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Attractors for non-compact semigroups via energy equations

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Abstract. The energy-equation approach used to prove the existence of the global attractor by establishing the so-called asymptotic compactness property of the semigroup is considered, and a general formulation that can handle a number of weakly damped hyperbolic equations and parabolic equations on either bounded or unbounded spatial domains is presented. As examples, three specific and physically relevant problems are considered, namely the flows of a second-grade fluid, the flows of a Newtonian fluid in an infinite channel past an obstacle, and a weakly damped, forced Korteweg–de Vries equation on the whole line.

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1. Introduction

Various physical phenomena, ranging from celestial mechanics to quantum mechanics, can be modelled by nonlinear evolutionary differential equations. Most of those equations are well posed and thus may be considered as dynamical systems on some appropriate phase space. If the 'permanent regime' is of interest, it is customary (and necessary) to take into account various kinds of dissipation mechanisms (such as friction, thermal diffusion, etc), which usually leads, if the system is autonomous, to the existence of an absorbing ball in the phase space and further leads to the existence of the *global attractor* (see section 2 for the definitions), which attracts all the orbits of the dynamical system. Then, the long time behaviour of the system is characterized by its behaviour on the global attractor, if it exists, so that from the point of view of either an analytical or numerical study and even for a possible control of the system, it is of great interest to study the existence and the properties of this global attractor for various kinds of equations arising in physical and mechanical applications.

There are many references on this topic; let us only mention the books [BV, Ha, He, Hr, La2, SY, T1].

There are essentially three apparently distinct properties that a semigroup may possess and such that each of them together with the existence of a bounded absorbing set leads to the existence of the global attractor. They are the *compactness* [BV, Ha, Hr, La2, SY, T1], the *asymptotic smoothness* [Ha, Hr], and the *asymptotic compactness* [La2, SY, T1] properties. The first condition is that a semigroup $\{S(t)\}_{t\geqslant 0}$ is such that $S(t_0)$ is a compact operator for some $t_0 \geqslant 0$; the second one is that for any closed, bounded, positively invariant set B there exists a compact set $K = K(B)$ which attracts B; and the third one is the precompactness of the sequence $\{S(t_j)u_j\}_{j\in\mathbb{N}}$ for every bounded sequence $\{u_j\}_{j\in\mathbb{N}}$ in the phase space and every sequence $\{t_j\}_{j\in\mathbb{N}}$ of positive numbers with $t_j \to \infty$. The first condition is stronger than the other two, but the last two are, in fact, quite related (see section 2 for more details). The major difference lies in the methods used in the applications in order to establish any one of those conditions. The choice of the proper method depends on the nature of each problem.

The compactness condition was the first one to be used. If the dynamical system is finite dimensional (corresponding to ordinary differential equations), this condition is a trivial consequence of the existence of an absorbing ball in the phase space, while for parabolic equations on bounded spatial domains this compactness property follows from a regularization of the solutions and some compact Sobolev embedding (i.e. one can obtain the existence of a *compact* absorbing set). However, the solution semigroup fails to be compact for most of the infinite-dimensional dynamical systems arising from weakly damped hyperbolic equations or parabolic equations on unbounded domains, even if there is an absorbing ball in the phase space. Thus, this method breaks down here.

The asymptotic smoothness and the asymptotic compactness properties are needed to handle those non-compact cases. One approach is to prove the so-called β-contraction property of the semigroup, which implies the asymptotic smoothness. This condition has been successfully exploited by Hale [Ha] and many other authors. Another approach is to decompose the solution semigroup into two parts: a (uniformly) compact part and a part which decays (uniformly) to zero as time goes to infinite (see, for instance, [Ha, Hr, T1], among many other references). Then, the proof of the existence of the global attractor using this splitting amounts to (either essentially or explicitly) proving either the asymptotic smoothness or the asymptotic compactness of the semigroup.

A more recent approach, which is the one we want to address here, is the use of energy equations to prove the asymptotic compactness property. For many physical systems there are energy equations (or their analogues) in the sense that the changing rate of energy equals the rate that energy is pumped into the system minus the energy dissipation rate due to various dissipation mechanisms. To our knowledge, it was first observed by Ball [B] (for weakly damped, driven semilinear wave equations) that such energy equations may be used to derive the asymptotic compactness of the solution semigroup. This technique was then applied to a weakly damped, driven Korteweg–de Vries (KdV) equation by Ghidaglia [G2]. Later, such technique was put into a systematic formulation suitable for applications to many weakly damped, driven hyperbolic-type equations by one of the authors [W] with a specific application to a weakly damped, driven nonlinear Schrödinger equation. Then, it was observed by another of the authors [R] that the same technique could be applied to parabolic-type problems, as well, with particular interest in equations on unbounded domains.

It has recently been observed (see [T1, 2nd edn]) that the splitting of the semigroup into a (uniformly) compact part and a (uniformly) decaying part mentioned previously is actually *necessary and sufficient* for the existence of the global attractor in the case where the phase space is a Hilbert space, and we note that the same equivalence holds if the phase space is a uniformly convex Banach space. This means that a decomposition of the solution semigroup must exist if the global attractor exists. However, it may be difficult to find such a decomposition in applications. In fact, no suitable decomposition has yet been found for the KdV equation or for the two-dimensional (2D) Navier–Stokes equations on unbounded domains when the forcing term does not belong to some weighted Sobolev spaces. The use of the energy-equation approach is relatively easy if it is not the only solution for those cases.

The purpose of this paper is to formulate in a systematic way the technique of applying the energy equation method in deriving the existence of the global attractor and to apply this formulation to three significant and essentially distinct problems which seem hard to be handled using other approaches.

The approach is relatively simple in that the assumptions are straightforward and may be verified directly from the equations. In most applications, the central part lies in establishing the energy-type equation (this may not be trivial, though, and is open for the Navier–Stokes equations in space dimension 3). For parabolic-type problems, the typical way is to establish enough regularity for the solutions, which then imply the energy equation. For hyperbolic-type problems, the typical way is to use the time reversibility to establish the energy equation. These techniques will be illustrated in section 4 via several examples.

This paper is organized as follows. In section 2 we give a brief review of some basic concepts on the dynamical system approach to evolution equations and study the interplay between those concepts. We recall, in particular, the properties leading to the existence of the global attractor. In section 3 we state our main theorem on the existence of the global attractor via energy equations. Finally, in section 4 we present several applications of our results to a number of physically relevant problems: (1) an equation for fluids of second grade (one of the simplest models for non-Newtonian fluids); (2) a simplified case of uniform flows past an obstacle in the plane; and (3) a weakly damped, driven KdV equation on the whole real line. The first example is hyperbolic, the second is parabolic and the third is dispersive.

2. Asymptotic compactness

Let E be a complete metric space (called the phase space) and let $\{S(t)\}_{t\geqslant 0}$ be a semigroup of continuous (nonlinear) operators in E, i.e. $\{S(t)\}_{t\geqslant 0}$ satisfies

$$\begin{cases} S(t+s) = S(t)\circ S(s), & \forall t, s \geqslant 0, \\ S(0) = I = \text{Identity in } E, \end{cases} \tag{2.1}$$

and

$$S(t) \text{ is a continuous (nonlinear) operator from } E \text{ into itself for any } t \geqslant 0. \tag{2.2}$$

In what follows, a semigroup for us will always mean a semigroup of continuous operators as defined by (2.1) and (2.2). For a set $B \subset E$, we define its ω-limit set by $\omega(B) = \cap_{s\geqslant 0}\overline{\cup_{t\geqslant s} S(t)B}$. It is easy to prove the following well known characterization of an ω-limit set:

$$\begin{gathered} w \in \omega(B) \Longleftrightarrow \exists \{w_j\}_{j\in\mathbb{N}} \subset B, \exists \{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+ \text{ such that} \\ t_j \to +\infty \text{ and } S(t_j)w_j \to w \text{ in } E. \end{gathered} \tag{2.3}$$

A set $\mathcal{B} \subset E$ is called an *absorbing set* for the semigroup $\{S(t)\}_{t\geqslant 0}$ if $\mathcal{B}$ 'absorbs' all the bounded sets of E, i.e. for every $B \subset E$ bounded, there exists a time $T = T(B) > 0$ such that $S(t)B \subset \mathcal{B}$, for all $t \geqslant T(B)$. The *global (or universal) attractor* of a semigroup $\{S(t)\}_{t\geqslant 0}$ is defined as the set $\mathcal{A} \subset E$ which is compact in E, invariant for $\{S(t)\}_{t\geqslant 0}$, i.e.

$S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geqslant 0$, and which attracts all the bounded sets of E, i.e. for any bounded set $B \subset E$, $\mathrm{dist}_E(S(t)B, \mathcal{A}) \to 0$ as $t \to +\infty$. Here dist_E is the usual semidistance in E between two sets. One can show that if the global attractor exists, it is unique. Moreover, the global attractor is minimal (with respect to the inclusion relation in E) among the closed sets that attract all the bounded sets and is maximal (idem) among the bounded, invariant sets.

For the concepts described above, as well as for the results stated below, we refer the reader to the works of [BV, Ha, Hr, La2, Sell, SY, T1].

In order to prove the existence of the global attractor one needs some kind of compactness of the semigroup together with the existence of a bounded absorbing set. For instance, if there exists a bounded absorbing set $\mathcal{B}$ and $S(t_0)$ is compact for some $t_0 > 0$, then $\mathcal{A} = \omega(\mathcal{B})$ is the global attractor. This condition is typical for parabolic equations on bounded spatial domains where the compactness follows from a regularization of the solutions and some compact Sobolev embedding. In those cases, one actually obtains the existence of a compact absorbing set.

However, many equations do not generate a compact semigroup in the above sense, so the compactness needed must be achieved in a different, weaker sense. We say that a semigroup $\{S(t)\}_{t\geqslant 0}$ is *asymptotically smooth*, or possesses the *asymptotic smoothness* property, if for any non-empty, closed, bounded subset $B \subset E$ for which $S(t)B \subset B, \forall t \geqslant 0$, there exists a compact set $K = K(B) \subset B$ which attracts B (see, e.g. Hale [Ha], and see also Babin and Vishik [BV] for a similar definition). If $\{S(t)\}_{t\geqslant 0}$ is asymptotically smooth and possesses a bounded absorbing set $\mathcal{B}$, then $\mathcal{A} = \omega(\tilde{\mathcal{B}})$ can be shown to be the global attractor, where $\tilde{\mathcal{B}} = \overline{\cup_{t\geqslant t_0} S(t)\mathcal{B}}$ for t_0 such that $S(t)\mathcal{B} \subset \mathcal{B}$ for any $t \geqslant t_0$.

A related concept is that of asymptotic compactness [La2, T1, SY]. One says that $\{S(t)\}_{t\geqslant 0}$ is *asymptotically compact* in E if the following condition holds:

$$\begin{cases} \text{If } \{u_j\}_{j\in\mathbb{N}} \subset E \text{ is bounded and } \{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+,\ t_j \to \infty \\ \text{then } \{S(t_j)u_j\}_{j\in\mathbb{N}} \text{ is precompact in } E. \end{cases} \tag{2.4}$$

This condition, together with the existence of a bounded absorbing set, implies the existence of the global attractor. Since this is the result we will be using in the rest of this work, we state it below in the form of a theorem.

Theorem 2.1. *Let E be a complete metric space and let $\{S(t)\}_{t\geqslant 0}$ be a semigroup of continuous (nonlinear) operators in E. If $\{S(t)\}_{t\geqslant 0}$ possesses a bounded absorbing set $\mathcal{B}$ in E and is asymptotically compact in E, then $\{S(t)\}_{t\geqslant 0}$ possesses the global attractor $\mathcal{A} = \omega(\mathcal{B})$. Moreover, if $t \mapsto S(t)u_0$ is continuous from $\mathbb{R}^+$ into E, for any $u_0 \in E$, and $\mathcal{B}$ is connected in E, then $\mathcal{A}$ is also connected in E.*

The proof of theorem 2.1 can be essentially found in [La2, theorem 3.4].

One can see from the characterization (2.3) that condition (2.4) of asymptotic compactness is a natural assumption associated to the ω-limit sets. In fact, *the asymptotic compactness property alone implies that the ω-limit set of any non-empty, bounded set is non-empty, compact, invariant, and attracts the corresponding bounded set*. This is a very important point which we want to stress and that shows the significance of this condition. The further existence of a bounded absorbing set implies then that the ω-limit set of this absorbing set attracts *any* bounded set.

Showing that a semigroup defined by an evolutionary equation is asymptotically smooth or asymptotically compact depends on the general properties of the equation. One way is to show in the case E is a Banach space that one has a splitting $S(t) = S_1(t) + S_2(t)$, where $S_1(t)$ is uniformly compact, i.e.

$$\forall B \subset E \text{ bounded } \exists\, t_0(B) \geqslant 0 \text{ such that } \overline{\bigcup_{t \geqslant t_0} S_1(t)B} \text{ is compact in } E, \quad (2.5)$$

while $S_2(t) : E \to E$ is continuous for each $t > 0$ and satisfies:

$$\lim_{t\to\infty} \sup_{u\in B} ||S_2(t)u||_E = 0, \qquad \forall B \subset E \text{ bounded.} \quad (2.6)$$

This splitting together with the existence of a bounded absorbing set leads directly to the existence of the global attractor, but the proof, actually, amounts essentially to showing either the asymptotic smoothness or the asymptotic compactness properties. This approach is typical for hyperbolic equations of second order in time on bounded spatial domains, where there is usually a regularization effect of the solution of the linear part of the equation with respect to the non-homogeneous term. It has also been used for some parabolic equations on unbounded domains by means of weighted functional spaces to recover some appropriate compact embeddings (see, e.g. [FLST]). We should also mention that a splitting of the semigroup involving the low- and high-frequency components of the solution has been used in order to prove a regularity result for the global attractor in the case of a weakly damped Schrödinger equation [Go] and a weakly damped KdV equation [MR].

There is also the notion of a β-contraction semigroup, given in [Ha]: we say that $\{S(t)\}_{t\geqslant 0}$ is a *β-contraction* with respect to a measure of non-compactness β on the metric space E, which is assumed to be complete, if there exists a continuous function $k : \mathbb{R}^+ \to \mathbb{R}^+$ with $k(t) \to 0$ as $t \to +\infty$ and such that for every $t > 0$ and every bounded subset $B \subset E$ the set $\cup_{0\leqslant s\leqslant t} S(s)B$ is bounded in E and $\beta(S(t)B) \leqslant k(t)\beta(B)$ as $t \to +\infty$. If $\{S(t)\}_{t\geqslant 0}$ is a β-contraction and there exists a bounded absorbing set, then the global attractor exists. We recall here that a measure of non-compactness β on a complete metric space E is a function β from the bounded sets of E into $\mathbb{R}^+$ such that:

(i) $\beta(A) = 0$ if and only if A is precompact;

(ii) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$; and

(iii) $\beta(A + B) \leqslant \beta(A) + \beta(B)$.

For example, the Kuratowski measure of non-compactness, α, is given by $\alpha(B) = \inf\{d > 0;\, B \text{ has a finite cover by sets of diameter less than } d\}$. For this measure, if $S(t) = T(t) + L(t)$ with $T(t)$ compact for $t > 0$, but not necessarily uniformly compact, and $L(t)$ linear with norm $||L(t)||$ continuous in t and vanishing for $t \to +\infty$, then $\{S(t)\}_{t\geqslant 0}$ is an α-contraction with $k(t) = ||L(t)||$. Many wave equations can also be put into this form (see e.g. [Ha]; see also [EK] for a recent survey article on the existence of the global attractor for a class of nonlinear wave equations). One can also exploit the existence of an absorbing set and obtain the α-contraction property with $k = k_B(t)$ depending on the bounded set B, or else only for subsets of the absorbing set; either one is enough for the existence of the global attractor even if $t \longmapsto k(t)$ is not continuous.

Another way which has recently been exploited is the energy equation approach [B, G2] described in more details (and extended to other cases) in section 3. It leads easily to the asymptotic compactness property. It essentially amounts to using the existence of a bounded absorbing set for extracting a weakly convergent sequence $\{S(t_{j'})u)\}_{j'}$ and then deducing from the energy equation that this weakly convergent sequence is also norm-convergent. The strong convergence and, hence, the precompactness of $\{S(t_j)u_j\}_j$ follow then from these two convergences provided the space is a uniformly convex Banach space.

Returning to the two concepts of asymptotic smoothness and asymptotic compactness, we remark that these two properties are actually quite related. It is not difficult to see that if $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact, then it is also asymptotically smooth with $K(B) = \omega(B)$. But the converse is not true. In particular, if the semigroup is asymptotically smooth, then in general only a non-empty, bounded set B which is further closed and positively invariant is such that its ω-limit set is non-empty, compact, invariant, and attracts B. However, the two concepts do bear a close relationship. Indeed, one can show that the asymptotic compactness property is equivalent to the property that for every non-empty, bounded set B (not necessarily closed or positively invariant) there exists a compact set $K = K(B)$ which attracts B.

Now, if one assumes the existence a of bounded absorbing set, then many of the concepts above turn out to be equivalent. For instance, both properties of asymptotic compactness and asymptotic smoothness become equivalent. They also turn out to be equivalent to the existence of the global attractor itself, as it can be easily checked. In fact, in the case E is a Hilbert space, Goubet and Moise (see remark I.1.5 of the second edition of [T1] remarked that even the splitting (2.5), (2.6) is equivalent to the existence of the global attractor (again, assuming the existence of a bounded absorbing set). This can be attained by writing $S_1(t) = PS(t)$ and $S_2(t) = S(t) - PS(t)$, where P is the projection onto the closed convex hull of the global attractor. Of course, this can also be achieved if E is a uniformly convex Banach space. Moreover, if one relaxes the continuity condition (which is actually not necessary in the proof) on $S_2(t)$ (but not on $S(t)$ itself), then a similar splitting can be obtained on an arbitrary Banach space E with P being one of the possibly many projections onto the global attractor $\mathcal{A}$ itself (Pu is such that $d(u, Pu) = \mathrm{dist}_E(u, \mathcal{A}), \forall u \in E$, which exists by the compactness of $\mathcal{A} \neq \emptyset$ and the Zorn lemma). This same decomposition shows that $\{S(t)\}_{t\geqslant 0}$ is an α-contraction with k depending on the bounded set and not necessarily continuous.

The major difference then turns out to be in the applications. Whether we use the splitting of the semigroup, the β-contraction property, the energy-equation approach, or some other method, the proper choice remains intimately related to the nature of each problem. In section 4 we present three examples for which the energy equation method discussed in section 3 is either the most suitable one or the only one available at the present moment.

3. Abstract energy equations

In this section we consider semigroups possessing a bounded absorbing set and satisfying some general abstract energy equation. We study under which conditions on the energy equation we can obtain the asymptotic compactness needed for the existence of the global attractor. The use of energy equations to prove the existence of the global attractor for weakly dissipative semigroups was first explored in [B], then followed by a number of other authors. The earliest works would first obtain the existence of a bounded 'weak' attractor (i.e. attracting the bounded sets in the weak topology) and then use the energy equation to show the actual attraction in the strong topology and the compactness of the attractor. Later [R], it was realized that the asymptotic compactness would follow directly from the energy equation and the global attractor could be obtained without mention to the weak attractor; this way would also avoid the assumption of the separability of the phase space needed for the existence of the weak attractor.

We present below some slight generalizations of the previous results of [B, G2, W, R, Go, MR, GM].

Let the phase space E be a reflexive Banach space, so that bounded sequences are weakly precompact. Let also $\{S(t)\}_{t\geqslant 0}$ be a semigroup of continuous (nonlinear) operators in E. Assume that $S(t)$ is weakly continuous in E for each $t \geqslant 0$, and that the trajectories of $\{S(t)\}_{t\geqslant 0}$ are continuous in E, i.e.

$$t \to S(t)u_0 \in \mathcal{C}(\mathbb{R}^+, E), \qquad \forall u_0 \in E. \tag{3.1}$$

The continuity condition (3.1) actually follows from the integral form of the energy equation, which is the form used in the proof of the asymptotic compactness, but we might very well assume, equivalently, (3.1) and the differential form of the energy equation. We also assume the existence of a bounded absorbing set $\mathcal{B}$ in E. For the energy equation, we assume that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\Phi(S(t)u_0) + J(S(t)u_0)) + \gamma(\Phi(S(t)u_0) + J(S(t)u_0)) + L(S(t)u_0)\\ = K(S(t)u_0) \qquad \forall u_0 \in E\end{aligned} \tag{3.2}$$

in the distribution sense in $\mathbb{R}^+$, where γ is a positive constant and Φ, J, K, and L are functionals satisfying the following hypotheses.

- $\Phi : E \to \mathbb{R}^+$, Φ is continuous, is bounded on bounded subsets of E and

$$\begin{aligned}&\text{if } \{u_j\}_j \text{ is bounded in } E,\ \{t_j\} \subset \mathbb{R}^+,\ t_j \to \infty,\ S(t_j)u_j \rightharpoonup w \text{ weakly in } E,\\ &\text{and } \limsup_{j\to\infty} \Phi(S(t_j)u_j) \leqslant \Phi(w),\ \text{then } S(t_j)u_j \to w \text{ strongly in } E.\end{aligned} \tag{3.3}$$

- $J : E \to \mathbb{R}$ is 'asymptotically weakly continuous' in the sense that

$$\begin{aligned}&\text{if } \{u_j\}_j \text{ is bounded in } E,\ \{t_j\} \subset \mathbb{R}^+,\ t_j \to \infty,\ \text{and } S(t_j)u_j \rightharpoonup w \text{ weakly in } E,\\ &\text{then } J(S(t_j)u_j) \to J(w).\end{aligned} \tag{3.4}$$

- $K : \cup_{t>0} S(t)E \to \mathbb{R}$ is 'asymptotically weakly continuous' in the sense that

$$\begin{aligned}&\text{if } \{u_j\}_j \text{ is bounded in } E,\ \{t_j\} \subset \mathbb{R}^+,\ t_j \to \infty,\ \text{and } S(t_j)u_j \rightharpoonup w \text{ weakly in } E,\\ &\text{then } \lim_{j\to\infty} \int_0^t \mathrm{e}^{-\gamma(t-s)} K(S(s+t_j)u_j)\,\mathrm{d}s = \int_0^t \mathrm{e}^{-\gamma(t-s)} K(S(s)w)\,\mathrm{d}s, \qquad \forall t > 0,\\ &\text{where it is assumed that } s \longmapsto K(S(s)u) \text{ belongs to } L^1(0,t), \forall t > 0, \forall u \in E.\end{aligned} \tag{3.5}$$

- $L : \cup_{t>0} S(t)E \to \mathbb{R}$ is 'asymptotically weakly lower semicontinuous' in the sense that if $\{u_j\}_j$ is bounded in E, $\{t_j\} \subset \mathbb{R}^+$, $t_j \to \infty$, and $S(t_j)u_j \rightharpoonup w$ weakly in E,

$$\begin{aligned}&\text{then } \int_0^t \mathrm{e}^{-\gamma(t-s)} L(S(s)w)\,\mathrm{d}s \leqslant \liminf_{j\to\infty} \int_0^t \mathrm{e}^{-\gamma(t-s)} L(S(s+t_j)u_j)\,\mathrm{d}s, \qquad \forall t > 0,\\ &\text{where it is assumed that } s \longmapsto L(S(s)u) \text{ belongs to } L^1(0,t),\ \forall t > 0,\ \forall u \in E.\end{aligned} \tag{3.6}$$

Note that we could have included the functional K in L, but for the sake of clarity in the applications, we keep them separate.

Usually, in applications, $\Phi(u)$ is just the square of the norm of u in E, so that (3.3) follows if E is uniformly convex, since in such spaces weak convergence plus norm convergence implies strong convergence (see e.g. [W]). It is also common to find in applications that J and K are weakly continuous in E with K bounded on bounded subsets of E, so that (3.4) and (3.5) follow.

For $L \equiv 0$ and J and K satisfying assumptions slightly stronger than (3.4) and (3.5), the proof that the asymptotic compactness follows under the assumptions above was essentially done by Wang [W, lemma A]. Also a specific case in which $L \not\equiv 0$ was considered by Rosa [R]. By putting those two cases together one can easily obtain the asymptotic compactness in the case above. For the sake of exposition, we outline this derivation below.

Let then $\{u_n\}_n \subset E$ be bounded and let $\{t_n\}_n \subset \mathbb{R}^+, t_n \to +\infty$. We need to show that $\{S(t_n)u_n\}_n$ is precompact in E. Since $\{S(t_n)u_n\}_n$ is bounded (due to the existence of a bounded absorbing set $\mathcal{B}$) and the space E is reflexive, it follows that

$$S(t_{n'})u_{n'} \rightharpoonup w \text{ weakly in } E. \tag{3.7}$$

for some $w \in \bar{co}\mathcal{B}$, the closed convex hull of $\mathcal{B}$, and some subsequence $\{n'\}$. Similarly, $\{S(t_{n'} - T)u_{n'}\}$ has a weakly convergent subsequence for each $T > 0$, so that if we restrict T to the countable set $\mathbb{N}$, we can obtain by a diagonalization process a further subsequence (still denoted $\{n'\}$) for which

$$S(t_{n'} - T)u_{n'} \rightharpoonup w_T \text{ weakly in } E, \qquad \forall T \in \mathbb{N}, \tag{3.8}$$

with $w_T \in \bar{co}\mathcal{B}$. Note then by the weak continuity of $S(T)$ that

$$w = S(T)w_T, \qquad \forall T \in \mathbb{N}. \tag{3.9}$$

Now, since the trajectories of $\{S(t)\}_{t\geqslant 0}$ are continuous, we obtain by integrating the energy equation (3.2) from 0 to T with $u_0 = S(t_{n'} - T)u_{n'}$ that

$$\begin{aligned}\Phi(S(t_{n'})u_{n'}) &+ J(S(t_{n'})u_{n'}) + \gamma \int_0^T \mathrm{e}^{-\gamma(T-s)} L(S(s)S(t_{n'} - T)u_{n'})\,\mathrm{d}s \\ &= [\Phi(S(t_{n'} - T)u_{n'}) + J(S(t_{n'} - T)u_{n'})]\mathrm{e}^{-\gamma T} \\ &\quad + \int_0^T \mathrm{e}^{-\gamma(T-s)} K(S(s)S(t_{n'} - T)u_{n'})\,\mathrm{d}s, \qquad \forall T \in \mathbb{N}, \qquad \forall t_{n'} \geqslant T.\end{aligned} \tag{3.10}$$

From (3.7), (3.8) and the assumptions on J, K and L, we can pass to the limit sup in (3.10) to find

$$\begin{aligned}\limsup_{n'}(\Phi(S(t_{n'})u_{n'})) &+ J(w) + \int_0^T \mathrm{e}^{-\gamma(T-s)} L(S(s)w_T)\,\mathrm{d}s \\ &\leqslant [c_{\mathcal{B}} + J(w_T)]\mathrm{e}^{-\gamma T} + \int_0^T \mathrm{e}^{-\gamma(T-s)} K(S(s)w_T)\,\mathrm{d}s, \qquad \forall T \in \mathbb{N},\end{aligned} \tag{3.11}$$

where $c_{\mathcal{B}} = \sup\{\Phi(v); v \in \bar{co}\mathcal{B}\} < \infty$.

By using again the energy equation now with $u_0 = w_T$, we find using (3.9) that

$$\begin{aligned}\Phi(w) + J(w) &+ \int_0^T \mathrm{e}^{-\gamma(T-s)} L(S(s)w_T)\,\mathrm{d}s = [\Phi(w_T) + J(w_T)]\mathrm{e}^{-\gamma T} \\ &+ \int_0^T \mathrm{e}^{-\gamma(T-s)} K(S(s)w_T)\,\mathrm{d}s, \qquad \forall T \in \mathbb{N}.\end{aligned} \tag{3.12}$$

Subtract (3.12) from (3.11) we find

$$\limsup_{n'}(\Phi(S(t_{n'})u_{n'})) - \Phi(w) \leqslant (c_{\mathcal{B}} + \Phi(w_T))\mathrm{e}^{-\gamma T} \leqslant 2c_{\mathcal{B}}\mathrm{e}^{-\gamma T}, \qquad \forall T \in \mathbb{N}. \tag{3.13}$$

By letting $T \to \infty$ we see that $\limsup_{n'} \Phi(S(t_{n'})u_{n'}) \leqslant \Phi(w)$, which together with the weak convergence (3.7) and assumption (3.3) implies that $S(t_{n'})u_{n'}$ converges strongly to w, which proves the asymptotic compactness of $\{S(t)\}_{t\geqslant 0}$. From theorem 2.1 we then deduce the existence of the global attractor.

Clearly, the same result follows if E is a closed convex subset of a reflexive Banach space, in which case the result could be applied to reaction–diffusion equations with invariant regions (see [S, T1]).

We have then the following result.

Theorem 3.1. *Let E be a reflexive Banach space or a closed, convex subset of such a space. Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup of continuous (nonlinear) operators in E which are also weakly continuous in E. Assume that $\{S(t)\}_{t\geqslant 0}$ possesses a bounded absorbing set and that its trajectories are continuous. Assume also that the energy equation (3.2) holds where γ is a positive constant and Φ, J, K, and L are functionals satisfying the hypotheses (3.3)–(3.6), respectively. Then $\{S(t)\}_{t\geqslant 0}$ possesses a global attractor which is connected if E is connected.*

The claim about the connectedness of the global attractor in theorem 3.1 is obvious.

In some applications, for example in equations in higher-order Sobolev spaces on unbounded domains, where the Sobolev embeddings are not compact, the functionals J and K in the energy equation (3.2) might not be weakly continuous. This is the case, for instance, with the weakly dissipative KdV equation on the whole line considered in section 4.3, for which the phase space is $H^2(\mathbb{R})$. In this case however, we can use one more energy equality to deduce first the asymptotic compactness with respect to the $L^2(\mathbb{R})$ strong topology, which is then used to show the 'asymptotic weak continuity' (see (3.4) and (3.6)) of J and K with respect to the $H^2(\mathbb{R})$ topology.

In view of such applications, we assume that we are given another reflexive Banach space F, $F \supset E$ with continuous injection (E as before). We assume that $\{S(t)\}_{t\geqslant 0}$ is a semigroup of continuous (nonlinear) operators in E which are also weakly continuous in E. We assume also that the energy equation (3.2) holds for a positive constant γ, for J, K, and L as before, and for Φ satisfying now the following assumption.

- $\Phi : E \to \mathbb{R}^+$, Φ is continuous, is bounded on bounded subsets of E and

$$\begin{aligned}&\text{if } \{u_j\}_j \text{ is bounded in } E,\ \{t_j\} \subset \mathbb{R}^+,\ t_j \to \infty,\ S(t_j)u_j \rightharpoonup w \text{ weakly in } E,\\ &\text{and } \limsup_{j\to\infty} \Phi(S(t_j)u_j) \leqslant \Phi(w),\ \text{then } S(t_j)u_j \to w \text{ strongly in } F.\end{aligned} \tag{3.14}$$

Then we can state the following lemma, whose proof is essentially the same as that for theorem 3.1 above.

Lemma 3.2. *Under the assumptions of theorem 3.1 except with (3.3) replaced by (3.14) with a reflexive Banach space $F \supset E$ with continuous injection, it follows that if $\{u_j\}_j$ is bounded in E and $\{t_j\}_j \subset \mathbb{R}^+$, $t_j \to \infty$, then $S(t_{j'})u_{j'} \to w$ strongly in F for some $w \in E$ and some subsequence $\{j'\}$.*

4. Applications

4.1. Fluids of second grade

The evolution of a second-grade incompressible fluid filling a bounded domain $\Omega \subset \mathbb{R}^2$ is described by the following equations:

$$\begin{cases} \dfrac{\partial}{\partial t}(u - \alpha\Delta u) - \nu\Delta u + \ \mathrm{curl}\ (u - \alpha\Delta u) \times u = f + \nabla p & \text{in } \Omega, \\ \mathrm{div}\ u = 0 & \text{in } \Omega. \end{cases} \tag{4.1.1}$$

Here $u = u(x, t)$ is the velocity and $p = p(x, t)$ is the modified pressure given by

$$p = -\tilde{p} - \frac{1}{2}|u|^2 + \alpha u \cdot \Delta u + \frac{\alpha}{4}\ \mathrm{tr}((\nabla u) + (\nabla u)^T)^2.$$

f is the external body force, the density of the fluid is $\rho = 1$ and the parameters ν and α are given positive constants. We assume that the fluid adheres to the boundary $\partial\Omega$, condition expressed by

$$u|_{\partial\Omega} = 0, \tag{4.1.2}$$

and we also consider that

$$u(x, 0) = u_0(x), \qquad x \in \Omega. \tag{4.1.3}$$

We also assume that Ω is a simply connected, bounded, open set with smooth (C^3) and connected boundary.

For a deeper understanding of the model of a second-grade fluid we refer the reader to [DF]†. Here we limit ourselves to continuing the mathematical study done by Cioranescu and Ouazar [CO] (see also [CG, CA]). We recall the mathematical setting of the problem. We consider the following functional spaces :

$$\mathcal{V} = \{u \in [C_0^\infty(\Omega)]^2,\ \operatorname{div} u = 0\}, \qquad V = \text{ the closure of } \mathcal{V} \text{ in } [H_0^1(\Omega)]^2.$$

We set

$$(f, g) = \int_\Omega f \cdot g \, \mathrm{d}x, |f| = (f, f)^{1/2}, \qquad \forall f, g \in [L^2(\Omega)]^2;$$
$$((f, g)) = \int_\Omega \operatorname{grad} f \cdot \operatorname{grad} g \, \mathrm{d}x, \qquad ||f|| = ((f, f))^{1/2}, \qquad \forall f, g \in V.$$

The space V is a Hilbert space with the scalar product

$$(u, v)_V = (u, v) + \alpha((u, v)). \tag{4.1.4}$$

We also consider the Hilbert space

$$W = \{u \in V,\ \operatorname{curl}\ (u - \alpha\Delta u) \in L^2(\Omega)\},$$

endowed with the scalar product

$$(u, v)_W = (u, v)_V + (\operatorname{curl}\ (u - \alpha\Delta u),\ \operatorname{curl}\ (v - \alpha\Delta v)), \qquad \forall u, v \in W. \tag{4.1.5}$$

The assumption on Ω allows us to prove that $W = \{u \in [H^3(\Omega) \cap H_0^1(\Omega)]^2,\ \operatorname{div} u = 0\}$ and that there exists a constant $C(\alpha)$ such that

$$|u|_{H^3} \leqslant C(\alpha)|\operatorname{curl}(u - \alpha\Delta u)|, \qquad \forall u \in W. \tag{4.1.6}$$

(For the proof see [CG].) If we identify V with its dual space V', we have $W \subset V \equiv V' \subset W'$, with continuous injections and each space being dense in the following one.

The weak formulation of the problem (4.1.1)–(4.1.3) is the following.

For u_0 and f given, find u such that

$$\begin{cases} (u', v)_V + \nu((u, v)) + b(u, u, v) - \alpha b(u, \Delta u, v) + \alpha b(v, \Delta u, u) = (f, v), & \forall v \in V, \\ u(0) = u_0, \end{cases} \tag{4.1.7}$$

where

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x.$$

† Although some authors criticize the physical interest of this model, it is not our intention here to judge its validity; we are only interested in studying its mathematical aspects.

A compilation of the previous results from [CO, CG, CA] implies that for $u_0 \in W$ and $f \in (H^1(\Omega))^2$ given, there exists a unique solution u of (4.1.7) satisfying

$$u \in L^\infty(\mathbb{R}^+; W), \qquad u' \in L^\infty(\mathbb{R}^+; V). \tag{4.1.8}$$

The proof of the existence relies on the Faedo–Galerkin method implemented with a special basis in V, namely the spectral basis $\{w_j\}_{j\geqslant 1}$ which satisfies

$$(w_j, v)_W = \lambda_j (w_j, v)_V, \qquad \forall v \in W, \quad \forall j \geqslant 1, \tag{4.1.9}$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, and $\lambda_j \to \infty$ as $j \to \infty$. Let us remark that $\{w_j\}_{j\geqslant 1}$ is also an orthogonal basis in W. Moreover, if Ω is of class C^3, then $w_j \in H^4(\Omega)^2$. The approximate solutions u_m satisfy

$$\{u_m\}_m \text{ is bounded in } L^\infty(\mathbb{R}^+; W), \qquad \{u'_m\}_m \text{ is bounded in } L^\infty(\mathbb{R}^+; V), \tag{4.1.10}$$

and we have the following energy equations in V and W respectively:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m|_V^2 + \nu||u_m||^2 = (f, u_m), \tag{4.1.11}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m|_W^2 + \frac{\nu}{\alpha}|u_m|_W^2 = \frac{1}{2}K(u_m), \tag{4.1.12}$$

where

$$\frac{1}{2}K(u_m) = \frac{\nu}{\alpha}|u_m|^2 + (f, u_m) + \Big(\frac{\nu}{\alpha}\mathrm{curl}\, u_m + \mathrm{curl} f,\ \mathrm{curl}\ (u_m - \alpha\Delta u_m)\Big). \tag{4.1.13}$$

We deduce from (4.1.10) that there exists a subsequence of $\{u_m\}$ (still denoted $\{u_m\}$) such that

$$\begin{aligned}
&u_m \overset{*}{\rightharpoonup} u \text{ star-weakly in } L^\infty(\mathbb{R}^+; W),\\
&u'_m \overset{*}{\rightharpoonup} u' \text{ star-weakly in } L^\infty(\mathbb{R}^+; V),\\
&u_m \to u \text{ strongly in } L^2(0, T; V \cap H^2), \qquad \forall T > 0.
\end{aligned}$$

The convergences above allow the passage to the limit when m goes to infinity to find that u is the solution of (4.1.7) (the uniqueness proof is standard).

Since $u \in L^\infty(\mathbb{R}^+; W)$ and $u' \in L^\infty(\mathbb{R}^+; V)$ we deduce that u is a.e. equal to a continuous function from $[0, T]$ into V for all $T > 0$. Moreover, $u \in L^\infty(\mathbb{R}^+; W) \cap C([0, T], V), \forall T > 0$, implies

$$u \in C_w([0, T]; W), \tag{4.1.14}$$

$$|u(t)|_W \leqslant |u|_{L^\infty(0,T;W)}, \qquad \forall t \in [0, T]. \tag{4.1.15}$$

(For details, see, e.g. [T2, ch3]). Note that (4.1.14) and (4.1.15) also hold for u_m. The results above are known and we now want to establish some new results. We first show that $u \in C([0, T], W), \forall T > 0$. Let us start by proving the following lemma.

Lemma 4.1.1. *$u_m(t) \rightharpoonup u(t)$ weakly in W, for all $t \geqslant 0$.*

Proof. Since $u_m \to u$ strongly in $L^2(0, T; V \cap H^2)$, $\forall T > 0$, we deduce that there exists a subsequence $\{u_{m'}\}$ such that

$$u_{m'}(t) \to u(t) \text{ strongly in } V, \text{ for a.e. } t \in [0, T], \qquad \forall T > 0. \tag{4.1.16}$$

For $0 \leqslant t \leqslant t+a \leqslant T$ we have

$$u_{m'}(t+a) - u_{m'}(t) = \int_t^{t+a} u'_{m'}(s)\,\mathrm{d}s, \qquad \text{in } V,$$

and thus

$$|u_{m'}(t+a) - u_{m'}(t)|_V \leqslant \int_t^{t+a} |u'_{m'}(s)|_V\,\mathrm{d}s \leqslant (\text{using (4.1.10)}) \leqslant ca, \qquad \forall m' \geqslant 1. \tag{4.1.17}$$

Using (4.1.16), (4.1.17), and the fact that $u_{m'}$ and u are in $C([0,T],V)$, we deduce that

$$u_{m'}(t) \to u(t) \text{ strongly in } V, \qquad \text{for all } t \in [0,T]. \tag{4.1.18}$$

From the previous convergence and using the spectral basis of V defined by (4.1.9), we obtain

$$(u_{m'}(t), w_j)_W \to (u(t), w_j)_W, \qquad \forall j \geqslant 1, \forall t \in [0,T], \tag{4.1.19}$$

and by the density of $\{w_j\}$ in W, we have

$$u_{m'}(t) \rightharpoonup u(t) \text{ weakly in } W, \quad \forall t \in [0,T]. \tag{4.1.20}$$

Then, by a contradiction argument, we deduce that the whole sequence $\{u_m(t)\}_m$ converges to $u(t)$ weakly in W for every $t \in [0,T]$. □

We now prove an energy inequality in W for the solution u. Integrating (4.1.12) between 0 and t we obtain

$$|u_m(t)|_W^2 = |u_{0m}|_W^2 \mathrm{e}^{-\frac{2\nu}{\alpha}t} + \int_0^t K(u_m(s))\mathrm{e}^{\frac{-2\nu}{\alpha}(t-s)}\,\mathrm{d}s, \qquad \forall t \in [0,T]. \tag{4.1.21}$$

Since $u_{0m} \to u_0$ strongly in W, $u_m \overset{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}^+; W)$, and K is weakly continuous on W, we can pass to the limit in (4.1.21) using the Lebesgue-dominated convergence theorem to find

$$\lim_{m\to\infty} |u_m(t)|_W^2 = |u_0|_W^2 \mathrm{e}^{-\frac{2\nu}{\alpha}t} + \int_0^t K(u(s))\mathrm{e}^{-\frac{2\nu}{\alpha}(t-s)}\,\mathrm{d}s, \qquad \forall t \in [0,T]. \tag{4.1.22}$$

From lemma 4.1.1 we have that

$$|u(t)|_W^2 \leqslant \liminf_{m\to\infty} |u_m(t)|_W^2, \qquad \forall t \in [0,T].$$

Thus, we conclude that

$$|u(t)|_W^2 \leqslant |u_0|_W^2 \mathrm{e}^{-\frac{2\nu}{\alpha}t} + \int_0^t K(u(s))\mathrm{e}^{-\frac{2\nu}{\alpha}(t-s)}\,\mathrm{d}s, \qquad \forall t \in [0,T]. \tag{4.1.23}$$

By reversing the time in equation (4.1.1) we find the following problem:

$$\begin{cases} \dfrac{\partial}{\partial t}(v - \alpha\Delta u) + \nu\Delta v - \operatorname{curl}(v - \alpha\Delta v) \times v = -f + \nabla q & \text{in } \Omega, \\ \operatorname{div} v = 0 \quad \text{in } \Omega, \\ v|_{\partial\Omega} = 0, \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{4.1.24}$$

We obtain in a similar way the finite time estimates in V and W for v, as well as the following energy inequality in W:

$$|v(t)|_W^2 \leqslant |v(0)|_W^2 e^{\frac{2\nu}{\alpha}t} - \int_0^t K(v(s)) e^{\frac{2\nu}{\alpha}(t-s)} \, ds, \qquad \forall t \in [0, T]. \tag{4.1.25}$$

If $v(0) = u(t_1) \in W$ for some $t_1 \in [0, T]$ (we recall from (4.1.14) that $u(t) \in W$ for $t \in [0, T]$), then by the uniqueness of the solutions we deduce that $v(t) = u(t_1 - t)$ for $t \in [0, t_1]$. Thus, we obtain from (4.1.25) for $t = t_1$ that

$$|u(0)|_W^2 \leqslant |u(t_1)|_W^2 \, e^{\frac{2\nu}{\alpha}t_1} - \int_0^{t_1} K(u(t_1 - s)) e^{\frac{2\nu}{\alpha}(t_1-s)} \, ds, \qquad \forall t_1 \in [0, T], \tag{4.1.26}$$

which gives

$$|u(0)|_W^2 \leqslant |u(t_1)|_W^2 e^{\frac{2\nu}{\alpha}t_1} - \int_0^{t_1} K(u(s)) e^{\frac{2\nu}{\alpha}s} \, ds, \qquad \forall t_1 \in [0, T]$$

or, equivalently,

$$|u(t_1)|_W^2 \geqslant |u(0)|_W^2 e^{-\frac{2\nu}{\alpha}t_1} + \int_0^{t_1} K(u(s)) e^{-\frac{2\nu}{\alpha}(t_1-s)} \, ds, \qquad \forall t_1 \in [0, T]. \tag{4.1.27}$$

From (4.1.23) and (4.1.27) we conclude the following.

Theorem 4.1.2. *For $u_0 \in W$ and $f \in [H^1(\Omega)]^2$ given, the solution u of the problem (4.1.7) satisfies the following energy equality:*

$$|u(t)|_W^2 = |u_0|_W^2 e^{-\frac{2\nu}{\alpha}t} + \int_0^t K(u(s)) e^{-\frac{2\nu}{\alpha}(t-s)} \, ds, \qquad \forall t \geqslant 0. \tag{4.1.28}$$

Moreover,

$$u \in C(\mathbb{R}^+, W). \tag{4.1.29}$$

The second statement of theorem 2 is obvious. Indeed, from (4.1.28) we deduce that $|u(t)|_W \to |u(t_0)|_W$ as $t \to t_0$, which together with $u \in C_w(\mathbb{R}^+, W)$ implies (4.1.29).

Thanks to theorem 4.1.2 we can define the semigroup $\{S(t)\}_{t \geqslant 0}$ in W by

$$S(t)u_0 = u(t), \qquad \forall t \geqslant 0. \tag{4.1.30}$$

We now state further properties of the semigroup $\{S(t)\}_{t \geqslant 0}$. More precisely, we prove the following.

Proposition 4.1.3. *The operators $S(t)$ are continuous and weakly continuous on W for all $t \geqslant 0$.*

Proof. Let us consider a sequence $u_{0n} \in W$ such that $u_{0n} \rightharpoonup u_0$ weakly in W. We set

$$u_n(t) = S(t)u_{0n}, \qquad u(t) = S(t)u_0, \qquad \forall t \geqslant 0.$$

From the *a priori* estimates in V and W we find that

$$\{u_n\}_n \text{ is bounded in } L^\infty(\mathbb{R}^+; W), \qquad \{u_n'\}_n \text{ is bounded in } L^\infty(\mathbb{R}^+; V), \tag{4.1.31}$$

and from theorem 4.1.2,

$$u, u_n \in C([0, T], W), \qquad \forall T > 0. \tag{4.1.32}$$

From (4.1.31) and (4.1.32) and with a reasoning as in lemma 4.1.1, we can extract a subsequence $\{u_{n'}\}$ such that

$$\begin{aligned}
&u_{n'} \overset{*}{\rightharpoonup} \tilde{u} \text{ weakly star in } L^\infty(\mathbb{R}^+; W),\\
&u'_{n'} \overset{*}{\rightharpoonup} \tilde{u}' \text{ weakly star in } L^\infty(\mathbb{R}^+; V),\\
&u_{n'} \to \tilde{u} \text{ strongly in } L^2(0, T; H^2 \cap V), \qquad \forall T > 0,\\
&u_{n'}(t) \to \tilde{u}(t) \text{ weakly in } W, \qquad \forall t \in [0, T],
\end{aligned} \tag{4.1.33}$$

for some $\tilde{u} \in L^\infty(\mathbb{R}^+; W) \cap C_w([0, T], W)$, $\forall T > 0$, with $\tilde{u}' \in L^\infty(\mathbb{R}^+; V)$.

The convergences (4.1.33) allow us to pass to the limit in the equation for $u_{n'}$ to find that $\tilde{u}$ is a solution of (4.1.7) with $\tilde{u}(0) = u_0$. Then by the uniqueness of the solutions we obtain $\tilde{u} = u$. Again, by a contradiction argument we deduce that the whole sequence $\{u_n\}$ converges to u in the sense of (4.1.33). In particular, we have

$$S(t)u_{0n} \rightharpoonup S(t)u_0 \quad \text{weakly in } W, \qquad \forall t \geqslant 0. \tag{4.1.34}$$

Now we consider $u_{0n} \to u_0$ strongly in W. The energy equation (4.1.28) for u_n reads

$$S(t)|u_{0n}|_W^2 = |u_{0n}|_W^2 \, \mathrm{e}^{-\frac{2\nu}{\alpha}t} + \int_0^t K(S(s)u_{0n}) \, \mathrm{e}^{-\frac{2\nu}{\alpha}(t-s)} \, \mathrm{d}s, \qquad \forall t \geqslant 0. \tag{4.1.35}$$

The weak convergence (4.1.34), the boundedness of K on bounded subsets of W, and the weak continuity of K on W allow us to pass to the limit in (4.1.35) to find that

$$\lim_{n\to\infty} |S(t)u_{0n}|_W^2 = |u_0|_W^2 \mathrm{e}^{-\frac{2\nu}{\alpha}t} + \int_0^t K(S(s)u_0)\mathrm{e}^{-\frac{2\nu}{\alpha}(t-s)} \, \mathrm{d}s \qquad \forall t > 0. \tag{4.1.36}$$

But from the energy equation for u, the right-hand side term in (4.1.36) is $|S(t)u_0|_W^2$. Thus,

$$\lim_{n\to\infty} |S(t)u_{0n}|_W^2 = |S(t)u_0|_W^2, \tag{4.1.37}$$

which together with (4.1.34) yields

$$S(t)u_{0n} \to S(t)u_0 \text{ strongly in } W \text{ as } n \to \infty, \qquad \forall t \geqslant 0. \tag{4.1.38}$$

□

Using the *a priori* estimates in V and W, we obtain the existence of bounded absorbing sets in V and, respectively, W. Combining theorem 4.1.2 (energy equation), proposition 4.1.3 (weak and strong continuity) and theorem 3.1, we deduce the existence of the global attractor:

Theorem 4.1.4. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded, open set with smooth (C^3) and connected boundary, and let $\nu > 0$, $\alpha > 0$, and $f \in [H^1(\Omega)]^2$ be given. Then the semigroup $\{S(t)\}_{t\geqslant 0}$ (which is actually a group) in W associated to the problem (4.1.1)–(4.1.3) possesses a global attractor in W.*

Remark. A similar model called the Navier–Stokes–Voigt system has been considered by Kalantarov [K], who proved the existence of the global attractor using a decomposition of the semigroup solution. We notice that the Navier–Stokes–Voigt system features a milder nonlinearity than that of the second-grade fluid model. As a consequence, the decomposition used in [K] cannot be transported to the present example, and this is because for the second-grade fluid model there is no regularization effect of the solution of the linear part of the system with respect to the non-homogeneous term, in opposition to the case of the Navier–Stokes–Voigt model.

4.2. Flows past an obstacle

In this section we study the long time behaviour of a uniform flow past an infinite long cylindrical obstacle. We will assume that the flow is uniform in the direction of the axis of the cylindrical obstacle and the flow approaches $U_\infty e_x$ farther away from the obstacle. In this respect we can consider a two-dimensional flow and assume the obstacle is a disk with radius r (more general obstacle can be treated in exactly the same way).

A further simplification is to observe that since the flow is uniform at infinity, we may assume that the flow is in an infinitely long channel with width $2L(L \gg r)$ and the obstacle is located at the centre, while the flow at the boundary of the channel is almost the uniform flow at infinity.

More precisely we assume that the flow is governed by the following Navier–Stokes equations in $\Omega = \mathbb{R}^1 \times (-L, L) \setminus B_r(0)(L \gg r)$:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f \qquad \text{in } \Omega, \tag{4.2.1a}$$

$$\text{div } u = 0 \qquad \text{in } \Omega, \tag{4.2.1b}$$

$$u = u_0 \qquad \text{at } t = 0, \tag{4.2.1c}$$

$$u = 0 \qquad \text{at } \partial B_r, \qquad u = \varphi \qquad \text{at } y = \pm L, \tag{4.2.1d}$$

with

$$\text{div } u_0 = 0 \qquad \text{in } \Omega, \qquad u_{02} = 0 \qquad \text{at } \partial\Omega, \qquad u_0 - U_\infty e_x \in \mathbb{L}^2(\Omega), \tag{4.2.2a}$$

$$\varphi - U_\infty e_x \in \mathbb{H}^2(\mathbb{R}^1 \times \{-L\} \cup \mathbb{R}^1 \times \{L\}), \qquad \varphi_2 \equiv 0, \tag{4.2.2b}$$

$$\text{div } f = 0, \qquad f_2 = 0 \qquad \text{at } \partial\Omega, \qquad f \in \mathbb{L}^2(\Omega). \tag{4.2.2c}$$

Remark 4.2.1. The simplest and physically interesting case is $f \equiv 0$ and $\varphi \equiv U_\infty e_x$.

The first simplification is to introduce the new variables

$$\widetilde{u} = u - U_\infty e_x, \qquad \tilde{\varphi} = \varphi - U_\infty e_x, \qquad \tilde{u}_0 = u_0 - U_\infty e_x. \tag{4.2.3}$$

Then $\tilde{u}$ satisfies the equations

$$\frac{\partial \tilde{u}}{\partial t} - \nu \Delta \tilde{u} + (\tilde{u} \cdot \nabla)\tilde{u} + U_\infty \partial_x \tilde{u} + \nabla p = f, \tag{4.2.4a}$$

$$\text{div } \tilde{u} = 0, \tag{4.2.4b}$$

$$\widetilde{u} = \widetilde{u}_0 \qquad \text{at } t = 0, \tag{4.2.4c}$$

$$\tilde{u} = -U_\infty e_x \qquad \text{at } \partial B_r, \qquad \tilde{u} = \tilde{\varphi} \qquad \text{at } y = \pm L. \tag{4.2.4d}$$

We observe that

$$\tilde{\varphi} \in \mathbb{H}^2(\mathbb{R}^1 \times \{\pm L\}). \tag{4.2.5}$$

Note that $\widetilde{u}$, $\tilde{u}_0$ and $\widetilde{\varphi}$ decay nicely near infinity. However, the boundary condition is not homogeneous and thus we apply a modified Hopf's technique (see [TW, T1]) to homogenize the boundary condition. More specifically we choose

$$\rho_i \in C^\infty([0,1]), \qquad \text{supp } \rho_i \subset [0, \tfrac{1}{2}], \tag{4.2.6a}$$

$$\int_0^1 \rho_1(s)\,\mathrm{d}s = 0, \qquad \rho_1(0) = 1, \tag{4.2.6b}$$

$$\rho_2(0) = 1, \qquad \rho_2'(0) = 0, \ |s\rho_2(s)| \leqslant \frac{\nu}{1001U_\infty}, \qquad |s\rho_2'(s)| \leqslant \frac{\nu}{1001U_\infty}, \tag{4.2.6c}$$

and we define, for $\varepsilon < 1$,

$$\Psi^1(x,y) = \begin{cases} -\tilde{\varphi}_1(x,L)\displaystyle\int_0^{L-y} \rho_1\left(\frac{s}{L\varepsilon}\right)\mathrm{d}s, & \text{for } \frac{L}{2} < y < L, \\ \tilde{\varphi}_1(x,-L)\displaystyle\int_0^{L+y} \rho_1\left(\frac{s}{L\varepsilon}\right)\mathrm{d}s, & \text{for } -L < y < -\frac{L}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.7}$$

$$\Psi^2(x,y) = \begin{cases} -U_\infty\rho_2\left(\dfrac{\sqrt{x^2+y^2}}{r} - 1\right)y, & \text{for } r < \sqrt{x^2+y^2} < 2r, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.8}$$

We then define

$$\phi^i(x,y) = \text{ curl } \Psi^i = (\partial_y\Psi^i, -\partial_x\Psi^i). \tag{4.2.9}$$

Observe that ϕ^1 matches $\tilde{\varphi}$ at $y = \pm L$ and ϕ^2 matches $-U_\infty e_x$ at ∂B_r. If we set

$$H = \{v \in \mathbb{L}^2(\Omega), \text{ div } v = 0, \ v \cdot \overrightarrow{n} = 0 \text{ at } \partial\Omega\}, \tag{4.2.10}$$

$$V = \{v \in \mathbb{H}_0^1(\Omega), \text{ div } v = 0, \ v = 0 \text{ at } \partial\Omega\}, \tag{4.2.11}$$

$$V' = \text{ the dual of } V, \tag{4.2.12}$$

where $\overrightarrow{n}$ denotes the unit outward normal at $\partial\Omega$, we have that

$$v = \tilde{u} - \phi^1 - \phi^2 \tag{4.2.13}$$

satisfies the equation

$$\begin{aligned} \frac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + (v\cdot\nabla)\phi^1 + (v\cdot\nabla)\phi^2 + (\phi^1\cdot\nabla)v + (\phi^2\cdot\nabla)v + U_\infty\partial_x v + \nabla p \\ = f + \nu\Delta\phi^1 + \nu\Delta\phi^2 - ((\phi^1+\phi^2)\cdot\nabla)(\phi^1+\phi^2) - U_\infty\partial_x(\phi^1+\phi^2) \\ = F(\varepsilon, \nu, U_\infty, r, L), \end{aligned} \tag{4.2.14a}$$

$$v \in V \qquad \text{for } t > 0, \qquad v = v_0 \qquad \text{at } t = 0, \tag{4.2.14b}$$

where

$$v_0 = u_0 - U_\infty e_x - \phi^1 - \phi^2 \in H. \tag{4.2.14c}$$

It is easy to check that for fixed $\varepsilon, \nu, U_\infty, r$, and L, the right-hand side of (4.2.14a), namely F, belongs to $\mathbb{L}^2(\Omega)$ thanks to our construction of ϕ^1 and ϕ^2.

We say that v is a weak solution of (4.2.14) if

$$v \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.2.15a}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(v,w) + \nu(\nabla v, \nabla w) + b(v,v,w) + b(v,\phi^1+\phi^2,w) + b(\phi^1+\phi^2,v,w) \\ + b(U_\infty e_x, v, w) = (F,w), \qquad \forall w \in V, \end{aligned} \tag{4.2.15b}$$

in the distributional sense, and

$$v(0) = v_0, \tag{4.2.15c}$$

where the trilinear term $b : \mathbb{H}_0^1 \times \mathbb{H}_0^1 \times \mathbb{H}_0^1 \to \mathbb{R}$ is defined by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x. \tag{4.2.15d}$$

The well posedness of (4.2.15) can be derived using a standard Faedo–Galerkin approach (see for instance [T2, ch 3]) and we may view it as a dynamical system of the form

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &+ \nu A v + B(v, v) + B(v, \phi^1 + \phi^2) + B(\phi^1 + \phi^2, v) \\ &+ B(U_\infty e_x, v) = PF \qquad \text{in } V', \ \text{for } t > 0, \end{aligned} \tag{4.2.16a}$$

$$v(0) = v_0, \tag{4.2.16b}$$

where $A : V \to V'$ is the Stokes operator defined by

$$\langle Av, w\rangle = (\nabla v, \nabla w), \qquad \forall v, w \in V, \tag{4.2.16c}$$

and $B(u, v)$ is a bilinear operator $\mathbb{H}_0^1 \times \mathbb{H}_0^1 \to V'$ defined by

$$\langle B(u, v), w\rangle = b(u, v, w), \qquad \forall u, v \in \mathbb{H}_0^1, \forall w \in V, \tag{4.2.16d}$$

and P is the Leray–Hopf projection from $\mathbb{L}^2(\Omega)$ onto H.

Our goal in this section is to show that (4.2.16) possesses a global attractor in H using theorem 3.1. Although dimension estimates can be obtained in the usual fashion using the global Lyapunov exponent technique (see for instance [T1]), we will refrain ourself from this topic as it is not our main concern here. In the special case of $U_\infty \equiv 0, \varphi \equiv 0$, such a problem was studied by Abergel [A] and Babin [Ba] for the case where the body force lies in some weighted Sobolev space, and by Rosa [R] for more general forces and domains.

Denoting the solution semigroup as $S(t)$, it is easily verified that $\{S(t), t \geqslant 0\}$ is a strongly continuous semigroup on H, and $S(t)$ is a continuous operator from H into H for each $t \geqslant 0$. Moreover, for $v_0 \in H$ and $T > 0$, there exists a constant $\kappa > 0$, such that for $v(t) = S(t)v_0$ we have

$$||v||_{L^\infty(0,T;H)} \leqslant \kappa, \qquad ||v||_{L^2(0,T;V)} \leqslant \kappa, \qquad ||v'||_{L^2(0,T,V')} \leqslant \kappa, \qquad v \in C(\mathbb{R}^+; H), \tag{4.2.17}$$

where $\kappa = \kappa(\nu, T, \varepsilon, |v_0|, |f|, ||\varphi||_{H^3(\mathbb{R}^1\times\{\pm L\})}, r, L, U_\infty)$.

This immediately implies that we have the following energy equation:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2 + \nu|\nabla v|^2 + \int_\Omega (v\cdot\nabla)(\phi^1 + \phi^2)\cdot v = (F, v). \tag{4.2.18}$$

A closer investigation into the well-posedness proof reveals that the solution set is compact in the sense that if $\{v_n, n \geqslant 1\}$ is a family of solutions on $[0, T]$ satisfying estimates (4.2.17) for a κ independent of n, then there exists a subsequence $\{v_{n'}, n' \geqslant 1\}$ and $v_{0\infty} \in H$, $v_\infty = S(t)v_{0\infty}$, such that

$$v_{n'} \rightharpoonup v_\infty \text{ weakly star in } L^\infty(0, T; H) \text{ and weakly in } L^2(0, T; V), \tag{4.2.19a}$$

$$v'_{n'} \rightharpoonup v'_\infty \text{ weakly in } L^2(0, T; V'). \tag{4.2.19b}$$

For a proof the reader is referred to [T2, ch 3, remark 3.2] or to [R] for more details. This actually implies the weak continuity of $S(t), \forall t \geqslant 0$. Indeed, let v_{0n} be a weakly convergent subsequence in H, then $v_n(t) = S(t)v_{0n}$ satisfies (4.2.17) with a constant κ independent of n. Let $v_{0\infty}$ be the weak limit of v_{0n}. Then each subsequence of $\{v_n, n \geqslant 1\}$ contains a sub-subsequence which converges to some v_∞ in the sense of (4.2.19). It is easy to check that $v_\infty(0) = v_{0\infty}$. Since this is true for each subsequence, we conclude that the whole sequence converges to $v_\infty(t) = S(t)v_{0\infty}$, i.e. $S(t)v_{0n} = v_n(t) \to v_\infty(t) = S(t)v_{0\infty}$, weakly in V' and then in H by density and (4.2.17). This completes the weak continuity proof.

Before we apply theorem 3.1, we need to verify the existence of a bounded absorbing set in H. This can be done via an appropriate choice of ε in (4.2.7) and using (4.2.18).

Observe that

$$\begin{aligned}
\left|\int_\Omega (v\cdot\nabla)\phi^2\cdot v\right| &= \left|\int_\Omega (v\cdot\nabla)v\cdot\phi^2\right| \\
&\leqslant r\left\|\frac{v}{\sqrt{x^2+y^2}-r}\right\|_{L^2(B_{2r\setminus Br})}|\nabla v|\left|\left(\frac{\sqrt{x^2+y^2}}{r}-1\right)\phi^2\right|_{L^\infty(B_{2r\setminus Br})} \\
&\leqslant 4r|\nabla v|^2\cdot|U_\infty|(|s\rho_2(s)|_{L^\infty}+4|s\rho_2'(s)|_{L^\infty}) \\
&\quad\text{(thanks to Hardy's inequality and (4.2.6)–(4.2.9))} \\
&\leqslant \frac{\nu}{5}|\nabla v|^2 \qquad \text{(thanks to (4.2.6))}
\end{aligned} \tag{4.2.20a}$$

$$\left|\int_\Omega (v\cdot\nabla)\phi^1\cdot v\right| = \left|\int_\Omega (v\cdot\nabla)v\cdot\phi^1\right| \tag{4.2.20b}$$

$$\begin{aligned}
&\leqslant \left(\left|\frac{v}{L-y}\right|_{L^2(1-\varepsilon<\frac{y}{L}<1)}+\left|\frac{v}{L+y}\right|_{L^2(-1<\frac{y}{L}<-1+\varepsilon)}\right)|\nabla v| \\
&\quad\times(|(L-y)\phi^1|_{L^\infty(1-\varepsilon<\frac{y}{L}<1)}+|(L+y)\phi^1|_{L^\infty(-1<\frac{y}{L}<-1+\varepsilon)})
\end{aligned} \tag{4.2.20c}$$

$$\begin{aligned}
&\quad\left(\text{since the support of } \phi^1 \text{ is in } \left\{1-\varepsilon<\frac{y}{L}<1\right\}\cup\left\{-1<\frac{y}{L}<-1+\varepsilon\right\}\right) \\
&\leqslant k|\nabla v|^2((L\varepsilon)^2|\varphi_{1x}|_{L^\infty(\mathbb{R}^1\times\{\pm L\})}+L\varepsilon|\varphi_1-U_\infty|_{L^\infty(R^1\times\{\pm L\})}) \\
&\quad\text{(by (4.2.7) and (4.2.9))} \\
&\leqslant \frac{\nu}{4}|\nabla v|^2
\end{aligned} \tag{4.2.20d}$$

provided we choose ε small enough:

$$\varepsilon \leqslant \min\left(\frac{\nu}{8kL|\varphi_1-U_\infty|_{L^\infty(\mathbb{R}^1\times\{\pm L\})}}, \frac{1}{L}\frac{\nu}{\sqrt{8k|\varphi_{1x}|_{L^\infty(\mathbb{R}^1\times\{\pm L\})}}}\right). \tag{4.2.21}$$

Combining (4.2.20) and (4.2.18) we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v|^2+\frac{\nu}{2}|\nabla v|^2 \leqslant (F, v), \tag{4.2.22}$$

which leads to the existence of a bounded absorbing ball in the usual way.

Now we rewrite (4.2.18) as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|S(t)v_0|^2+\frac{\lambda_1\nu}{2}|S(t)v_0|^2+2\nu|\nabla(S(t)v_0)|^2-\frac{\lambda_1\nu}{2}|S(t)v_0|^2 \\
+2b(S(t)v_0, \phi^1+\phi^2, S(t)v_0) = 2(F, S(t)v_0)
\end{aligned} \tag{4.2.23}$$

where λ_1 is the first eigenvalue of the Stokes operator on Ω. In the notation used in section 3, we identify the separable reflexive Banach space E with H, $\{S(t)\}_{t\geqslant 0}$ as above,

$$\Phi(v) = |v|^2, \qquad J(v) = 0, \qquad \gamma = \frac{\lambda_1\nu}{2}, \tag{4.2.24a}$$

$$L(v) = 2\nu|\nabla v|^2 - \frac{\lambda_1 \nu}{2}|v|^2 + 2b(v, \phi^1 + \phi^2, v), \qquad K(v) = 2(F, v). \tag{4.2.24b}$$

Thus, all assumptions of theorem 3.1 are satisfied except we need to verify the asymptotic weak lower semicontinuity of L. For this purpose we first notice that L is a quadratic form in V and that V contains $\cup_{t>0} S(t)H$. Then, thanks to (4.2.20),

$$L(v) \geqslant \nu|\nabla v|^2 - \frac{\lambda_1 \nu}{2}|v|^2 \geqslant \frac{\nu}{2}|\nabla v|^2 \qquad \text{(by Poincaré inequality)}. \tag{4.2.25}$$

Thus

$$\left(\int_0^t \mathrm{e}^{-\gamma(t-s)} L(\cdot)\, \mathrm{d}s \right)^{1/2}$$

is a norm in $L^2(0, t; V)$ equivalent to the usual one, so that its square is weakly lower semicontinuous in $L^2(0, t; V)$, which together with the weak continuity of $S(t)$ gives in particular the desired asymptotic weak lower semicontinuity of L in the sense of (3.6).

Hence, the existence of the global attractor follows from theorem 3.1 and we have the following result.

Theorem 4.2.1. *Under the assumptions above, in particular (4.2.2) and (4.2.21), the semigroup* $\{S(t)\}_{t\geqslant 0}$ *associated to (4.2.16) possesses a connected global attractor in* H.

4.3. Weakly damped, forced Korteweg–de Vries equation

We consider the KdV equation with weak damping and an external time independent force:

$$u_t + uu_x + u_{xxx} + \gamma u = f, \tag{4.3.1}$$

where $u = u(x, t), \gamma > 0$ and $f = f(x)$. This equation was proposed by Ott and Sudan [OS] as a model for the propagation of ion-sound waves damped by ion-neutral collisions. We take $E = H^2(\mathbb{R})$ to be the phase space of this equation and supplement it with the initial condition

$$u(x, 0) = u_0(x), \tag{4.3.2}$$

for $u_0 \in H^2(\mathbb{R})$. We assume that f is in $H^2(\mathbb{R})$.

Equation (4.3.1) with space periodicity L and a time-independent force $f \in H^2_{\text{per}}(0, L)$ generates a group in $H^2_{\text{per}}(0, L)$ for which the existence of the global attractor has been proved by Ghidaglia [G1, G2]. The same holds in $H^m_{\text{per}}(0, L), m \geqslant 3$, provided $f \in H^k_{\text{per}}(0, L), k \geqslant m$, in which case the global attractor is compact in $H^k_{\text{per}}(0, L)$, as proved by Moise and Rosa [MR]. The whole space case has been treated by Laurençot [L], who also used the energy equation approach but with the drawback of using a splitting of the group and weighted spaces in a complicated intermediate step. We avoid this intermediate step by using a second energy equation, namely that in $L^2(\mathbb{R})$ besides the one in $H^2(\mathbb{R})$, which makes the proof much simpler. The $H^1(\mathbb{R})$-case can also be treated by this approach and will be presented in a forthcoming paper.

For the well posedness, we have the following result.

Theorem 4.3.1. *Let* $\gamma \in \mathbb{R}$ *and* $f \in H^2(\mathbb{R})$ *be given. Then, for every* $u_0 \in H^2(\mathbb{R})$ *there exists a unique solution* $u = u(t)$ *of (4.3.1), (4.3.2) satisfying*

$$u \in C([0, T], H^2(\mathbb{R})), \qquad \forall T > 0. \tag{4.3.3}$$

Moreover, the following energy equations hold in the distribution sense on $(0, \infty)$*:*

$$\frac{\mathrm{d}}{\mathrm{d}t} I_m(u(t)) + 2\gamma I_m(u(t)) = K_m(u(t)), \qquad m = 0, 1, 2, \tag{4.3.4}$$

where

$$I_0(u) = \int u^2 \mathrm{d}x, \qquad K_0(u) = \int 2fu\,\mathrm{d}x, \tag{4.3.5}$$

$$I_1(u) = \int \left[u_x^2 - \frac{1}{3}u^3\right] \mathrm{d}x, \qquad K_1(u) = \int \left[\frac{\gamma}{3}u^3 + 2f_x u_x - fu^2\right] \mathrm{d}x, \tag{4.3.6}$$

$$\begin{cases} I_2(u) = \displaystyle\int \left[u_{xx}^2 - \frac{5}{3}u_x^2 + \frac{5}{36}u^4\right] \mathrm{d}x, \\ K_2(u) = \displaystyle\int \left[\frac{5\gamma}{3}uu_x^2 - \frac{5\gamma}{18}u^4 + 2f_{xx}u_x x + \frac{5}{3}u_x^2 f + \frac{10}{3}uu_{xx}f + \frac{5}{9}u^3 f\right] \mathrm{d}x, \end{cases} \tag{4.3.7}$$

with all the integrals over $\mathbb{R}$*. Finally for every* $R,\ T > 0$ *there exists a constant* $C = C(R, T)$ *such that*

$$\sup\{||u(t)||_{H^2(\mathbb{R})}, 0 \leqslant t \leqslant T, ||u_0||_{H^2(\mathbb{R})} \leqslant R\} \leqslant C(R, T). \tag{4.3.8}$$

The proof of theorem 4.3.1 follows as in the case of $\gamma = 0$ and $f = 0$. The existence of solutions in $L^\infty((0, T); H^2(\mathbb{R})) \cap C([0, T], L^2(\mathbb{R}))$ and an inequality ($\leqslant$) in (4.3.4) can be obtained by parabolic regularization [T3, BS, MR]. The uniqueness is straightforward. The equality in (4.3.4) and, as a consequence, the regularity $u \in C([0, T], H^2(\mathbb{R}))$ can be obtained by using the time reversibility of the solutions as done in section 4.3.1 for the second-grade fluids.

Thanks to theorem 4.3.1, one can define for $\gamma > 0$, which is the case of interest for us, the semigroup $\{S(t)\}_{t\geqslant 0}$ in $H^2(\mathbb{R})$ by $S(t)u_0 = u(t)$, where $u = u(t)$ is the solution of (4.3.1), (4.3.2). The continuity of the trajectories $t \to S(t)u_0$ follows from (4.3.3). Thus, most of the conditions of theorem 3.1 hold, and we need to verify the remaining conditions. We have the following.

Lemma 4.3.2. *The semigroup* $\{S(t)\}_{t\geqslant 0}$ *possesses a bounded absorbing set in* $H^2(\mathbb{R})$*.*

Proof. The existence of a bounded absorbing set can be obtained just like in the autonomous space periodic case treated by Ghidaglia [G1], the differences being that the Agmon inequality has a different constant. We do not develop any details here. □

Lemma 4.3.3. $\{S(t)\}_{t\geqslant 0}$ *is a semigroup of continuous and weakly continuous operators in* $H^2(\mathbb{R})$*.*

Proof. For the weak continuity, let $u_{0j} \rightharpoonup u_0$ weakly in $H^2(\mathbb{R})$. We fix T and and we set $u_j(t) = S(t)u_{0j}$ for $0 \leqslant t \leqslant T$. Note that $\{u_{0j}\}_j$ is bounded in $H^2(\mathbb{R})$ since it has a weak limit in that space. Then, thanks to the long time estimates given by the existence of a uniformly absorbing set (lemma 4.3.2) and thanks to the local in time estimates given by (4.3.8), it follows that

$$\{u_j\}_j \text{ is bounded in } L^\infty(0, T; H^2(\mathbb{R})). \tag{4.3.9}$$

Then, from equation (4.3.1) itself, we deduce that

$$\{u_j'\}_j \text{ is bounded in } L^\infty(0,T;H^{-1}(\mathbb{R})), \tag{4.3.10}$$

where $H^{-1}(\mathbb{R})$ is the dual of $H^1(\mathbb{R})$ when we identify $L^2(\mathbb{R})$ with its dual. From (4.3.10) it follows that for $0 < a < T$ and $v \in H^1(\mathbb{R})$, the following estimate holds:

$$\begin{aligned}(u_j(t+a) - u_j(t), v)_{L^2(\mathbb{R})} &= \int_t^{t+a} \langle u_j'(s), v\rangle_{H^{-1}(\mathbb{R}),H^1(\mathbb{R})}\, \mathrm{d}s \\ &\leqslant a\|u_j'\|_{L^\infty(0,T;H^{-1}(\mathbb{R}))}\|v\|_{H^1(\mathbb{R})}, \qquad \forall 0 \leqslant t \leqslant T-a,\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{H^{-1}(\mathbb{R}),H^1(\mathbb{R})}$ denotes the duality product between the two spaces $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$. By taking $v = u_j(t+a) - u_j(t)$ for each $t \in [0, T-a]$, which is possible since $u_j \in C([0,T], H^2(\mathbb{R}))$, we find that

$$\|u_j(t+a) - u_j(t)\|^2_{L^2(\mathbb{R})} \leqslant 2a\|u_j'\|_{L^\infty(0,T;H^{-1}(\mathbb{R}))}\,\|u_j\|_{L^\infty(0,T;H^1(\mathbb{R}))}, \qquad \forall t \in [0, T-a].$$

Taking (4.3.9) and (4.3.10) into account, we obtain

$$\|u_j(t+a) - u_j(t)\|_{L^2(\mathbb{R})} \leqslant c_T a^{1/2}, \qquad \forall t \in [0,T-a],\ \forall a \in (0,T). \tag{4.3.11}$$

Now, for each $r > 0$, consider the sequence $\{u_{j,r}\}_j$, where $u_{j,r}(t) = \rho_r u_j(t)$, for $\rho_r = \rho_r(x) = \rho(x/r)$, with $\rho \in C^\infty(\mathbb{R})$, $\rho \geqslant 0$, $\rho(\xi) = 1$ for $|\xi| \leqslant 1$, and $\rho(\xi) = 0$ for $|\xi| \geqslant 2$. Thus, from (4.3.9) and (4.3.11), it follows that for each $r > 0$, the sequence $\{u_{j,r}\}_j$ is equibounded and equicontinuous in $C([0,T], L^2(-2r,2r))$. Moreover, from (4.3.9) and the fact that each u_j is continuous from $[0,T]$ to $H^2(\mathbb{R})$, it follows that for each $t \in [0,T]$, the set $\{u_{j,r}(t)\}_j$ is bounded in $H^2_0(-2r,2r)$, hence precompact in $L^2(-2r,2r)$. Therefore, we can apply the Arzela-Ascoli theorem to $\{u_{j,r}\}_j$ to deduce that this sequence is precompact in $C([0,T], L^2(-2r,2r))$. It is then clear that for each r, the sequence $\{u_j|_{(-r,r)}\}_j$ is precompact in $C([0,T], L^2(-r,r))$, where $u_j|_{(-r,r)}$ is the restriction of u_j in space to $(-r,r)$. Then, by a diagonalization process, we can find a subsequence $\{u_{j'}\}_{j'}$ and an element $\tilde{u} \in C([0,T], L^2_{\mathrm{loc}}(\mathbb{R}))$ such that $\{u_{j'}|_{(-r,r)}\}_{j'}$ converges to $\tilde{u}|_{(-r,r)}$ in $C([0,T], L^2(-r,r))$, which is to say that $\{u_{j'}\}_{j'}$ converges to $\tilde{u}$ in the topology of the Frechet space $C([0,T], L^2_{\mathrm{loc}}(\mathbb{R}))$. On the other hand, from (4.3.9) one can also assume that (passing to a further subsequence if necessary) $\{u_{j'}\}_{j'}$ converges to $\tilde{u}$ weakly star in $L^\infty(0,T;H^2(\mathbb{R}))$, which gives in particular that $\tilde{u} \in L^\infty(0,T;H^2(\mathbb{R}))$. Thus we have that

$$\begin{cases} u_{j'} \to \tilde{u} & \text{weakly star in } L^\infty(0,T;H^2(\mathbb{R})) \\ & \text{and strongly in } C([0,T], L^2_{\mathrm{loc}}(\mathbb{R})). \end{cases} \tag{4.3.12}$$

The convergence (4.3.12) allows us to pass to the limit in the weak form of the equation for u_j (the weak form of (4.3.1), (4.3.2) with $u_0 = u_{0j}$) to find that $\tilde{u}$ solves (the weak and the strong form of) the equations (4.3.1), (4.3.2). By the uniqueness of the solutions, we must have $\tilde{u}(t) = S(t)u_0$. Then, by a contradiction argument, one can deduce that in fact the whole sequence $\{u_j\}$ converges to $\tilde{u}$ in the sense of (4.3.12), and hence that

$$\begin{aligned} S(t)u_{0j} \to S(t)u_0 &\text{ weakly star in } L^\infty(0,T;H^2(\mathbb{R})) \\ &\text{and strongly in } C([0,T], L^2_{\mathrm{loc}}(\mathbb{R})). \end{aligned} \tag{4.3.13}$$

Now, from the strong convergence in (4.3.13), we find that for every t such that $0 \leqslant t \leqslant T$ and for every $v \in C_c^\infty(\mathbb{R})$, the space of C^∞ functions with compact support,

$$(S(t)u_{0j}, v)_{H^2(\mathbb{R})} = (S(t)u_{0j}, Lv)_{L^2(\mathbb{R})} \to (S(t)u_0, Lv)_{L^2(\mathbb{R})} = (S(t)u_0, v)_{H^2(\mathbb{R})},$$

where $Lv = v - v_{xx} + v_{xxxx} \in C_c^\infty(\mathbb{R})$. Then from (4.3.9) and the density of $C_c^\infty(\mathbb{R})$ in $H^2(\mathbb{R})$, it follows that for every $t, 0 \leqslant t \leqslant T$ and hence for every $t \geqslant 0$ since T was arbitrary,

$$(S(t)u_{0j}, v)_{H^2(\mathbb{R})} \to (S(t)u_0, v)_{H^2(\mathbb{R})}, \qquad \forall v \in H^2(\mathbb{R}), \forall t \geqslant 0, \tag{4.3.14}$$

which proves the weak continuity of $S(t)$ in $H^2(\mathbb{R})$.

For the strong continuity, assume that $u_{0j} \to u_0$ strongly in $H^2(\mathbb{R})$. In particular $u_{0j} \rightharpoonup u_0$ weakly in $H^2(\mathbb{R})$, so that the above convergences (4.3.13) and (4.3.14) hold. From the energy equation (4.3.4) for $m = 0$ it follows that

$$|u_j(t)|^2_{L^2(\mathbb{R})} = |u_{0j}|^2_{L^2(\mathbb{R})} \mathrm{e}^{-2\gamma t} + 2\int_0^t \mathrm{e}^{-2\gamma(t-s)}(f, u_j(s))_{L^2(\mathbb{R})}\,\mathrm{d}s, \qquad \forall t \geqslant 0, \tag{4.3.15}$$

where $u_j(t) = S(t)u_{0j}$ and $u(t) = S(t)u_0$. From the weak continuity (4.3.14), and the uniform boundedness (4.3.9), together with the strong convergence $u_{0j} \to u_0$ in $H^2(\mathbb{R})$, we can pass to the limit in (4.3.15) to find that for any $t \geqslant 0$

$$\lim_{j\to\infty} |u_j(t)|^2_{L^2(\mathbb{R})} = |u_0|^2_{L^2(\mathbb{R})} \mathrm{e}^{-2\gamma t} + 2\int_0^t \mathrm{e}^{-2\gamma(t-s)}(f, u(s))_{L^2(\mathbb{R})}\,\mathrm{d}s = |u(t)|^2_{L^2}. \tag{4.3.16}$$

From the weak continuity (4.3.14) and the L^2-norm convergence (4.3.16) it follows, since $L^2(\mathbb{R})$ is a Hilbert space, that

$$u_j(t) \to u(t) \text{ strongly in } L^2(\mathbb{R}), \qquad \forall t \geqslant 0. \tag{4.3.17}$$

Using interpolation, it follows from (4.3.17) and (4.3.14) that

$$u_j(t) \to u(t) \text{ strongly in } H^1(\mathbb{R}), \qquad \forall t \geqslant 0. \tag{4.3.18}$$

Now, from the energy equation (4.3.4) for $m = 2$, we have

$$I_2(u_j(t)) = I_2(u_{0j})\mathrm{e}^{-2\gamma t} + \int_0^t \mathrm{e}^{-2\gamma(t-s)} K_2(u_j(s))\,\mathrm{d}s, \qquad \forall t \geqslant 0. \tag{4.3.19}$$

As above, using also (4.3.18), we can pass to the limit in (4.3.19) to find that

$$\lim_{j\to\infty} I_2(u_j(t)) = I_2(u(t)), \qquad \forall t \geqslant 0. \tag{4.3.20}$$

Using (4.3.18) again, it follows from (4.3.20) and the definition of I_2 given by (4.3.7) that

$$||u_j(t)||_{H^2(\mathbb{R})} \to ||u(t)||_{H^2(\mathbb{R})}, \qquad \forall t \geqslant 0. \tag{4.3.21}$$

Then, (4.3.21) together with the weak continuity (4.3.14) implies finally that

$$S(t)u_{0j} \to S(t)u_0 \text{ strongly in } H^2(\mathbb{R}), \qquad \forall t \geqslant 0 \tag{4.3.22}$$

which proves the strong continuity of $S(t)$ in $H^2(\mathbb{R})$. □

In order to apply theorem 3.1, it remains to verify the corresponding conditions (3.3), (3.4) and (3.5) for the energy equation with $m = 2$. In order to do that, we first need the following result.

Lemma 4.3.4. *Let $\{u_j\}_j$ be bounded in $H^2(\mathbb{R})$ and $\{t_j\}_j \subset \mathbb{R}^+$ with $t_j \to \infty$. Then there exist $w \in H^2(\mathbb{R})$ and a subsequence $\{j'\}$ such that $S(t_{j'})u_{j'} \to w$ strongly in $H^1(\mathbb{R})$.*

Proof. We apply lemma 3.2 with the energy equation (4.3.4) for $m = 0$. In the notations of lemma 3.2, the terms of this energy equation are

$$\Phi(u) = I_0(u) = |u|_{L^2}^2, \qquad J(u) = 0, \qquad K(u) = K_0(u) = 2\int f u \,\mathrm{d}x \qquad L(u) = 0,$$

and obviously $F = L^2(\mathbb{R})$. The hypothesis (3.14) is trivially satisfied. In order to verify (3.5), let $\{u_j\}_j$ be bounded in $H^2(\mathbb{R})$, $\{t_j\}_j \subset \mathbb{R}^+$, $t_j \to \infty$ such that $S(t_j)u_j \rightharpoonup w$ weakly in $H^2(\mathbb{R})$. Since the operators $S(t)$ are weakly continuous (from lemma 4.3.3), we deduce that

$$S(s)S(t_j)u_j \rightharpoonup S(s)w \text{ weakly in } H^2(\mathbb{R}), \qquad \forall s \geqslant 0. \tag{4.3.23}$$

The map $s \mapsto K_0(S(s)u)$ belongs to $L^1(0,t)$, $\forall t > 0$, $\forall u \in H^2(\mathbb{R})$. Taking into account (4.3.23) and the definition of K_0, we have that $K_0(S(s)S(t_j)u_j) \to K_0(S(s)w)$ as $j \to \infty$, for $s \in (0,t)$. Moreover, $s \mapsto K_0(S(s)S(t_j)u_j)$ is uniformly bounded on $\mathbb{R}^+$ thanks to the existence of a bounded absorbing set for $\{S(t)\}_{t\geqslant 0}$. Then, by the Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty}\int_0^t \mathrm{e}^{-2\gamma(t-s)}K_0(S(s)\,S(t_j)u_j)\,\mathrm{d}s = \int_0^t \mathrm{e}^{-2\gamma(t-s)}K_0(S(s)w)\,\mathrm{d}s, \qquad \forall t > 0,$$

so that (3.5) holds true.

Consider now $\{u_j\}_j$ bounded in $H^2(\mathbb{R})$, and $\{t_j\}_j \subset \mathbb{R}^+$ with $t_j \to \infty$. Since the semigroup $\{S(t)\}_{t\geqslant 0}$ has a bounded absorbing set in $H^2(\mathbb{R})$, we deduce that there exists a subsequence $\{j'\}$ such that

$$S(t_{j'})u_{j'} \rightharpoonup w \text{ weakly in } H^2(\mathbb{R}), \tag{4.3.24}$$

for some $w \in H^2(\mathbb{R})$. Now, we apply lemma 3.2 to deduce (passing to a further subsequence and then using a contradiction argument) that

$$S(t_{j'})u_{j'} \to w \text{ strongly in } L^2(\mathbb{R}). \tag{4.3.25}$$

By interpolation, we finally deduce from (4.3.24) and (4.3.25) that

$$S(t_{j'})u_{j'} \to w \text{ strongly in } H^1(\mathbb{R}), \tag{4.3.26}$$

which completes the proof of the lemma. □

We now apply theorem 3.1 with the energy equation (4.3.4) with $m = 2$. In the notation of theorem 3.1, the terms of the energy equation are

$$\Phi(u) = \int u_{xx}^2 \,\mathrm{d}x, \qquad J(u) = \int [-\tfrac{5}{3}u_x^2 + \tfrac{5}{36}u^4]\,\mathrm{d}x, \qquad K(u) = K_2(u), \qquad L(u) = 0.$$

Consider again $\{u_j\}_j$ bounded in $H^2(\mathbb{R})$ and $\{t_j\}_j \subset \mathbb{R}^+$, $t_j \to \infty$ such that $S(t_j)u_j \rightharpoonup w$ weakly in $H^2(\mathbb{R})$. Then using lemma 4.3.4 there exists a subsequence $\{j'\}$ such that

$$S(t_{j'})u_{j'} \to w \text{ strongly in } H^1(\mathbb{R}),$$

and, by a contradiction argument, the whole sequence convergences to w strongly in $H^1(\mathbb{R})$. Then, (3.3) and (3.4) are trivially satisfied, and for (3.5) we use the same arguments as in lemma 4.3.4. Thus, we can apply theorem 3.1 to conclude that $\{S(t)\}_{t\geqslant 0}$ possesses a global attractor in $H^2(\mathbb{R})$.

Theorem 4.3.5. *Let $\gamma > 0$ and f in $H^2(\mathbb{R})$. Then the semigroup $\{S(t)\}_{t\geqslant 0}$ possesses a global attractor $\mathcal{A}$ in $H^2(\mathbb{R})$.*

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