that (X, S) is strongly scattering and not weakly mixing.

Let $U = \{y \in X : y_0 = 1\}$. Then

$$N(x, U) = \bigcup_{l=1}^{\infty} W_l^0 \qquad \text{and} \qquad N(U, U) = \bigcup_{l=1}^{\infty} B_l^0.$$

As $B_1^0 \subset B_2^0 \subset B_3^0 \ldots$ and $(1)_l$, is satisfied by all l, we know that $N(U, U)$ is not thick, and consequently that (X, S) is not weakly mixing.

We now check that (X, S) satisfies property $(*)$, and then by lemma 2.2 (X, T) is strongly scattering. For each neighbourhood V of x, there is $l \in \mathbb{N}$ such that the cylinder $V_l = [C_l] \subset V$. It is easy to see that

$$N(x, V_l) = \bigcup_{i=l}^{\infty} W_i^l \qquad \text{and} \qquad N(V_l, V_l) = \bigcup_{i=l}^{\infty} B_i^l. \tag{3.5}$$

For $r \in \mathbb{Z}_+$ and $k \in \mathbb{N}$ there are infinitely many s with $\phi(s+1) = r$ (see lemma 2.2 for the role of r and k). Thus we may take $s \geqslant k + l$ such that $\phi(s+1) = r$. Then by (3.5) and (3.4)

$$\begin{aligned} N(V_l, V_l) &\supset B_{s+1}^l \\ &\supset \bigcup_{1 \leqslant i \leqslant j \leqslant s(s+1)-1} \{a_i^s + \cdots + a_j^s\} \qquad (\text{as } 0 \in B_s^l) \\ &\supset \bigcup_{1 \leqslant i < j \leqslant s+1} \{q_{i,j}^s + k_s\} \qquad (\text{as } q_{i,j}^s + k_s = a_t^s \text{ for some } t \ (3.3)). \end{aligned}$$

Recall that (see (3.2))

$$q_{i,j}^s = p_i^s + \cdots + p_{j-1}^s + (j - (i+1))k_s - \phi(s+1) = p_i^s + \cdots + p_{j-1}^s + (j - (i+1))k_s - r.$$

Let $p_i = p_i^s + k_s$. Then

$$q_{i,j}^s + k_s = p_i + \cdots + p_{j-1} - r$$

for $1 \leqslant i < j \leqslant s+1$. Set $p_{i,j} = \sum_{l=i}^{j} p_l$, $1 \leqslant i \leqslant j \leqslant s$. Then we have $p_{i,j} - r \in N(V_l, V_l)$ and hence (X, S) is strongly scattering by lemma 2.2.

To finish the proof we must choose $p_1^l, p_2^l, \ldots, p_l^l, p_{i,j}^l$ $(1 \leqslant i < j \leqslant l+1)$ such that B_{l+1}^0 satisfies $(1)_{l+1}$. There are many ways to do this; for example, we take $p_1^l = 2k_l + 2 + \phi(l+1)$ and inductively we take $p_i^l \geqslant \sum_{j=1}^{i-1}(p_j^l + k_l) + 2k_l + 2$ for $2 \leqslant i \leqslant l$. At the same time we let

$$p_{1,2}^l \geqslant \left(\sum_{1 \leqslant i < j \leqslant l+1} (q_{i,j}^l + k_l) \right) + 2k_l + 2.$$

For $1 \leqslant i < j \leqslant l+1$ let

$$p_{i,j}^l \geqslant \sum_{1 \leqslant i_1 < i} \sum_{j_1 = i_1 + 1}^{l+1} (p_{i_1, j_1}^l + k_l) + \sum_{i < j_1 < j} (p_{i,j_1}^l + k_l) + \sum_{1 \leqslant i < j \leqslant l+1} (q_{i,j}^l + k_l) + 2k_l + 2. \tag{3.6}$$

The reason we take $p_{i,j}^l$ in the above form is that we hope (3.8) holds, i.e. if $1 \leqslant i_1 \leqslant j_1 \leqslant l(l+1)$, $1 \leqslant i_2 \leqslant j_2 \leqslant l(l+1)$ and $\{i_1, j_1\} \neq \{i_2, j_2\}$ then the four intervals

$$[(a_{i_1}^l + \cdots + a_{j_1}^l) \pm B_l^i] \qquad \text{and} \qquad [(a_{i_2}^l + \cdots + a_{j_2}^l) \pm B_l^i]$$

are disjoint with the gaps $\geqslant 2$ (see (3.4)) as $\max B_l^i \leqslant k_l$, where $[A]$ is the convex hull of a finite set A of $\mathbb{Z}_+$ in $\mathbb{Z}_+$.

Now we check that B_{l+1}^0 satisfies $(1)_{l+1}$.

By the choosing of $p_1^l, \ldots, p_l^l$, we know that if $\{i_1, j_1\} \neq \{i_2, j_2\}$, where $i_1 < j_1$ and $i_2 < j_2$,

$$|q_{i_1,j_1}^l - q_{i_2,j_2}^l| \geqslant 2k_l + 2. \tag{3.7}$$

We claim that if $1 \leqslant i_1 \leqslant j_1 \leqslant l(l+1)$, $1 \leqslant i_2 \leqslant j_2 \leqslant l(l+1)$ and $\{i_1, j_1\} \neq \{i_2, j_2\}$, then

$$|(a_{i_1}^l + \cdots + a_{j_1}^l) - (a_{i_2}^l + \cdots + a_{j_2}^l)| \geqslant 2k_l + 2. \tag{3.8}$$

Proof of the claim: Set $I_1 = \{i_1, \ldots, j_1\}$, $I_2 = \{i_2, \ldots, j_2\}$ and $I = (I_1 \Delta I_2) \cap \{2, 4, \ldots, l(l+1)\}$.

Case 1 ($I \neq \emptyset$). Let $2m = \max I$; then by (3.2), (3.3) and (3.6)

$$\begin{aligned} |(a_{i_1}^l + \cdots + a_{j_1}^l) - (a_{i_2}^l + \cdots + a_{j_2}^l)| &\geqslant a_{2m}^l - \left(\sum_{1 \leqslant i < m} a_{2i}^l + \sum_{1 \leqslant j \leqslant l(l+1)/2} a_{2j-1}^l \right) \\ &= a_{2m}^l - \left(\sum_{1 \leqslant i < m} a_{2i}^l + \sum_{1 \leqslant i < j \leqslant (l+1)} (q_{i,j}^l + k_l) \right) \geqslant 2k_l + 2. \end{aligned}$$

Case 2 ($I = \emptyset$). If $I_1 \cap \{2, 4, \dots, l(l+1)\} = \emptyset$, then $I_1 = \{i_1\}$, $I_2 = \{i_2\}$ and $i_1 \neq i_2$ are odd. From (3.7) we have $|a^l_{i_1} - a^l_{i_2}| \geqslant 2k_l + 2$.

If $I_1 \cap \{2, 4, \dots, l(l+1)\} = \{2j, 2(j+1), \dots, 2s\}$, then $I_1, I_2 \in \{\{2j-1, 2j, 2j+1, \dots, 2s\}, \{2j, \dots, 2s\}, \{2j-1, \dots, 2s+1\}, \{2j, \dots, 2s+1\}\}$. Then

$$|(a^l_{i_1} + \cdots + a^l_{j_1}) - (a^l_{i_2} + \cdots + a^l_{j_2})| = a^l_{2s+1} \text{ or } |a^l_{2s+1} - a^l_{2j-1}| \text{ or } a^l_{2j-1} \geqslant 2k_l + 2.$$

This ends the proof of the claim.

Assume that there is s such that $\{s, s-1\} \subset B^0_{l+1}$. By $(1)_l$, $s, s-1$ do not belong to B^0_l simultaneously. As $\max B^0_l \leqslant k_l$ and $a^l_i \geqslant 2k_l+2$ $(1 \leqslant i \leqslant l(l+1))$, for $1 \leqslant i \leqslant j \leqslant l(l+1)-1$

$$\min\{(a^l_i + \cdots + a^l_j) \pm B^0_l\} \geqslant k_l + 2 \geqslant \max B^0_l + 2,$$

we have (see (3.4))

$$\{s, s-1\} \subset \bigcup_{1 \leqslant i \leqslant j \leqslant l(l+1)-1} ((a^l_i + \cdots + a^l_j) \pm B^0_l).$$

By the claim there must exist i_s, j_s such that

$$\{s, s-1\} \subset (a^l_{i_s} + \cdots + a^l_{j_s}) \pm B^0_l.$$

From $(1)_l$ there is no s with $\{s, s-1\} \subset B^0_l$; we get that

$$s \in (a^l_{i_s} + \cdots + a^l_{j_s}) + B^0_l \qquad \text{and} \qquad s - 1 \in (a^l_{i_s} + \cdots + a^l_{j_s}) - B^0_l. \tag{3.9}$$

Since

$$\min\{(a^l_{i_s} + \cdots + a^l_{j_s}) + B^0_l\} = \max\{(a^l_{i_s} + \cdots + a^l_{j_s}) - B^0_l\} = a^l_{i_s} + \cdots + a^l_{j_s},$$

(3.9) is impossible as $1 \notin B^0_l$. That is, there is no s with $\{s, s-1\} \subset B^0_{l+1}$. Hence $1 \notin B^0_{l+1}$ since $0 \in B^0_{l+1}$. □

4. The relation

In this section, we will introduce several other notions which are comparable to scattering, and discuss the relationship among them. First we recall a result of Weiss, Akin and Glasner.

A subset $\mathcal{F}$ of $2^{\mathbb{Z}_+}$ is called a *family* when it is hereditary upwards, i.e. $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. We say $\mathcal{F}$ is *proper* if $\mathcal{F}$ is neither empty nor $2^{\mathbb{Z}_+}$. If $\mathcal{F}$ is a family then its *dual*

$$k\mathcal{F} = \{F : F \cap F_1 \neq \emptyset \text{ for all } F_1 \in \mathcal{F}\}$$

is a family and we have $k(k\mathcal{F}) = \mathcal{F}$. For $i \in \mathbb{Z}_+$ let $g^i : \mathbb{Z}_+ \to \mathbb{Z}_+$ be the translation map defined by $g^i(j) = i + j$. A family $\mathcal{F}$ is called *translation invariant* if for every $i \in \mathbb{Z}_+$, $F \in \mathcal{F}$ iff $g^{-i}(F) \in \mathcal{F}$. A family $\mathcal{F}$ is a *thick family* if for each $F \in \mathcal{F}$ and $i_1, \dots, i_n \in \mathbb{Z}_+$ we have $g^{-i_1}(F) \cap \cdots \cap g^{-i_n}(F) \in \mathcal{F}$. Finally we say a dynamical system (X, T) is *$\mathcal{F}$-transitive* if $N(U, V) \in \mathcal{F}$ for each pair of non-empty open subsets U, V of X. For example if $\mathcal{F}$ is the family of thick sets, then $k\mathcal{F}$ is the family of syndetic sets. So $\mathcal{F}$-transitive is weakly mixing and $k\mathcal{F}$-transitive is TE.

The following theorem was first proved by Weiss in some special case ($\mathcal{F}$ is the family of thick subsets) and was then generalized by Akin and Glasner [AG, theorem 4.15] in the present form.

W-AG theorem. *Let $\mathcal{F}$ be a proper, translation-invariant, thick family of subsets of $\mathbb{Z}_+$. A dynamical system (X, T) is $k\mathcal{F}$-transitive iff it is weakly disjoint from any $\mathcal{F}$-transitive system.*

Before further discussions, we make several observations. Let WM and TE denote weakly mixing and topologically ergodic, respectively. It is easy to see that the properties of extreme, strong and weak scattering are preserved by factor maps. For the product we have the following proposition.

Proposition 4.1.

(1) For a TE system extreme scattering implies WM.
(2) For an E-system strong scattering implies WM.
(3) For a minimal system weak scattering implies WM.
(4) If (X, T) is WM and (Y, S) is WM $\cap$ TE, then $T \times S$ is WM. If (X, T) is TE, and (Y, S) is WM $\cap$ TE, then $T \times S$ is TE. Thus, if both (X, T) and (Y, S) are WM $\cap$ TE, so is $T \times S$.
(5) If (X, T) is extremely scattering and (Y, S) is extremely scattering which is also TE, then $T \times S$ is extremely scattering.
(6) If (X, T) is strongly scattering and (Y, S) is strongly scattering which is also an E-system, then $T \times S$ is strongly scattering.
(7) If (X, T) is scattering and (Y, S) is scattering with dense set of minimal points, then $T \times S$ is scattering.

Proof. (1) and (2) are obvious. Now we assume that (X, T) is a minimal weakly scattering system and (Y, S) is the maximal equicontinuous factor of (X, T). As $T \times S$ is transitive and (Y, S) is a factor of (X, T), we have that $S \times S$ is transitive. Thus (Y, S) is trivial. Hence T is WM. This proves (3).

To prove (4) we suppose (X, T) is WM and (Y, S) is WM $\cap$ TE.

First we note that $S \times S$ is WM $\cap$ TE. As S is WM, $S \times S$ is transitive. Let U and V be two non-empty open subsets of Y. Assume $n > 0$ such that $V_1 = S^{-n}U \cap V \neq \emptyset$. Then $N(V_1, V_1) \subset N(U, U)$. Thus, $N(U \times V, U \times V) = N(U, U) \cap N(V, V) \supset N(V_1, V_1)$ is syndetic.

As $T \times T$ is WM, $S \times S$ is TE we have that $(T \times T) \times (S \times S)$ is transitive. That is, $T \times S$ is WM.

Now assume (X, T) is TE, (Y, S) is WM $\cap$ TE. We want to show that $T \times S$ has this property. Let (X', T') be WM. Then $T' \times (S \times T) = (T' \times S) \times T$ is transitive, as $T' \times S$ is WM and T is TE. By the W-AG theorem, $S \times T$ is TE and so is $T \times S$.

(5) can be proved in the same fashion. (6) follows from the fact that (Y, S) is weakly mixing, and its product with any E-system is also an E-system. To prove (7) we use the following facts: (a) if (Y', S') and (Y, S) are two systems having a dense set of minimal points, so is $S' \times S$; (b) if a system is scattering, then it is weakly disjoint from any transitive system with a dense set of minimal points (see [AG]). □

We do not know how to prove a similar result for a weakly scattering system. Note that one can find (4) and (7) in [A] and [AG], respectively.

In the measure theoretical setting, the product of an ergodic system with a WM system is ergodic and the product of a WM system with a WM system is WM. The facts are not valid in topological setting. However, we have the following corollary.

Corollary 4.2. *Let (Y, S) be a dynamical system. Then:*

(a) The product of (Y, S) with any WM system is WM iff (Y, S) is WM $\cap$ TE.
(b) The product of (Y, S) with any TE system is TE iff (Y, S) is WM $\cap$ TE.
(c) The product of (Y, S) with any WM $\cap$ TE system is WM $\cap$ TE iff (Y, S) is WM $\cap$ TE.

Proof. (a) follows from proposition 4.1(4) and the W-AG theorem. (b) follows from proposition 4.1(4) and the fact that the topological ergodicity is preserved by factor maps. (c) follows from proposition 4.1(4) and the fact that both TE and WM are preserved by factor maps. □

Motivated by the above corollary, we have the following definition. Let P be a dynamical property stronger than transitivity and stable under factor maps. We use $\lambda(P)$ to denote the property such that a dynamical system having $\lambda(P)$ iff its product with any system having P has P. Let $\lambda^2(P) = \lambda\lambda(P)$. We have the following theorem.

Theorem 4.3. *Let P be a dynamical property stronger than transitivity and stable under factor maps. Then:*

(1) $P \leqslant \lambda(P)$ and $\lambda(P)$ is stable under factor maps.
(2) $\lambda(P) \times \lambda(P) \subset \lambda(P)$. Particularly, $\lambda(P) \geqslant WM$.
(3) $\lambda^2(P) = \lambda(P)$.

Proof. (1) is obvious. To show (2) we note that $P \times (\lambda(P) \times \lambda(P)) \subset P \times \lambda(P) \subset P$. That is, $\lambda(P) \times \lambda(P) \subset \lambda(P)$. By (1), $\lambda^2(P) \geqslant \lambda(P)$, and by (2), $\lambda(P) \subset \lambda^2(P)$ (as (X, T) has $\lambda^2(P)$ iff $\lambda(P) \times (X, T) \subset \lambda(P)$), i.e. $\lambda^2(P) \leqslant \lambda(P)$. Thus, $\lambda^2(P) = \lambda(P)$ and (3) is proved. □

A dynamical system is *almost equicontinuous* if it is transitive and there is a transitive point x such that for each $\epsilon > 0$ there is $\delta > 0$ such that if $d(y, x) < \delta$ then $d(T^i(y), T^i(x)) < \epsilon$ for each $i \in \mathbb{Z}_+$ (see [AAB]). Moreover, the set of almost equicontinuous points is the set of transitive points. We first show that a dynamical system satisfying property $(*)$ (see lemma 2.2) cannot be almost equicontinuous. We start with the following lemma.

Lemma 4.4. *Suppose that (X, T) is almost equicontinuous (non-trivial) with an equicontinuous point x. Assume for each i, U_i is an open neighbourhood of x with $\bigcap_i U_i = \{x\}$. Then there is i such that for any $p, q \in \mathbb{N}$, $N(U_i, U_i) \supset \{p - 1, q - 1, p + q - 1\}$ does not hold.*

Proof. Assume that for each i there are p_i, q_i such that $N(U_i, U_i) \supset \{p_i - 1, q_i - 1, p_i + q_i - 1\}$.

Take an open neighbourhhod U of x such that $diam(U) < \epsilon$, $diam(T(U)) < \epsilon$ and $4\epsilon < d(U, T(U))$ for some $\epsilon > 0$. As x is an equicontinuous point we can assume that $diam(T^n(U)) < \epsilon$ for each $n \in \mathbb{N}$. Let $V = T(U)$. Then $T^n(U) \cap U \neq \emptyset$ and $T^n(U) \cap V \neq \emptyset$ cannot hold at the same time.

It is easy to see that there is i such that $N(x, U) \supset N(U_i, U_i)$. Thus

$$N(U, U) = N(x, U) - N(x, U) \supset N(U_i, U_i) - N(U_i, U_i) \supset \{p_i, p_i - 1\}.$$

This implies that $T^{p_i}(U) \cap U \neq \emptyset$ and $T^{p_i}(U) \cap V \supset T(T^{p_i - 1}(U) \cap U) \neq \emptyset$, a contradiction. □

Corollary 4.5. *A dynamical system satisfying property $(*)$ cannot be almost equicontinuous.*

Let S be a subset of $\mathbb{Z}_+$. The *upper Banach density of S* is

$$BD^*(S) = \limsup_{|I| \to +\infty} \frac{|S \cap I|}{|I|},$$

where I ranges over intervals of $\mathbb{Z}_+$. The *upper density* of S is

$$D^*(S) = \limsup_{n\to\infty} \frac{|S \cap [0, n-1]|}{n}.$$

We say that *the lower Banach density of S is one* if for each $a < 1$ there is N such that for any subinterval I of $\mathbb{Z}_+$ with $|I| \geqslant N$ we have $|S \cap I| \geqslant a|I|$.

A dynamical system is *D-transitive* if the lower Banach density of $N(U, V)$ is one for each pair of non-empty open subsets U, V of X, and it is *Banach-transitive* if the upper Banach density of $N(U, V)$ is positive for each pair of non-empty open subsets U, V of X.

In [GW1, Y] the authors prove that an almost equicontinuous system satisfying that the upper density of $N(U, V)$ is positive should be minimal. In fact, we have the following theorem.

Theorem 4.6. *If (X, T) is almost equicontinuous and is Banach-transitive, then it is minimal.*

Proof. As (X, T) is almost equicontinuous, there is an equicontinuous point p which is a transitive point. For $\epsilon > 0$ let $U = B_\epsilon(p)$. Then, there is $\delta > 0$ such that $N(p, U) \supset N(U_\delta, U_\delta)$, where $U_\delta = B_\delta(p)$. For example, we take $0 < \delta < \epsilon/2$ such that $diam(T^n(U_\delta)) < \epsilon/2$ for $n \in \mathbb{Z}_+$. At the same time, there is $\delta_1 > 0$ such that $N(p, U_\delta) \supset N(U_{\delta_1}, U_{\delta_1})$, where $U_{\delta_1} = B_{\delta_1}(p)$. Thus, we have

$$N(p, U) \supset N(U_\delta, U_\delta) \supset N(U_{\delta_1}, U_{\delta_1}) - N(U_{\delta_1}, U_{\delta_1}).$$

By [F2, proposition 3.19] $N(p, U)$ is syndetic, and thus p is an almost periodic point. This implies that (X, T) is minimal. □

A subset $S \subset \mathbb{Z}_+$ is *thickly syndetic* if $g^{-i_1}(S) \cap \cdots \cap g^{-i_n}(S)$ is syndetic for each $i_1, \ldots, i_n \in \mathbb{Z}_+$, and *piecewise syndetic* if it is the intersection of a thick set and a syndetic set. We remark that S is thickly syndetic iff for each $n \in \mathbb{N}$ there is a syndetic subset $S^n = \{s_1^n < s_2^n < \cdots\}$ such that $\{s_j^n, s_j^n + 1, \ldots, s_j^n + n\} \subset S$ for each j. Recall that two dynamical properties P_1 and P_2 are symmetrically dual if $P_1^\curlywedge = P_2$ and $P_2^\curlywedge = P_1$. By theorem 2.1 for a property P stronger than transitivity $P^\curlywedge$ and $P^{\curlywedge\curlywedge}$ are symmetrically dual. In the following theorem, we determine a property for which both $P^\curlywedge$ and $P^{\curlywedge\curlywedge}$ can be described explicitly using family notion (this is the first property we know). We remark that (1)–(3) are due to Glasner by personal communications.

Theorem 4.7.

(1) The class WM $\cap$ TE $= \mathcal{F}$-transitive, with $\mathcal{F}$ being the family of thickly syndetic subsets of $\mathbb{Z}_+$.
(2) $k\mathcal{F}$-transitive = piecewise-syndetic-transitive = ($\mathcal{F}$-transitive)$^\curlywedge$ = (WM $\cap$ TE)$^\curlywedge$.
(3) Extremely scattering system is piecewise-syndetic-transitive.
(4) (piecewise-syndetic-transitive)$^\curlywedge$ = WM $\cap$ TE.

Consequently there is no non-trivial almost equicontinuous extremely scattering system.

Proof. (1) $\mathcal{F}$-transitive implies WM $\cap$ TE by definition. For the other direction, observe that for $(X, T) \in$ WM $\cap$ TE, by (4) of proposition 4.1, it follows that for each $n \in \mathbb{N}$, $(X^n, T) \in$ WM $\cap$ TE. This implies that $N(U, V)$ is thickly syndetic for each non-empty open set U, V of X.

(2) It is easy to see that $\mathcal{F}$ is a translation-invariant, thick family. Thus by the W-AG theorem, the result follows.

(3) By definition extremely scattering $=$ (TE)$^{\curlywedge}$. According to (2), (TE)$^{\curlywedge} \subset$ (WM $\cap$ TE)$^{\curlywedge}$ $=$ piecewise-syndetic-transitive.

(4) Let $P =$ piecewise-syndetic-transitive. As WM $\subset P$, we have (by the W-AG theorem)

$$(P)^{\curlywedge} \subset (WM)^{\curlywedge} = TE \subset P.$$

This implies that if $(X, T) \in P^{\curlywedge}$, then (X, T) is TE and $(X \times X, T \times T)$ is transitive. Thus, $(X, T) \in$ WM $\cap$ TE.

Now assume that (X, T) is extremely scattering. If (X, T) is minimal, then it is WM by proposition 4.1(3). If it is also almost equicontinuous, then it is equicontinuous [AAB], a contradiction as it is well known that the maximal equicontinuous factor of a minimal WM system is trivial.

If (X, T) is not minimal, then it cannot be almost equicontinuous as there is no non-minimal almost equicontinuous system which is Banach transitive by what we just proved and lemma 4.6. $\square$

Corollary 4.8. *There is a strongly scattering and not extremely scattering system.*

Proof. In the construction of theorem 3.1, it is clear that we can choose $N(U, V)$ to have zero upper Banach density for some non-empty open sets U, V. By theorem 4.7 (X, T) is not extremely scattering. $\square$

Definition. *$W \subset \mathbb{Z}_+$ is called a Poincaré sequence if for any m.p.s. $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$, we have $\mu(A \cap T^{-n}(A)) > 0$ for some $n \in W$, $n \neq 0$.*

A subset $W \subset \mathbb{Z}_+$ is called a recurrence set if for any dynamical system (Y, ρ, T) and any $\epsilon > 0$ there are $y_0 \in Y$ and $d \in W$ $(d \neq 0)$ with $\rho(T^d(y_0), y_0) < \epsilon$.

The following facts are known [F1, W].

Facts. *(X, T) is WM iff $N(U, V) \cap S \neq \emptyset$ for each syndetic $S \subset \mathbb{Z}_+$.*

W is a Poincaré sequence iff for each $S \subset \mathbb{Z}_+$ with positive upper Banach density, $W \cap (S - S) \neq \emptyset$.

W is a recurrence set iff for each syndetic $S \subset \mathbb{Z}_+$, $W \cap (S - S) \neq \emptyset$.

Theorem 4.9. *(X, T) is strongly scattering iff $N(U, V) \cap (S - S) \neq \emptyset$ for each S with positive upper Banach density and each pair of non-empty open subsets U and V of X iff $N(U, V)$ is a Poincaré set.*

Proof. Assume that $N(U, V) \cap (S - S) \neq \emptyset$ for each S with positive upper Banach density and each pair of non-empty open subsets U and V of X. It is clear that $N(U, V)$ is infinite and thus (X, T) is transitive.

Let (Y, W) be an E-system and U_1, V_1 be non-empty open subsets of Y. Assume that y is a transitive point in V_1. Then there is $n_0 \in \mathbb{N}$ such that $y \in V_1$ and $W^{n_0}(y) \in U_1$. Thus there is an open neighbourhood Q of y contained in V_1 with $W^{n_0}(Q) \subset U_1$. This implies that $N(V_1, U_1) \supset N(Q, Q) + n_0$ and $N(Q, Q) = N(y, Q) - N(y, Q)$. We show now that $N(y, Q)$ has positive upper Banach density. Let μ be an invariant measure with full support. Then $\mu(Q) > 0$ and thus there is an ergodic measure ν with $\nu(Q) > 0$. As y is a quasi-generic point for ν [F2, proposition 3.9], we see easily that $N(y, Q)$ has positive upper Banach density. Thus we have $N(U, T^{-n_0}(V)) \cap N(Q, Q) \neq \emptyset$ by our assumption, and hence $(n_0 + N(U, T^{-n_0}(V))) \cap (n_0 + N(Q, Q)) \neq \emptyset$. Since $N(U, T^{-n_0}(V)) \subset N(U, V) - n_0$, we have $N(U, V) \cap N(U_1, V_1) \neq \emptyset$. Then $N(U \times U_1, V \times V_1) = N(U, V) \cap N(U_1, V_1) \neq \emptyset$. That is, (X, T) is strongly scattering.

Now assume that (X, T) is strongly scattering. Then if (Y, W) is an E-system, we have $N(U \times B, V \times B) = N(U, V) \cap N(B, B) \neq \emptyset$, where U, V are non-empty open subsets of X and B is a non-empty open subset of Y.

For a given S with positive upper Banach density, define a sequence $x \in \Sigma = \{0, 1\}^{\mathbb{N}}$ such that $x_i = 1$ iff $i \in S$. Let Y_1 be the orbit closure of x under the shift σ and $A(1) = \{y \in Y_1 : y(0) = 1\}$. Then by lemma 3.17 of [F2], there is an invariant measure μ with $\mu(A(1)) > 0$. By the ergodic decomposition we know that there is an ergodic measure ν such that $\nu(A(1)) > 0$. Let Y be the support of ν and $B = Y \cap A(1)$. Note that B is open in Y. We have

$$S - S \supset \{n \in \mathbb{Z}_+ : \nu(\sigma^{-n}(B) \cap B) > 0\} \supset N(B, B).$$

As (Y, σ) is an E-system, we have $N(U, V) \cap N(B, B) \neq \emptyset$, and hence $N(U, V) \cap (S - S) \neq \emptyset$. □

Similarly to theorem 4.9 we have the following result which gives another characterization of scattering.

Theorem 4.10. *For a dynamical system (X, T) the following statements are equivalent:*

(1) (X, T) is scattering.
(2) $N(U, V) \cap (S - S) \neq \emptyset$ for any syndetic subset S and each pair of non-empty open subsets U and V of X.
(3) $N(U, V)$ is a recurrence set.
(4) For any finite open cover $\mathcal{U}$ by non-dense open subsets $N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}) \to \infty$, where $N(\alpha)$ is the minimal cardinality of subcovers of α.

Proof. (1) and (4) are equivalent by [BHM], and (2) and (3) are equivalent by the facts stated before.

Assume $N(U, V) \cap (S - S) \neq \emptyset$ for any syndetic subset S of $\mathbb{N}$ and each pair of non-empty open subsets U and V of X.

Let (Y, W) be a minimal system. It is well known that if $y \in Y$ and Q is an open neighbourhood of y, then $N(y, Q)$ is syndetic. Thus following the proof of the first part of theorem 4.9, we have that $N(U \times U_1, V \times V_1) \neq \emptyset$ for any pair of non-empty open subsets U, V of X and any pair of non-empty open subsets U_1, V_1 of Y. That is, (X, T) is scattering.

Now assume that (X, T) is scattering. Then if (Y, W) is a minimal system, we have $N(U \times B, V \times B) = N(U, V) \cap N(B, B) \neq \emptyset$, where U, V are non-empty open subsets of X and B is a non-empty open subset of Y.

Let S be a syndetic subset of $\mathbb{N}$ and let x denote the indicator function of S in $\{0, 1\}^{\mathbb{N}}$, then if Y is any minimal set in the orbit closure of x under the shift σ, and $B = \{y \in Y : y(0) = 1\}$, the fact that S is syndetic implies that B is not empty and $N(B, B) \subset S - S$. As (Y, σ) is minimal, we have $N(U, V) \cap N(B, B) \neq \emptyset$, and hence $N(U, V) \cap (S - S) \neq \emptyset$. □

We know that any weakly scattering system is totally transitive as it is weakly disjoint from any periodic system. Two dynamical systems (X, T) and (Y, S) are *disjoint* if the only non-empty closed invariant subset of $X \times Y$ which projects onto both X and Y is $X \times Y$. Moreover, we have the following theorem.

Theorem 4.11. *For an E-system weak scattering is equivalent to scattering.*

Proof. Let (X, T) be a weakly scattering E-system. By [AG, theorem 3.7] (X, T) is disjoint from each minimal equicontinuous system. Then following proposition A.1 of [BHM], (X, T) is scattering. □

For weak scattering we have the following theorem.

Theorem 4.12. *(X, T) is weakly scattering iff $N(U, V) \cap (S_1 - S_1) \neq \emptyset$, where U, V are non-empty open sets of X and S_1 has the form of $N(y_0, B)$, where (Y, S) is minimal and equicontinuous, $y_0 \in Y$ and $B \subset Y$ is open.*

Proof. A direct consequence of the definition. □

It is an open problem if for each syndetic set S, $S - S$ contains $S_1 - S_1$ for some S_1, which has the form of $N(y_0, B)$ in some minimal equicontinuous system. If this problem has an affirmative answer, then by theorems 4.10 and 4.12, weak scattering and scattering are the same properties. Thus, if weak scattering and scattering are different properties, then the above problem will have a negative answer.

Remark 4.13. It is known that there is a recurrence set which is not a Poincaré set [W]. This fact implies that there is $S \subset \mathbb{Z}$ having positive upper Banach density and there is no syndetic set R with $R - R \subset S - S$. A subset $S \subset \mathbb{Z}$ is a dynamical Poincaré set (resp. recurrence set) if it has the form $N(U, V)$, where U, V are non-empty open sets of some system (X, T). Thus if strong scattering and scattering are different properties, then there is a dynamical recurrence set which is not a Poincaré set.

Before stating some open questions we summarize the results obtained in this paper using figure 1. The entries in the diagram appear as names of classes with their $\mathcal{E}$-transitive characterization below (when one is available). $A^{\curlywedge}$ denotes the class of systems which are weakly disjoint from the class A, —⊂ is just ⊂, and → means taking $^{\curlywedge}$ of a class. The k above an arrow means that in addition passage is to a dual family (by the W-AG theorem).

The following questions remain open:

Question 1. Is there an almost equicontinuous system which is strongly scattering?

Remark. We know that extreme scattering cannot be almost equicontinuous. If (X, T) is strongly scattering and almost equicontinuous (not minimal), then every invariant measure is supported on a minimal set which is a fixed point. The proof is almost the same as the proof that almost equicontinuous scattering system implies that each minimal set is trivial [BHM, remark 4.5]. If the above question has a negative answer, then strongly scattering and scattering are different properties.

Question 2. We know that strong scattering, extreme scattering and weak mixing are different properties. What about weak scattering, scattering and strong scattering?

Question 3. Is it true that (X, T) is extremely scattering iff $(S - S) \cap N(U, V) \neq \emptyset$ for each S with $S - S$ syndetic?

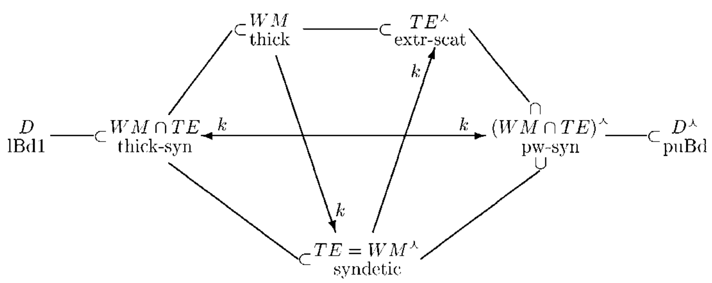

Figure 1. Results of this paper.

Question 4. Is it true that weak scattering implies that the regionally proximal relation is the Cartesian square?

Remark. If (X, T) is 2-scattering (see [BHM]), then (X, T) is disjoint from all equicontinuous minimal systems. Thus 2-scattering implies weak scattering. It is known [HY] that if (X, T) is 2-scattering then the regionally proximal relation is the Cartesian square.

Let (X, T) be a weakly scattering system and (Y, S) be its maximal equicontinuous factor. Then (Y, S) is weakly disjoint from any equicontinuous system and hence is trivial. That is, the smallest closed invariant equivalence relation generated by the regionally proximal relation is the Cartesian square. If question 4 has a positive answer then the weakly scattering system will be chaotic in many senses; see [HY, theorem 3.5] and [BGKM, theorem 2.1].

Acknowledgments

We thank F Blanchard for a careful reading of the first version and many useful suggestions. We thank E Akin and E Glasner for sending us various versions of [AG] from which we benefitted a lot, and E Glasner for his useful remarks which improved theorem 4.7 significantly. The careful reading and useful suggestions of the referees are greatly appreciated. The research of XY was supported by project 973.

References

[A] Akin E 1997 *Recurrence in Topological dynamics: Furstenberg Families and Ellis Actions* (New York: Plenum)

[AAB] Akin E, Auslander J and Berg K 1996 When is a transitive map chaotic? *Convergence in Ergodic Theory and Probability* (*Ohio State Univ. Math. Res. Inst. Publ.* 5) (Berlin: de Gruyter) pp 25–40

[AG] Akin E and Glasner E 2001 Residual properties and almost equicontinuity *J. Anal. Math.* **84** 243–86

[BHM] Blanchard F, Host B and Maass A 2000 Topological complexity *Ergod. Th. Dynam. Sys.* **20** 641–62

[BGKM] Blanchard F, Glasner E, Kolyada S and Maass A 2000 On Li–Yorke pairs *Preprint*

[DGS] Denker M, Grillenberger C and Sigmund K 1975 *Ergodic Theory on Compact Spaces* (*Lecture Notes* vol 527) (Berlin: Springer)

[F1] Furstenberg H 1967 Disjointness in ergodic theory, minimal sets and a problem in Diophantine approximations *Math. Sys. Th.* **1** 1–49

[F2] Furstenberg H 1981 *Recurrence in Ergodic Theory and Combinatorial Number Theory* (Princeton, NJ: Princeton University Press)

[G] Glasner E 1998 On minimal actions of Polish groups *Topology Appl.* **85** 119–25

[GW1] Glasner E and Weiss B 1993 Sensitive dependence on initial conditions *Nonlinearity* **6** 1067–75

[GW2] Glasner E and Weiss B 2000 Locally equicontinuous dynamical systems *Colloq. Math.* **84–85** 345–61

[HY] Huang W and Ye X 2002 Devaney's chaos and 2-scattering imply Li–Yorke's chaos *Topology Appl.* **117** 259–72

[W] Weiss B 2000 *Single Orbit Dynamics* (*AMS Regional Conference Series in Mathematics* No 95) (New York: AMS)

[Y] Yang R 2000 Topologically ergodic maps *Preprint*